Oblivious Single Access Machines are Concretely Efficient

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Abstract

Oblivious algorithms allow a space-constrained client program to securely outsource storage to an untrusted server. Any program can be compiled to an oblivious form via Oblivious RAM (ORAM), but this is asymptotically and concretely expensive. Recent work (Appan et al., CCS'24) proposed a weakening of ORAM called Oblivious Single Access Machine (OSAM), which offers asymptotically-improved oblivious compilation for many programs, including those that manipulate graph data structures. While of theoretical interest, OSAM graph algorithms were worse than generic ORAM, even for large graphs (tested on graphs of size up to 2^{25}).

This work improves the concrete costs of OSAM-based oblivious algorithms. In short, the original work on OSAM proposed algorithms for manipulating objects with *pointers* to other objects, but their management of pointers involves non-trivial and concretely-expensive algorithms. Our work greatly simplifies and improves the efficiency of OSAM-based pointer handling by co-designing (1) pointer-friendly modifications to the underlying Path ORAM algorithm and (2) new algorithms for managing pointers and building graphs from pointers.

Our work provides generic and easy-to-use oblivious tools with concretely better performance than state-of-the-art generic tools. Natural graph algorithms can now be automatically compiled to an oblivious form while enjoying up to a 4× improvement in performance as compared to generic Path ORAM (and at least 8× as compared to the original OSAM).

1 Introduction

Oblivious data structures and algorithms are powerful tools that enable a space-constrained client to securely outsource its storage to an untrusted server. These tools are the workhorse underlying many privacy-preserving applications, especially those that involve non-trivial quantities of structured data. Oblivious data structures have been leveraged in the design of symmetric searchable encryption [GMP16, FVY$^+$17], secure processor design [RFK$^+$17], and outsourced secure computation [GGH$^+$13, HHWW19].

The key technical challenge in achieving oblivious data structures/algorithms is the need to obfuscate the client program's *access pattern*. The order in which the client uploads/downloads memory slots should reveal nothing about the client's data. Roughly, this is achieved by accessing data in some data-independent pattern, or by rearranging data to make access patterns *appear* data-independent.

Oblivious RAM. *Any* program can be automatically made oblivious by means of an Oblivious RAM (ORAM) compiler [GO96]. An ORAM compiler serves as a virtual layer between the client's program and the server, translating each of the client's *logical* memory queries into *physical* data-oblivious queries that are

sent to the server. The generic nature of this approach makes ORAM compilers both theoretically interesting and of considerable practical value [LHS+14, LWN+15a, ZDB+17].

Unfortunately, ORAM compilers incur a high cost. In particular, they can incur (1) a high number of roundtrips between client and server and (2) a high *blow-up*, which is defined as the ratio between the amount of data entering/leaving the client's cleartext program and the amount of data entering/leaving the same program after ORAM compilation. As one representative example, for a memory of size N, the influential Path ORAM incurs $O(\log^2 N)$ blow-up and $O(\log N)$ roundtrips per access [SvDS+13].

Since fully generic compilers incur a high cost, it becomes natural to ask: Can we design efficient compilers targeted at *certain useful classes* of client programs?

Oblivious SAM. *Oblivious single access machine* (OSAM) compilers were one such proposal [AHR24]. An OSAM compiler works for any client program that writes to each memory slot at most once and reads from each memory slot at most once. Appan et al. [AHR24] showed that an OSAM compiler obviates the need for Path ORAM recursion, improving both bandwidth blow-up and roundtrips by a logarithmic factor to $O(\log N)$ and $O(1)$, respectively.

This single access restriction may seem prohibitive, but one can introduce additional non-cryptographic compilers that further translate client code from some easy-to-use source language to the restrictive OSAM interface. Appan et al. [AHR24] showed that arbitrary *pointer machine* programs—programs that manipulate objects and pointers to those objects, but that do not support random access arrays—can be automatically made oblivious with asymptotically lower cost than via ORAM. Since pointers and objects naturally represent directed graphs, this formalism and its improved cost led to various state-of-the-art oblivious graph algorithms. Formally, each time a pointer is dereferenced, [AHR24]'s approach incurs $O(\log N \log d)$ blow-up and $O(\log d)$ roundtrips per pointer dereference, where d is the number of existing pointers to the dereferenced object. When $d \ll N$, this can yield asymptotic improvement as compared to ORAM.

The OSAM formalism is thus of theoretical value, but its practical value remained unclear. The problem is that the second layer of compilation—from pointers/objects to SAM-friendly operations—incurs high constant factors. Since SAM memory cells can only be read once, when the client dereferences a pointer, the compiler must save the retrieved object back to a fresh memory cell such that it can be dereferenced again later. This becomes expensive when a program generates two or more pointers to the same object. When an object is moved to a new memory cell, the compiler must somehow "alert" all pointers as to the identity of that new memory cell. In general, these pointers might themselves be stored in objects on the server, which makes handling such alerts relatively difficult.

The difficulty of arranging such alerts is the source of [AHR24]'s $\log d$ scaling and large constant factors (we discuss them in detail in §2). Looking ahead to §7, we implemented and tested the original OSAM compiler and were *not able to find a single graph algorithm configuration where OSAM outperformed the standard Path ORAM.* Further techniques are needed to extract concrete improvements from OSAM's asymptotic advantage.

1.1 Contribution

We develop and implement improved techniques for achieving OSAM. The total effect of our nontrivial optimizations makes OSAM superior to Path ORAM for natural graph processing problems.

Improved Algorithms. We relax the SAM interface to allow programs that write memory cells more than once (we still enforce at most one read per memory cell). This relaxed version of SAM with multiple writes is called SAM+. We modify Path OSAM (OSAM instantiated with non-recursive Path ORAM) into Path OSAM+ that supports *multiple writes*. While this modification is small, the proof that the resulting OSAM does not "overflow" is nontrivial.

Equipped with the multiple-write capabilities, we design greatly simplified and improved algorithms for managing shared pointers. Our new algorithms are based on a modification of the splay tree data structure [ST85]. In addition, we redesign graph emulation algorithms from pointers.

One of the main concrete inefficiencies of [AHR24]'s handling of pointers is that the approach eagerly creates *copies* of pointers, which can be expensive in the OSAM paradigm. We develop a new "caching" technique that substantially reduces the number of copies created.

Oblivious graph library. To provide evidence of the utility of our improvements, we provide a generic

oblivious graph library written in Python and a Rust implementation of Path OSAM$^+$ (§ 3). The graph library is intentionally written in a high-level language to free programmers from having to interface with cryptographic algorithms. Our entire graph library is built through a pointer machine interface and contains several common algorithms. Specifically, we implement BFS, DFS, Dijkstra's, Prim's, Random Walk, PageRank, Contact Discovery, and Directed Triangle Count. These algorithms are augmented by standard helper procedures. None of our algorithms perform any task-specific optimizations.

With our OSAM compiler, a developer can write algorithms in a standard pointer-based style, and our compiler will automatically convert them into oblivious ones. The compiled program enjoys asymptotic *and concrete* performance improvement as compared to the same program compiled with a baseline ORAM compiler. Our experiments on graphs up to size $\approx 2^{25}$ show that our OSAM-based compilation can be up to 4× more efficient in blowup and roundtrips than the Path ORAM baseline, and up to 8× more efficient than the original OSAM.

1.2 Related Work

Any ORAM compiler must incur at least $\Omega(\log N)$ bandwidth blow-up [GO96, LN18], and popular tree-based ORAMs incur $O(\log^2 N)$ blow-up and $O(\log N)$ roundtrips per access [SvDS$^+$13]. The theoretically-remarkable OptORAMa construction [AKL$^+$20] achieves optimal $\Theta(\log N)$ blow-up, but it either involves practically-infeasible constant factors [DO20] or that the client keep very large (though still sublinear) local state [AKM23], and it still incurs $O(\log N)$ roundtrips per access.

Because ORAM is costly, the literature has considered special-purpose handling for stacks, queues, search trees, and priority queues [ZE13, WNL$^+$14, Shi20, JLS21]. OSAM generalized many such structures by capturing them as SAM programs and using simplified tree-based ORAM to achieve matching asymptotics [AHR24]. As discussed, OSAM also supports general pointer machine programs.

Perhaps OSAM's most attractive feature is its ability to compile graph algorithms. Prior works on oblivious graph algorithms took non-generic approaches such as using trusted hardware enclaves [ZDB$^+$17, CDP$^+$24], focusing on the multiparty computation setting [KS14, LWN$^+$15b, Ost24], and/or specializing to specific algorithms [CK10, MKNK15, GKT21, FGPT24, ZPZP24].

2 Technical Overview

This section sketches our improvements to OSAM and its handling of pointer-based programs, as compared to [AHR24]. We will describe our constructions as programs in the SAM and SAM$^+$ models. As we will see in §3, such programs can be made oblivious efficiently.

[AHR24]'s SAM program that implements pointer handling consists of three parts: (1) a solution that allows a constant number of pointers to share a value, (2) an inverted pointer tree that builds on the constant-pointer solution to allow more pointers to share the same value, and (3) a pointer interface that allows the client to write natural-looking pointer programs that are *valid* SAM programs.

The constant factors incurred in each part of [AHR24]'s solution *multiply* to produce the final constant factor. Our improvements bring down the final constant factor by roughly 8×, resulting in 4× improvement over Path ORAM. We next describe [AHR24]'s solution for each part, point out sources of inefficiency, and describe how we address them.

2.1 Sharing Constant Numbers of Pointers

To understand [AHR24] SAM pointer management, we start with a simple case where each value in SAM memory has only a single pointer pointing to it (i.e., no shared pointers), such as when implementing a tree. In this case, each pointer is a raw SAM address, which can be dereferenced by reading this address. Due to restrictions imposed by SAM, this address cannot be read a second time. To ensure the same pointer can be dereferenced again, we can write the value to a new address and update the pointer to point to this new address [WNL$^+$14]. (This section measures pointer cost in terms of SAM/SAM$^+$ operations, which both incur factors of 1 for roundtrips and $\log N$ for bandwidth blowup.)

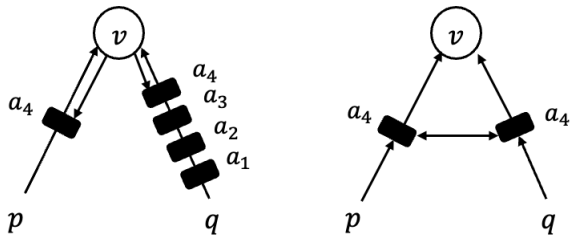


Figure 1: The constant-pointer solution of [AHR24] (on the left) compared with ours (on the right) when a pointer p is dereferenced three times. [AHR24] uses an address queue to store *all* addresses that v was written to since the last time q was dereferenced. In our solution, p and q point to locations that store only the *latest* address of v.

The difficulty of shared pointers. When *multiple* pointers point to the same value, the problem becomes more challenging because the two pointers must somehow coordinate. Consider two pointers p and q both pointing to value v. If p and q are simply the same raw SAM address, then when the program dereferences p, q is invalidated. To ensure p and q both remain valid after one is dereferenced, more work is needed.

[AHR24] first uses SAM to implement a *queue*. This queue is characterized by two addresses: head and tail. The client can use tail to add things to the queue, and use head to read things from the queue. Let us again consider two pointers p and q that point to the same value v. p and q hold the heads of queues that store a_1, the address of v. The value v stores the tails of these queues. Each queue stores *all* the addresses that v had been written to since the last time the pointer was dereferenced. For instance, suppose that v is fetched by dereferencing p three times. Each time p is dereferenced, v is written to a new address, and the client appends this address to q's queue using the tail stored in v. When q is dereferenced, the head stored at q is used to read all four addresses from q's queue, and the last (and latest) address in the queue is used to fetch v. Figure 1 (left) illustrates this process.

Queues allow p and q to share access to v while obeying SAM restrictions. Appending to a queue writes a SAM address; chasing through the queue eventually yields the v's current address. By implementing queues carefully, one can arrange that no SAM address is ever written to/read from more than once.

SAM with multiple writes. We eliminate the need for address queues by *relaxing* the SAM model into the SAM$^+$ model. Instead of enforcing that each address be read from *and* written to at most once, we only enforce single read and allow *multiple writes* to the same address. We adapt the underlying OSAM compiler into an OSAM$^+$ compiler that can support multiple writes. The resulting compiler is called Path OSAM$^+$. When a value is rewritten to the same address, the Path OSAM$^+$ server stores not only the latest version of the value, but also stale versions. Path OSAM$^+$ ensures that stale versions are eventually overwritten by later versions.

Our key innovation in designing Path OSAM$^+$ is not the construction itself. Rather, it is showing that, despite this change, client storage does not grow too large (i.e., the stash does not overflow, for those familiar with Path ORAM). The proof for Path ORAM crucially relies on the fact that each value is written to a *uniformly random* address. This no longer holds when multiple writes to the same address are allowed, so we must revisit the proof from the ground up. There are two main new steps in our proof: (1) show that the number of values with j versions on the server decreases exponentially in j in expectation, and (2) bound client storage via a Chernoff bound by carefully choosing random variables that are either independent or negatively correlated. §3 presents details of our construction and its proof. This technique has been used in the past with an incorrect proof [GKW18]. We show a counterexample to the prior proof in Appendix A.

An improved solution from multiple writes. With SAM$^+$, we can implement a two-pointer solution without growing queues. Each pointer holds the address of a location in server memory that stores (1) v's latest address and (2) the address of the location of the other pointer. The client can dereference a pointer, say p, by fetching and reading v's latest address from p's location. This requires reading p's location, and also reading v. Due to single read constraints, the client writes v to a new address, and stores this new address at a new location using which p can be dereferenced later. To ensure that q can also be dereferenced

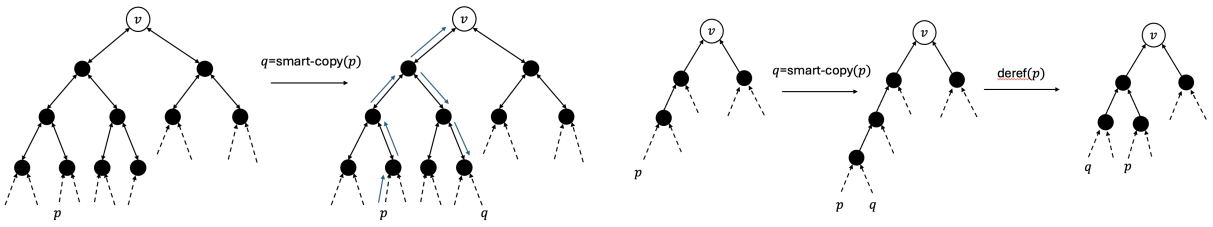


Figure 2: [AHR24]'s smart copy (left) compared with ours (right). [AHR24]'s copy adds a new pointer to the rightmost leaf of the tree by traversing up and down the tree. Our copy simply extends the path to the pointer's leaf.

to fetch v, the address of q's location (which was written in p's location) is updated to hold (1) the new address of v and (2) the address of p's new location. This is possible only because we can write *multiple* values to q's (and p's) location. Figure 1 illustrates (right) illustrates this process.

2.2 Handling Multiple Shared Pointers

[AHR24] generalizes their 2-pointer solution to d pointers by building an "inverted" binary-tree storing v at the root: pointers point to leaves of this tree, and each tree node points to its parent. Hence, each node has only a constant number of incoming pointers. The value v can be fetched by chasing pointers from leaf to root.

Pointers are copied by adding a new pointer at the *rightmost* leaf of the tree. This is done by traversing to the root, then traversing down the tree to the right. This ensures the tree is always balanced, guaranteeing that each pointer operation can be performed using $O(\log d)$ SAM operations. To differentiate this operation from a verbatim (bitwise) copy of a pointer, [AHR24] call it a *smart copy*.

This implementation of smart copy increases cost in two ways. First, it requires each node to additionally store pointers to their children. Second, it requires the tree to be traversed *twice:* once up (to fetch the root), and once down (to add a new pointer). We reduce both of these costs.

Our smart copy. In the SAM$^+$ setting, a simple way to smart copy a pointer that points to a leaf is to extend the path from the root to that leaf. To do this, the client can create a new leaf that points to the old leaf, create a new pointer that points to this new leaf, and make the original pointer also point to this new leaf. When implemented this way, smart copy now incurs only $O(1)$ SAM$^+$ operations. On the other hand, paths in the inverted tree may be *unbalanced*. Dereferencing a pointer, which requires traversing a path in this tree, could require $O(d)$ SAM$^+$ operations.

We reduce the cost of performing a pointer dereference by *amortizing* its cost over the copy operations. Our solution is heavily inspired by *splay trees* [ST85]. A splay tree is a binary search tree in which paths may be unbalanced. On traversing the path to a node, the splay tree balances this path by tree rotations, similar to other structures such as the famous AVL tree. By the potential method ([CLRS22, Section 17.3]), the *amortized cost* of each splay tree operation can be shown to be $O(\log d)$ SAM$^+$ operations.

When the client dereferences a pointer, the client traverses a path from leaf to root, and rebuilds that path, performing tree rotations to better balance the tree. Unsurprisingly, we can adapt the amortized analysis used for splay trees to show that this yields $O(\log d)$ SAM$^+$ operations. In our rotation-based strategy, the client touches only those nodes along the path it anyway needed to traverse, a far simpler strategy than the double traversal on copy of [AHR24]. These improvements compounds with our removal of queues: in the [AHR24] solution, traversing an edge in a pointer tree involved chasing down a queue of pointers; in ours, it is a single address read. § 4 presents full details of our splay-tree-based solution.

2.3 Avoiding Smart Copies

When a pointer machine program dereferences a value, the dereferenced value might itself contain further pointers, such as when implementing a data structure. When this happens, [AHR24] cautiously *smart copies* the dereferenced value, which smart copies all contained pointers, then writes one smart copy of the value

back to server. This is done to ensure no two identical copies of a pointer reside anywhere in the system. If this were not enforced, it might be possible for a client's pointer machine program to (1) read a value x from memory, (2) dereference a pointer p in x, (3) read x from memory a second time, and (4) dereference the identical pointer p a second time. This would yield two reads to the same SAM address, which is disallowed by SAM (and SAM$^+$). Even with our improved smart copy operation, this design is expensive, as the *amortized cost* of smart copy is $O(\log d)$ SAM$^+$ operations.

In many cases, the above strategy is overly pessimistic. As one example, consider a pointer machine program that traverses a single path through some d-ary tree. Each time a node `node` is accessed, the above strategy cautiously copies `node`'s child pointers, then writes `node` back to the server, just in case the above error case might arise. But because we know that the program will simply read `node`, then read one of its children recursively, this caution is unnecessary and wasteful. Indeed, $d-1$ of the pointers in `node` need not be smart copied.

We instead propose a solution that fetches values to client memory *without* smart copying. We do this by introducing a notion of so-called *moves*. When dereferencing a pointer, a client program can *move* the value from the server, which involves a simple pointer dereference without a smart copy. The result of moving from a pointer is (1) the dereferenced value v and (2) a write-back SAM address where the client program should write a new version v (this write-back address is the new root of an inverted pointer tree). By moving from a pointer p, the client program takes on a promise that it will *not* dereference v a second time until it itself writes a new version back. With some care, the client program can therefore minimize the number of smart copies that are created. This optimization is particularly important for local algorithms which often access only a single neighborhood of nodes.

We automate away this promise of not deferencing an object a second time by introducing a client-side cache whose size is proportional to the number of local variables in the client program that automatically handles moves. §5 presents more details.

3 OSAM with Multiple Writes

The SAM model restricts each address to be written to at most once *and* read from at most once. In this section, we show OSAM can be modified to handle a stronger version of SAM that allows an address to be written to *multiple* times. We restrict each address to be read at most once.

Definition 1 (Single Access Machine$^+$ (SAM$^+$))**.** *A* ***Single Access Machine***$^+$ *(SAM*$^+$*) (with multi-write) is an algorithm implementing a memory storing polynomial number of addressable memory cells, each of some specified bit-width w. The machine responds to three types of memory requests:*

- `addr` $\leftarrow$ `Alloc`()*: The machine responds with a fresh memory address (i.e., an address that has not been chosen before).*
- `Write`(`addr`, `val`)*: The machine writes value* `val` $\in \{0,1\}^w$ *to address* `addr`*. If* `addr` *was not allocated by the machine, then the machine instead halts and outputs $\perp$.*
- `val` $\leftarrow$ `Read`(`addr`)*: The machine responds with the latest value written to* `addr`*, or it responds with* `None` *if nothing is written. If (1)* `addr` *was not allocated or (2)* `addr` *was already read from, then the machine instead halts and outputs $\perp$.*

A ***SAM***$^+$ ***program*** *is an interactive, randomized algorithm that issues memory requests to the machine. A program is* ***valid*** *if it never issues a request that causes the machine to output $\perp$.*

An oblivious SAM$^+$ (OSAM$^+$) hides the addresses used in memory requests of any valid SAM program.

Definition 2 (Oblivious SAM$^+$ (OSAM$^+$))**.** *A SAM*$^+$ *compiler Π implements the SAM*$^+$ *interface by interacting with a random access memory. Π is an* ***oblivious*** *SAM*$^+$ *(OSAM*$^+$*) compiler if there exists a polynomial-time simulator* Sim *such that for any polynomial-length sequence of requests $\mathcal{R}$ made by a* **valid SAM$^+$ program***, the following ensembles are statistically close in the security parameter λ, $\Pi(1^\lambda, \mathcal{R}) \stackrel{s}{=} \mathsf{Sim}(1^\lambda, \mathcal{L}(\mathcal{R}))$. Above, $\mathcal{L}(\mathcal{R})$ denotes the number of* `Read`/`Write` *requests (i.e., non-*`Alloc` *requests).*

To state, informally, the requests issued by Π can be simulated given only the *total number* of `Read`/`Write` requests in the underlying SAM$^+$ program.

3.1 Path OSAM$^+$ Construction

We start by explaining the Path OSAM construction of [AHR24]. For readers familiar with Path ORAM [SvDS$^+$13], this is simply Path ORAM with deterministic order eviction, but without the position map.

Values are written to and read from *addresses* in the address space of the client's outsourced memory. Let N denote the size (in words) of the client's outsourced memory (we assume that N is a power of 2. The server memory is organized in a complete binary tree with N leaves. Each tree node contains a *bucket* that holds a constant number (Z) of values. The client locally stores a *stash* that contains values downloaded from the server.

Path OSAM allocates a new address by sampling a leaf uniformly at random. When the SAM program issues a request to write a value to an address, Path OSAM maintains the invariant that the value lies in some bucket on the path to the corresponding leaf. When the SAM program issues a read request for an address, Path OSAM fetches the value by downloading (to the stash) all buckets on the path towards that address's assigned leaf.

To hide addresses being written to, the client always writes values to the stash, and performs *eviction* operations that move values towards their assigned leaves. The eviction operation is performed on a *path* by (1) downloading the path (2) greedily filling buckets from the leaf to the root with values from the stash[1] (3) uploading the path back to server memory. A variant of Path OSAM [RFK$^+$15] evicts paths in a deterministic *reverse-lexicographic* order.[2] This allows the server's view to be simulated: `Write`s and `Read`s can be simulated by reading a uniformly random path, then evicting a deterministic path.

Values may overflow from server memory, and the client stores these values in the stash. Path ORAM (and our Path OSAM$^+$) ensures that a stash of size $S = O(\lambda)$ will not overflow, except with probability negligible in λ [SvDS$^+$13, RFK$^+$15].

Modifying eviction to allow multiple writes. The Path OSAM construction of [AHR24] does not reveal addresses during `Write`, so it is oblivious even when multiple writes are allowed. What is not clear is whether *correctness* holds with multiple writes to the same address. This is because existing stash size analysis assumes that each write picks a leaf *uniformly at random*, which does not hold when multiple writes are allowed. We show that the stash size remains bounded when multiple writes are allowed.

First, we make a few natural modifications to the eviction algorithm of Path OSAM to support multiple writes. Notice that now, a path may contain *multiple* values written to the *same* address. When any path is evicted/read, our modified Path OSAM$^+$ *deletes from that path stale* values written to the same address, and only keeps the *latest* value written to that address. When this happens, we say that Path OSAM$^+$ *merges* values repeatedly written to the same address. This does not affect correctness, since older versions of these values are no longer useful.

[GKW18] claim by the same reasoning that multiple versions of a value does not impact stash size. However, their argument overlooks the fact that before an overwritten value is deleted, it might have prevented other values from moving down the ORAM tree, which could increase stash size. We explain in more detail why their claim is incorrect in Appendix A.

We present our Path OSAM$^+$ construction in Figure 3. `Read` and `Alloc` are exactly the same as the Path OSAM of [AHR24]. Specifically, `Alloc` allocates an address by sampling a leaf uniformly at random. `Read`, given an address, reads the path to the corresponding leaf.

We modify `Write` and `Evict` to handle multiple writes. While writing a value to the stash, we increment `ctr` and write `ctr` along with the value. This identifies the latest version of a value. `Evict` reads values on a path to the stash, *merges* values in the stash to keep only the latest value written to each address, and then as before, writes back values as deep as possible from the leaf to the root. `Write` performs a read to a random dummy address to prevent the construction from leaking whether the program is performing a read or a write operation.

[1] A value is placed in a bucket if the bucket lies on the path to the value's assigned leaf.

[2] The reverse-lexicographic order [GGH$^+$13] counts from 0 to N, but reverses each number's bits. For instance, if $N = 7$, this would be $0, 4, 2, 6, 1, 5, 3, 7$

```
evictCtr ← 0
stash ← empty-list

def Alloc( ) → address :
    leaf ←$ [N]                    // Uniformly sample a leaf
    return leaf

def Read(a : address) → val | None :
    v ← ReadAndRm(a)               // None if no such address
      written to
    Evict( )
    return v

def Write(a : address, v : val) :
    ReadAndRm(Alloc())             // Read a dummy address
    stash ← stash \ {{a, _, _} ∈ stash}   // Remove any
      old version of a if it exists
    stash.append({a, 0, v})
    Evict( )

def ReadAndRm(a : address) → val | None :
    interpret a as leaf
    // Load the path to leaf from server along with the
      level it was found, perform merge, then search the
      stash for the element labeled with a; see [SvDS+13]

def Merge(stash) :
    M ← { }
    for {a, ℓ, v} ∈ stash do
        if a ∈ M then
            ℓ', v' ← M[a]
            if ℓ < ℓ' then M[a] ← {ℓ, v}     // Keep the
              copy closer to the root
        else M[a] ← {ℓ, v}
    stash ← empty-list
    for a ∈ M do
        stash.append(M[a])

def Evict( ) :
    ℓ ← bitReverse(evictCtr mod 2^L)
    evictCtr ← evictCtr + 1
    for j ← 0 to L do
        stash ← stash ∪ ReadBucket(P(ℓ, j), j)
    Merge(stash)
    for j ← L to 0 do
        WriteBucket(P(ℓ, j), stash)
```

Figure 3: Path OSAM with multiple writes. $\mathcal{P}(\ell, i)$ denotes the bucket at level i on the path to leaf ℓ.

Lemma 1 (Latest version is closest to the root)**.** *Suppose multiple versions of the same address a are present in the union of the stash and the tree. After any* `ReadAndRm` *call the latest version of a is returned.*

Proof. First note that the path an item may reside in is fixed. Consider the path eviction that inserts the newer version into the tree. If the newer version were stored strictly below the older version, then the two versions would lie on the same path that was evicted before writing, and thus the two items would be merged, deleting the older one. Therefore, whenever both versions exist in the tree at the same time, the new version cannot be deeper than the older version. Since `Merge` always keeps the copy with smaller depth, the surviving version of i is the latest one and is closest to the root. □

3.2 Stash Size Analysis

Theorem 1. *In Path OSAM$^+$ with multiple writes and when bucket size $Z \geq 15$ and total memory size $N \geq 16$, the probability that the number of values in the stash exceeds S is at most* $\exp(-O(S))$.

Our formal analysis confirms that constant-sized buckets are sufficient. The analysis is somewhat loose, requiring a value of $Z \geq 15$. In practice, we and prior work [GKW18] never observed multiple writes hurting stash size at all, and the same value of Z (e.g., $Z = 4$) from Ring ORAM works experimentally.

Our proof follows the structure of the proof presented in [RFK$^+$15] for a variant of Path ORAM called Ring ORAM. Consider an imaginary Path OSAM$^+$ with *infinite* bucket capacity, denoted $\texttt{OSAM}^\infty$. Let T be a rooted subtree, $n(T)$ be the number of nodes in T, and $X(T)$ be the number of values in T. The proof consists of two steps. (1) Prove that the stash of Path OSAM$^+$ overflows only if $X(T)$ for some rooted subtree T in $\texttt{OSAM}^\infty$ exceeds $n(T) \cdot Z + S$. (2) Derive a Chernoff-style bound on $\Pr[X(T) > n(T) \cdot Z + S]$ in $\texttt{OSAM}^\infty$. Our proof for step (1) follows largely from [RFK$^+$15] (after minor modifications). Our proof for step (2) requires a new analysis. This is because step (2) of [RFK$^+$15] assumes that each value is written to a uniformly random leaf, which no longer holds as we allow multiple writes to reuse the same leaf.

Our extension to the Ring ORAM proof is non-trivial. In fact, a prior work [GKW18] erroneously claimed that the proof with multiple writes is immediate, in the sense that overwritten values in the ORAM tree cannot harm stash size, since those overwritten elements will be eventually deleted anyway. However, their

argument overlooks the fact that before an overwritten value is deleted, it might have prevented other values from moving down the ORAM tree, which could increase stash size. We explain in detail why the proof of [GKW18] fails in Appendix A.

Our proof instead carefully accounts for each possible number of copies of each value and shows that eviction succeeds even in the presence of multiple writes. Note that our proof retroactively shows the correctness of the construction of [GKW18], which may be of independent interest.

1. When a post-processing procedure G is performed on $\texttt{OSAM}^\infty$, after any sequence of accesses, the stash size of $G(\texttt{OSAM}^\infty)$ equals the stash size of $\texttt{OSAM}^Z$;
2. The stash size of $G(\texttt{OSAM}^\infty)$ exceeds S only if the "occupancy" of some rooted subtree of $\texttt{OSAM}^\infty$ exceeds its capacity by S; and
3. The probability that the occupancy of some rooted subtree of $\texttt{OSAM}^\infty$ exceeds its capacity by S is exponentially small.

We now proceed to introduce our new notation and then present the formal claims that lead to Theorem 1.

Notation. We label buckets linearly with the children of bucket b_i being b_{2i} and b_{2i+1}. b_1 is the root bucket and b_0 represents the stash. b_i^∞ and b_i^Z denote the i-th buckets in $\texttt{OSAM}^\infty$ and $\texttt{OSAM}^Z$. L denotes the number of levels in the Path OSAM tree. We use $\texttt{val}_a^t$ to denote a value written to address a during query t. $\texttt{val}_a^{t_1}, \texttt{val}_a^{t_2}, \texttt{val}_a^{t_3}, \ldots$ represent values repeatedly written to the same address a.

O denotes an OSAM *state* after a sequence of accesses: this is the state of all buckets and the stash in Path OSAM, including the values contained in each bucket / stash. We use O^∞ to denote a state of $\texttt{OSAM}^\infty$ and O^Z to denote a state of $\texttt{OSAM}^Z$. We consider OSAM states after a sequence of m accesses. $G_{O_Z}(O_\infty)$ is a post-processing procedure that takes as input states O_Z and O_∞, and outputs an OSAM state. As we show below, the output of G is a valid O^Z state. The function $\texttt{st}(O)$ denotes the stash size for OSAM state O.

For any sub-tree $T \in \texttt{OSAM}^\infty$, n denotes the number of nodes in T, and $c(T) = n \cdot Z$ denotes its capacity in $\texttt{OSAM}^Z$. We use $X(T)$ to denote T's occupancy, i.e, the number of values stored in T.

Equivalence between $G(\texttt{OSAM}^\infty)$ and $\texttt{OSAM}^Z$. When a path is evicted with multiple values written to the same address, older versions of the value get deleted, and only the latest version remains. After any sequence of access requests, if a value is in some bucket of O^Z, then the value must be also be in some bucket of O^∞. This is because if a value $\texttt{val}_a^t$ is replaced by a later value $\texttt{val}_a^{t'}$ $(t' > t)$ in O^∞, then it is also replaced in O^Z, since a finite bucket capacity does not prevent merging.

Lemma 2. *Let O^∞ and O^Z be the states of $\texttt{OSAM}^\infty$ and $\texttt{OSAM}^Z$ after an access sequence $\texttt{val}_{a_1}^1, \texttt{val}_{a_2}^2, \texttt{val}_{a_3}^3, \ldots, \texttt{val}_{a_m}^m$. If a value $\texttt{val}_a^t$ is not in any bucket of O^∞, then $\texttt{val}_a^t$ is also not in any bucket of O^Z.*

Proof. For contradiction, suppose that some $\texttt{val}_a^t$ is in O^Z but not in O^∞. This happens when another value $\texttt{val}_a^{t'}$, where $t' > t$, merges with $\texttt{val}_a^t$ in O^∞. When this happens, suppose that $\texttt{val}_a^t$ is present in O^∞ in bucket b_i^∞, and in O^Z in bucket b_j^Z that is an ancestor of b_i^Z. This means that some path passing through b_i^∞, and thereby through b_j^Z, was evicted. Since $t' > t$, $\texttt{val}_a^t$ is deleted in O^Z as well: a contradiction. □

By Lemma 2, all values present in some bucket of O^Z are also in O^∞. Note that there may still be values that are present in O^∞ but *not* present in O^Z. Our post-processing algorithm G (defined below) re-assigns blocks in buckets in O^∞ to match O^Z while *deleting* extra blocks in O^∞. More precisely, G processes each bucket b_i^∞ as follows for $i = 2^{L+1} - 1$ down to 1 (from leaves to root): for each value b_i^∞ that is 1) also in b_i^Z remains in b_i^∞, 2) in an ancestor of b_i^Z is moved to the parent $b_{i/2}^\infty$, and 3) not in any ancestor of b_i^Z is *deleted*.

$G_{O^Z}(O^\infty) = O^Z$ if each b_i^∞ after G contains the same blocks as b_i^Z for each $i = 0, 1, \ldots, 2^{L+1} - 1$. The below Lemma then follows from Lemma 2 and Lemma 1 of [RFK+15].

Lemma 3. *For the same access sequence and randomness, the post-processing procedure G ensures $G_{O^Z}(O^\infty) = O^Z$.*

Bounding stash size. With equivalence between O^Z and $G(O^\infty)$ established, we bound the stash size of any state O^Z by bounding the stash size of $G_{O^Z}(O^\infty)$ (denoted by $\texttt{st}(G_{O^Z}(O^\infty))$). A *rooted subtree* is a subtree $T \in \texttt{OSAM}^\infty$ that contains the root of $\texttt{OSAM}^\infty$. The Lemma below, which largely follows from [RFK+15], states that the stash of $G(O^\infty)$ overflows only if some rooted subtree of $\texttt{OSAM}^\infty$ (before post-processing) exceeds its capacity by S.

Lemma 4. *If* $\texttt{st}(G_{O^Z}(O^\infty)) > S$*, then* $\exists\ T \in \texttt{OSAM}^\infty$ *such that* $X(T) > c(T) + S$ *before post-processing.*

Proof. Consider the maximal rooted subtree T such that all buckets in T have exactly Z values after post-processing. We prove that all values in the stash of $G_{O^Z}(O^\infty)$ originate from T. For contradiction, suppose that some value originates from a bucket $b \notin T$. By the maximality of T, some bucket b' that is an ancestor of b (or b itself) contains less than Z values after post-processing, so no value from b can go to the stash of $G_{O^Z}(O^\infty)$.

Consider the values that occupy T before post-processing. These values have three outcomes: (1) deleted since they are not in O^Z, (2) remaining in T after post-processing, or (3) moved to the stash after post-processing. If $\texttt{st}(G_{O^Z}(O^\infty)) > S$, then the number of values of type (2) and (3) alone exceed the capacity of T by S. Hence, $X(T) > c(T) + S$ before post-processing. □

By Lemmas 3 and 4, we have

$$\begin{aligned}\Pr\big[\texttt{st}(O^Z > S)\big] &= \Pr[\texttt{st}(G_{O^Z}(O^\infty)) > S] \\ &\le \sum_{T \in \texttt{OSAM}^\infty} \Pr[X(T) > c(T) + S] \\ &\le \sum_{n=0}^{N} 4^n \cdot \Pr[X(T) > c(T) + S]\end{aligned}$$

The above follows from a union bound, and a bound on the n-th Catalan number (there are fewer than 4^n binary trees of size n).

We now bound $\Pr[X(T) > c(T) + S]$ by deriving a bound on the *moment generating function* $\mathbb{E}\big[e^{tX(T)}\big]$, where $t > 0$ is a parameter that we will fix later. Our proof relies on a Lemma that bounds the number of values that could *possibly* be written to some bucket b that belongs to T.

Lemma 5. *Consider some sequence* $A = \{\texttt{op}_i \mid \texttt{op}_i = \texttt{Write}(\texttt{addr}, \texttt{val})$ *or* $\texttt{Read}(\texttt{addr})\}$ *of* m *accesses. A value* $\texttt{val}$ *is a* **candidate write** *to bucket* b *of a subtree* T *of* $\texttt{ORAM}^\infty$ *if there exists some address* $\texttt{addr}$ *for which* v *belongs to* b *after* A *is applied, when deterministic reverse-lexicographic order eviction is used. The number of candidate writes to any bucket* $b \in O_\infty$ *at level* L *is upper bounded by* 2^L*.*

Proof. (From Lemma 3 of Ring ORAM [RFK+15]) For a bucket b_i at level L, we use t_1 and t_2 to denote the last time paths through b's left child (b_{2i}) and right child (b_{2i+1}) were evicted. Without loss of generality, assume that $t_1 < t_2$. Any value $\texttt{val}_a^t$ for which $t \le t_1$ will not be present in b as they will be in either b_{2i} or b_{2i+1}. Values with $t > t_2$ will not be in b since, after t_2, no path through b is evicted. Thus, any value $\texttt{val}_a^t$ present in b must have $t_1 \le t < t_2$. By deterministic reverse-lexicographic order eviction, there are at most 2^L such values since $t_2 - t_1 = 2^L$. □

We can bound $\mathbb{E}\big[e^{tX(T)}\big]$ by carefully choosing *random variables* that are independent of or negatively correlated with each other. Since we allow multiple writes, each value might appear more than once in any given subtree T. If a particular value i appears exactly j times in some subtree T, we say that i contributes *weight* j to T. We write w_i^j to denote a random variable that indicates whether or not value i contributes *weight* j to $X(T)$. Note that variables w_a^*, w_b^* for $a \ne b$ are independent random variables, and that w_i^j, w_i^k for $j \ne k$ are *negatively correlated*. In the latter case, if a block i appears exactly j times in T, then it cannot also appear exactly $k \ne j$ times.

We write $w^j = \sum_i w_i^j$ to denote the number of values appearing j times in T. We show that the expected number of values that contribute weight j to subtree T drops exponentially in j.

Lemma 6. *For all rooted subtree* $T \in \texttt{OSAM}^\infty$ *of size* n *and all* j*,* $\mathbb{E}\big[w^j\big] \le (3n+1)2^{-j}$*.*

Proof. We use $Y(b)$ to denote the number of (possibly non-contiguous) repeatedly written sequences of j values that *end* in bucket $b \in T$. For each such sequence, the j repeatedly written values could lie on any node on the path from the root to b. We use $Y(b, b')$ to denote the number of (possibly non-contiguous) repeated-write sequences of j values that start in bucket b' and end in bucket b.

Since b is a non-leaf bucket with infinite capacity, a value *remains* in b only if it can *not* go to the *child* on that evicted path, which happens with $\frac{1}{2^{L+1}}$ probability. This allows us to get a tighter bound on $\mathbb{E}[Y(b)]$:

$$\mathbb{E}[Y(b)] = \sum_{b'} \mathbb{E}[Y(b, b')] \leq \sum_{L'=0}^{L-j+1} \frac{2^{L'}}{2^{L+1}} \leq \frac{2^{L-j+2}}{2^{L+1}} = \frac{2}{2^j}.$$

Suppose that n_ℓ is the number of leaf buckets in T and $n_{n\ell}$ is the number of non-leaf buckets in T. We can bound $\mathbb{E}\big[w^j\big]$ using a union bound. Note that $n_\ell \leq (n+1)/2$.

$$\begin{aligned}
\mathbb{E}\big[w^j\big] &\leq n_\ell \cdot \frac{4}{2^j} + n_{n\ell} \cdot \frac{2}{2^j} \\
&\leq n_\ell \cdot \frac{2}{2^j} + (n_\ell + n_{n\ell}) \cdot \frac{2}{2^j} \\
&\leq \frac{n+1}{2} \cdot \frac{2}{2^j} + n \cdot \frac{2}{2^j} = \frac{3n+1}{2^j}
\end{aligned}$$

□

Using Lemma 6, we can apply a Chernoff-style bound to bound $\mathbb{E}\big[e^{tX(T)}\big]$ when $t = \ln(4/3)$.

Lemma 7. *After any sequence of accesses to* $\texttt{OSAM}^\infty$*, the following holds for any rooted subtree* $T \in \texttt{OSAM}^\infty$ *of size* n*, and for memory size* N*:*

$$\mathbb{E}\Big[e^{\ln\left(\frac{4}{3}\right)X(T)}\Big] < e^{n\cdot\left(\frac{8}{3}+\frac{3}{N}\right)+O(1)}$$

Proof. Let p_{ij} be the probability that values at an address i appear j times. Note that $X(T) = \sum_{j=1}^{\log N} j \cdot w^j$ and $w^j = \sum_i w_i^j$. We can use a Chernoff-style bound on $\mathbb{E}\big[e^{tX(T)}\big]$.

$$\begin{aligned}
&\mathbb{E}\Big[e^{tX(T)}\Big] = \mathbb{E}\Big[e^{\sum_i \sum_j tjw_i^j}\Big] = \mathbb{E}\left[\prod_{i,j} e^{tjw_i^j}\right] \\
&\leq \prod_{i,j} \mathbb{E}\Big[e^{tjw_i^j}\Big] \text{ (independence/negative correlation)} \\
&= \prod_{i,j} \left(p_{ij} \cdot e^{tj\cdot 1} + (1-p_{ij}) \cdot e^{tj\cdot 0}\right) \text{ (by defn. of } \mathbb{E}[\cdot]) \\
&= \prod_{i,j} \left(p_{ij}(e^{tj}-1)+1\right) \\
&\leq \prod_{i,j} e^{p_{ij}(e^{tj}-1)} \text{ (for all } x,\ x+1 \leq e^x) \\
&= \prod_j e^{(e^{tj}-1)\sum_i p_{ij}} = \prod_j e^{(e^{tj}-1)\,\mathbb{E}[w^j]} \text{ (by defn. of } \mathbb{E}\big[w^j\big]) \\
&= e^{\sum_{j=1}^{\log N}(e^{tj}-1)\,\mathbb{E}[w^j]} = e^{(e^t-1)\,\mathbb{E}[w^1]+\sum_{j=2}^{\log N}(e^{tj}-1)\,\mathbb{E}[w^j]}
\end{aligned}$$

By Lemma 3 of Ring ORAM [RFK+15], the expected weight of T for sequences of length 1, i.e., $\mathbb{E}\big[w^1\big]$,

is at most $n/2$. By Lemma 6, $\mathbb{E}\left[w^j\right] < \frac{3n+1}{2^j}$. Hence, we can bound the exponent above as follows:

$$\begin{aligned}&(e^t-1)\,\mathbb{E}\left[w^1\right] + \sum_{j=2}^{\log N}(e^{tj}-1)\,\mathbb{E}\left[w^j\right]\\ \leq&(e^t-1)\cdot\frac{n}{2} + \sum_{j=2}^{\log N}(e^{tj}-1)\cdot\frac{3n+1}{2^j}\end{aligned}$$

Plugging in $t = \ln(4/3)$, we have

$$\begin{aligned}\sum_{j=2}^{\log N}(e^{tj}-1)\cdot\frac{1}{2^j} &= \sum_{j=2}^{\log N}\left(\frac{2}{3}\right)^j - \sum_{j=2}^{\log N}\frac{1}{2^j}\\ &< \frac{4}{3} - (\frac{1}{2} - \frac{1}{N}) = \frac{5}{6} + \frac{1}{N}.\end{aligned}$$

The exponent from earlier is thus upper-bounded by:

$$\frac{n}{6} + \left(\frac{5}{6} + \frac{1}{N}\right)(3n+1) = \frac{8n}{3} + \frac{3n}{N} + \frac{5}{6} + \frac{1}{N}.$$

We can now bound $\mathbb{E}\left[e^{\ln(4/3)X(T)}\right] < e^{n\cdot(\frac{8}{3}+\frac{3}{N})+O(1)}$. This completes the proof of Lemma 7. □

We now return to the proof of Theorem 1. Now, we can bound $\Pr[X(T) > c(T) + S]$ as:

$$\begin{aligned}\Pr[X(T) > c(T)+S] &= \Pr\left[e^{tX(T)} > e^{t(nZ+S)}\right]\\ &\leq \mathbb{E}\left[e^{tX(T)}\right]\cdot e^{-t(nZ+S)}\end{aligned}$$

Setting $t = \ln(4/3)$:

$$\Pr[X(T) > c(T)+S] \leq \left(\frac{3}{4}\right)^S \cdot e^{O(1)} \cdot e^{-n\left(\ln\left(\frac{4}{3}\right)Z - \left(\frac{8}{3}+\frac{3}{N}\right)\right)}$$

If we choose Z such that $q = \ln\left(\frac{4}{3}\right)Z - (\frac{8}{3} + \frac{3}{N}) - \ln 4 > 0$, then we can bound the probability of stash overflow:

$$\Pr[\texttt{st}(O_Z) > S] \leq \left(\frac{3}{4}\right)^S \cdot e^{O(1)} \cdot \sum_{n\geq 1} e^{-nq} < \left(\frac{3}{4}\right)^S \cdot O(1)$$

$Z = 15$ suffices as long as $N \geq 2^4$. This completes the proof of Theorem 1.

4 Improved Pointer Operations

This section discusses how we implement the semantics of a standard pointer machine as a SAM$^+$ program. Namely, our handling makes it possible to create values, make pointers to those values, then copy, dereference and delete those pointers. Any such program can be made oblivious via our Path OSAM$^+$ construction. As discussed in §2, our solution relies on an inverted pointer tree that enables multiple pointers to point to a shared value.

Structure of the tree. Our inverted pointer tree contains the shared value at its root. Each node of the tree, except for the root, holds the address of its parent and its sibling. Hence, each item stored in SAM$^+$ memory is of one of two types: `item` ::= `Root(userT)` | `Inner(addr, addr)`. `userT` is any user-defined type for the value at the root. Each pointer points either to the `Root` node (if no other pointer points to the value) or to an `Inner` (leaf) node. A pointer can be dereferenced by chasing parents until the root is reached

```
def link(a : addr, b : addr, p : addr) :
    Write(a, Inner(b, p))
    Write(b, Inner(a, p))

def splay(a : addr, b : addr, p : addr) → (addr, userT) :
    switch Read(p) do
        case Root(v) do
            out ← Alloc()
            link(a, b, out)
            return (out, v)
        case Inner(c, p′) do
            switch Read(p′) do
                case Root(v) do
                    // zig splay and return
                    out ← Alloc(); mid ← Alloc()
                    link(b, c, mid); link(a, mid, out)
                    return (out, v)
                case Inner(d, p″) do
                    // zig-zag splay, and recurse
                    ℓ ← Alloc() ; r ← Alloc()
                    link(a, b, ℓ); link(c, d, r)
                    return splay(ℓ, r, p″)

def new(v : userT) → addr :
    a ← Alloc()
    Write(a, Root(v))
    return a

def copy(a : addr) → (addr, addr) :
    b ← Alloc()
    c ← Alloc()
    link(b, c, a)
    return (b, c)

def move(a : addr) → (addr, addr, userT) :
    switch Read(a) do
        case Root(v) do
            a′ ← Alloc()
            return (a′, a′, v)
        case Inner(s, p) do
            a′ ← Alloc()
            (root′, v) ← splay(a′, s, p)
            return (a′, root′, v)

def delete(a : addr) :
    switch Read(a) do
        case Root(v) do
            // do nothing
        case Inner(s, p) do
            switch Read(p) do
                case Root(v) do
                    Write(s, Root(v)
                case Inner(s′, p′) do
                    link(s, s′, p′)
```

Figure 4: Implementing pointer operations using SAM$^+$.

and the value can be fetched. However, addresses on this path, once read, cannot be read again, so nodes on this path must be re-written to new addresses, and the siblings of nodes on this path must be updated with these new addresses. This is why each node also stores the address of its sibling: when a node is re-written to a new address, the client (re-)writes to the address of the node's sibling (1) the node's updated address and (2) the updated address of the node's parent. Figure 5 illustrates.

Figure 4 presents a SAM$^+$ program that implements our inverted pointer tree. We next describe our construction in detail.

Creating the first pointer to a value. The `new` operation is used to write a value for the first time to server memory, and returns a pointer to this value. This is done by creating a `Root` item that contains the value, writing this value to an address in SAM$^+$ memory, and returning this address as the new pointer.

Linking addresses. Once a value is created, subsequent pointer operations re-structure the inverted pointer tree. `link` is a useful helper procedure that arranges two addresses a and b (of `Inner` nodes) as siblings that each point to some parent p.

Smart copying a pointer The `copy` operation takes a pointer (the address a of a node n) as input, and outputs two new addresses of `Inner` nodes: one is an updated address of the `Inner` node of the input pointer, and the other is an address of the `Inner` node of the pointer's copy. `copy` simply extends the path to the node n. This is done by creating two new addresses, and linking these with each other as siblings that point to the address of a. See Figure 2 in §2.

Deleting a pointer. `delete` takes as input a pointer (the address of a node n) and deletes this pointer by deleting n from the tree. `delete` does the opposite of `copy` by *truncating* the path to the pointer's leaf. Deleting n should not invalidate the pointer's sibling, so the pointer's sibling is linked with n's parent's sibling by making them both point to n's grandparent. Our implementations of `copy` and `delete` may cause tree paths to be $O(d)$ length in the worst-case, and cause the tree to become un-balanced. This makes

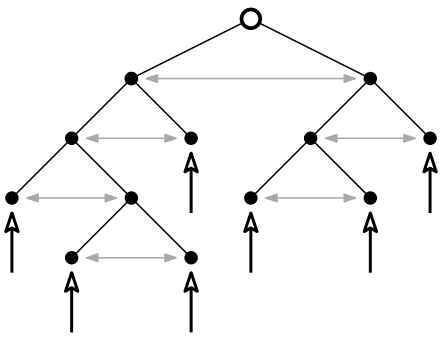


Figure 5: An illustration of our inverted splay tree. Pointers are addresses of *leaf* nodes of the tree.

pointer dereference costly. We bound the *amortized* cost of pointer dereference by borrowing techniques used in *splay* trees [ST85].

The `splay` operation `splay` traverses and balances a tree path, taking as input (1) the address p of a node n on this path and (2) addresses a and b of n's children, and outputs the value at the root along with a new address for this value. `splay` balances the path from n to the root by performing rotations[3] used in splay trees. These rotations can be performed by linking a node with the sibling of its parent or grandparent. We present more details in Appendix B.

Dereferencing a pointer. The dereference operation is called `move`, which takes as input the address of a node n, and outputs a new address for n, a new address for the root node, and the value at the root. Conceptually, `move` moves the pointed-to value from the server to client's local memory. When the client program invokes `move`, it assumes responsibility to write back (a possibly updated value) to the new address for n. The `move` operation is primarily a `splay` traversal.

The theorem below follows using the properties of splay trees. We present a formal proof in Appendix B.

Theorem 2. *Each pointer tree operation implemented by Figure 4 incurs amortized $O(\log d)$ SAM$^+$ operations, where d is the number of leaves of the pointer tree.*

5 Client-Side Cache

The main operation for dereferencing a pointer is `move`. As we have discussed, `move` is more efficient than the approach of [AHR24], which eagerly smart copied values on dereferences. The trade-off is that `move` imposes a burden on the programmer in the sense that they must pair the call to `move` with a call to `Write` on some write-back address. Failing to do so can result in illegal SAM$^+$ programs. While it is not too difficult to use `move` directly properly (indeed, our Python implementation does this), we present a mechanism to automatic this management using a constant-sized client-side *cache* and an associated programming model which manages pointer operations. The cache introduces a further level of indirection that allows the client to write natural pointer programs, properly inserting calls to `move`, `Write`, and `copy` only as needed. When the client program reads a value, that value is placed in and managed by the cache. Whenever the client program queries a SAM$^+$ address, it first checks if that SAM$^+$ address is in cache, and if it is, it need not send a query to the server. Because values in cache are on the client's local machine, *it is safe to repeatedly read the same SAM$^+$ address.*

The cache's programming model involves two types—`CacheVal` and `CachePtr`—which act as handles respectively to values and pointers. The client program is only allowed to refer to the value and its contents through `CacheVal`s and `CachePtr`s. As a result, the cache programming model *completely hides* values and pointers, and their management, from the programmer. This allows the client to locally use `CachePtr`s and `CacheVal`s, without needing to worry about invalidating the underlying SAM$^+$ program.

[3]We note that the splay tree uses zig-zag and zig-zig rotations in order to ensure that the resulting tree is a binary *search* tree. Since the inverted pointer tree is not used to perform search, we only use zig-zag, which is more efficient than zig-zig

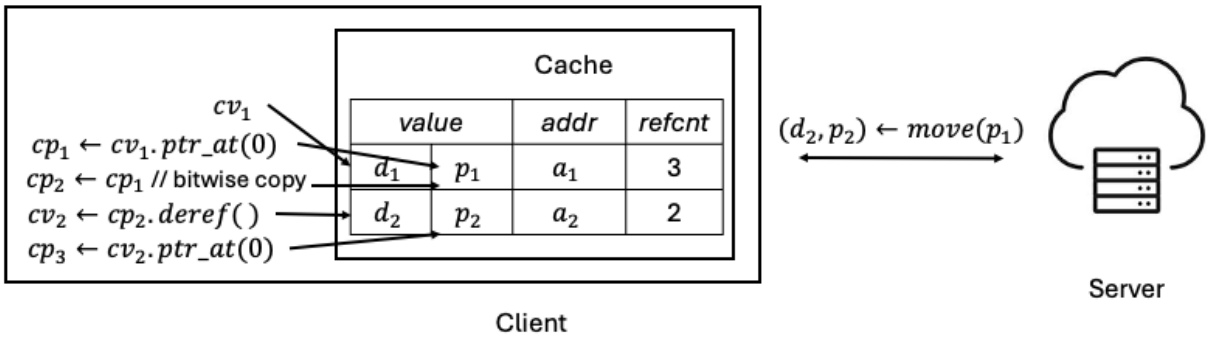


Figure 6: An illustration of how the cache works using a simple client-side program as an example. The client-program uses cache values ($cv_\star$) and cache pointers ($cp_\star$) to fetch values from the server. Operations performed on cache values and cache pointers are translated into pointer calls that result in legal SAM$^+$ programs.

More precisely, the cache is a map from SAM$^+$ addresses to values. Each value in cache also stores a *reference count*, which is the number of current handles referring to that value. This is the sum of the number of `CacheVal`s referencing the value, and the number of `CachePtr`s referencing pointers contained in the value. Figure 6 illustrates.

Dereferencing `CachePtr`s. When the client dereferences a `CachePtr`, the cache performs a `move` operation on the referenced pointer, with one small change: On each call to `Read`, we first check if the read address is already in cache. Once `move` reaches the root of the pointer tree, if the dereferenced value is not in cache, then we store it in cache, indexed by its write-back address, say a. If it is already in cache, then no change to the cache is made. Notice that the `move` operation, via `splay`, already arranges that the children of the root point to the write-back address a. Hence, when the client dereferences some other pointer to v and reaches the penultimate level of the inverted splay tree, it first checks if the cache stores a value mapped to a, and only fetches v from the server if there is no such mapping.

Copying `CachePtr`s. The client is also allowed to create bitwise copies (i.e., verbatim, not smart copies) of `CachePtr`s; such bitwise copies reference the same location in the cache and increment the reference count of the value stored in that location. Repeatedly dereferencing bitwise copies does not cause invalid SAM$^+$ programs as, each time a `CachePtr` is dereferenced, the *move* operation is called on an *updated* version of the referenced pointer in the cache.

Updating `CachePtr`s. The client can also use `CachePtr`s to update pointers in server memory. When the client updates a `CachePtr` p to some `CachePtr` q, the cache (1) *deletes* the referenced pointer and (2) updates the referenced pointer to be a *smart copy* of the pointer referenced by q. This causes the pointer referenced by p to point to the value that the pointer referenced by q points to.

Evicting values to server memory. A value is written back to server memory (using the address it is mapped to) when no references to it are alive in the client program. The cache checks this using the value's reference count. Each time a `CachePtr`/`CacheVal` falls out of scope or is deleted, the cache decrements the corresponding reference count. The cache writes the value to server memory when its reference count falls to 0. This limits cache size to the size of the client's program, since the number of unique `CachePtr`/`CacheVal` that reference values in the cache is upper bounded by the number of *variables* used in the client's program.

6 Oblivious Graph

This section describes our oblivious graph library that is written in Python and connected to our Path OSAM$^+$ implementation written in Rust which builds on Meta's implementation. As mentioned in the Introduction, our graph library just interfaces with cryptography through the pointer interface and the pointer methods described in Figure 4. This allows us to quickly do ablation experiments, implement

our OSAM$^+$, implement the [AHR24] version of OSAM, toggle move semantics, and directly implement a traditional ORAM program using a position map. The codebase is available in our GitHub repository titled `ConcreteOSAM`.

6.1 Framework

Given an arbitrary degree graph, referred to as the *original graph* that contains vertices and edges, we create a so-called *emulating graph*. The emulating graph is built on top of the pointer interface, and it is a directed graph of constant out-degree. Our emulating graphs consist of three types of objects as nodes in the graph: *vertices*, *edges*, and *fanout nodes*.

1. A vertex node is a vertex in the original graph.
2. An edge $u \to v$ in the original graph is emulated by an edge node that contains the edge weight and a pointer to the vertex v.
3. Ideally, vertices would have pointers to all of their edges. The emulating graph has a constant out-degree bf. To handle arbitrary degree original graphs, edges out of each vertex are managed as trees with branching factor bf. We call the non-leaf nodes in this graph *fanout nodes* and the tree as an *edge tree*.

The edges in the emulated graph are managed by pointers between the objects, as discussed in Section 4. Edges and fanout nodes have a fixed in-degree of 1. We build the emulating graph as follows:

1. Each vertex is assigned a unique ID, which are consecutive integers starting from 0. We create two oblivious maps that map names u to vertex IDs id, and from vertex IDs id to a pointer p to the vertex. These maps allow vertex lookup during graph algorithms. Both maps are implemented using oblivious AVL trees. The second oblivious map increases the number of references to each vertex by 1.
2. Considering all of u's outgoing edges, as necessary create an *edge tree*. If a vertex in the original graph has an out-degree of d, the height of the edge tree is $\lceil \log_{bf}(d) \rceil$.

Review of [AHR24] graph emulation techniques Emulating graphs as described by [AHR24] consist of three object types: *vertex* nodes, *outgoing edge* nodes, and *incoming edge* nodes. Outgoing edge nodes are analogous to our fanout nodes.

An incoming edge node represents a destination in a directed edge. This incoming edge vertex maintains visited status and points to v's vertex. An incoming edge tree is built to handle an arbitrary in degree and keep the number of queue-based OSAM pointers to the same object small. Incoming edge nodes do not exist in our graph emulation; we directly handle multiple references to a vertex through pointers. If "visited" is necessary for graph algorithms, it is stored in the vertex object.

In [AHR24], "visited" was stored in incoming edges to prevent having to "chase" the pointer to the corresponding vertex to see if a node has been visited. The removal of incoming edge vertices was done for two reasons: 1) splay trees are faster to chase than the queue structures of [AHR24] and 2) the incoming edge tree created a fixed a binary structure on top of the splay trees that kept splay trees from being able to balance effectively.

6.2 Graph Algorithms

[AHR24] gave constructions for graph traversal algorithms such as BFS, DFS, Dijkstra's, and Prim's. These are *global* algorithms, operating on the entire graph. We expand to *local* algorithms such as Random Walk, Contact Discovery, PageRank, and Directed Triangle Count.

Helper procedures. We first give two useful helper procedures for traversing an emulating graph. We adapt `addNeighbors` from [AHR24], and give a new `getRandomNeighbor` procedure. See Figure 9 for implementation details.

`addNeighbors`: Given a pointer to a vertex, conducts BFS on its edge tree. Adds edge pointers or other vertex metadata like ID or label to a specified data structure.

```
entryPoints ← oblivious-map   // Maps IDs to pointers

def RandomWalk(p : addr, id : int, l : int) → list :
    path ← empty-list
    path.append(id)
    for i ← 1 to l do
        id, p ← getRandomNeighbor(p)
        if !p then
            break
        path.append(id)
    return path

def ContactDiscovery(p1 : addr, p2 : addr) → set :
    n1 ← empty-set, n2 ← empty-set, n3 ← empty-set
    addNeighbors(p1, SET, n1, None, None)
    addNeighbors(p2, SET, n2, None, None)
    for n ∈ n1 do
        if n ∈ n2 then
            n3 ← n3 ∪ {n}
    return n3

def PageRank(entryPoints, l : int, df : float) → map :
    visits ← empty-map, p ← None, id ← None
    for i ← 1 to l do
        r ←$ [0, 1]   // Uniformly sample a float between
          0 and 1
        if r ≤ df then
            id, p ← getRandomNeighbor(p)
        if r > df or !p then
            // Reset if df threshold is reached or a next
              vertex was not found
            id ←$ |V|
            p ← entryPoints[id]
        if id ∈ visits then
            visits[id] ← visits[id] + 1
        else
            visits[id] ← 1
    ratios ← empty-map
    for id ∈ visits do
        ratios[id] ← visits[id]/l
    return ratios
```

Figure 7: Local oblivious graph algorithms.

`getRandomNeighbor`: Given a pointer to a vertex, randomly traverses down its edge tree and returns a destination vertex ID and original edge. Outputs nothing if the given vertex has no edge vertices to choose from.

Local algorithms. We describe Random Walk, Contact Discovery, and PageRank in Figure 7. Random Walk simply performs a random walk until l steps are made or it arrives at a node with no outgoing edges. Contact Discovery outputs the set of neighbors shared by two entry points. PageRank assesses the importance of vertices by performing several random walks of total length l throughout the graph. A new walk occurs (1) naturally with a probability of $1 - df$ or (2) if it reaches a node with no out edges. `entryPoints` is an oblivious map matching vertex IDs to vertex pointers. It is used to periodically pick starting points.

Additional procedures. We provide an overview of the data structures and additional graph functions not discussed earlier. See Figures 8 for the implementation specifics on the following algorithms.

Vertices also store out-degree and height (of the edge tree) to assist with `getRandomNeighbor`. Edge nodes, which have their functionality folded into fanout nodes, store the destination vertex ID. This allows us to save one pointer access during certain procedures. `DirectedTriangleCount` outputs all unique 3-vertex cycles containing a given vertex node. We define a directed triangle as a cycle of three vertices (u, v, w) that are connected by three edges $u \to v$, $v \to w$, and $w \to u$. `getLabel` is a helper function that processes what label should be used when adding an object to a data structure. `getPriority` computes the priority used to rank items in a priority queue.

In [AHR24] emulating graphs, when a vertex node is processed, the helper method `visit` traverses up to the vertex node, then traverses down to the leaf incoming edge nodes and flips *visited*. The visited status is kept at the leaf nodes for quicker checking. [AHR24] toggle a global bit *globalVisited* to check against *visited* for each traversal. This means the bits for *visited* can fall out of sync if only a subgraph is traversed. We now store *visited* at each vertex node, which necessarily makes `visit` a much simpler algorithm that only marks a vertex node as visited. To avoid using a global bit, we track which vertex nodes were visited and reset them later with a new method `unvisit`.

```
struct vertex :
    id : int
    type : enum
    outDegree : int
    outChildren : list                    // List of addrs
    height : int
    label : bit
    visited : int | None

vertexIDs ← oblivious-map   // Maps names to IDs

entryPoints ← oblivious-map        // Maps IDs to
 pointers to vertices

def DirectedTriangleCount(p1 : addr, id1 :
 int) → set :
    triangles ← empty-set
    q1 ← initQueue()
    addNeighbors(p1, QUEUE, q1, None, None)
    while q1.head do
        id2, p2 ← dequeue(q1)
        q2 ← initQueue()
        addNeighbors(p1, QUEUE, q2, None, None)
        while q2.head do
            id3, p3 ← dequeue(q2)
            n3 ← empty-set
            addNeighbors(p3, SET, n3, None, None)
            if id1 ∈ n3 then
                triangles ←
                  triangles ∪ {(id1, id2, id3)}
    return triangles

def
 getLabel(getL : enum, v : vertex, f : fanout) → int
 :
    switch getL do
        case ID do
            label ← v.label
        case LW do
            label ← v.label + f.weight
        case None do
            label ← None
    return label
```

```
struct fanout :
    type : enum
    children : list
    edge : addr weight : int
    isLeaf : bit
    dstID : int
    label : int | None

def getPointer(name : string) → (addr, int) :
    id ← vertexIDs[name]
    p ← entryPoints[id]
    return (p, id)

def visit(p : addr, label : int | None) → addr :
    v ← get(p)
    v.visited ← 1
    v.label ← label
    put(p, v)
    return (p, id)

def unvisit(qVisited : queue) :
    while qVisited.head do
        p ← dequeue(qVisited)
        if p then
            v ← get(p)
            v.visited ← 0
            put(p, v)

def getPriority(getP : enum, v : vertex, f :
 fanout) → int :
    switch getP do
        case LW do
            pty ← v.label + f.weight
        case W do
            pty ← f.weight
        case None do
            pty ← None
    return pty
```

Figure 8: Additional emulating graph functions (1).

```
def addNeighbors(p : addr, ds : enum, pDS :
 queue | addr | None | empty-set | empty-list, getL :
 enum, getP : enum) → addr | None :
    v ← get(p)                                  // vertex node
    q ← initQueue()
    for p′ ∈ v.outChildren do
        if p′ then
            enqueue(q, p′)
    while q.head do
        p′ ← dequeue(q)
        f ← get(p′)                             // fanout node
        if !f.isLeaf then
            for p″ ∈ f.children do
                if p″ then
                    enqueue(q, p″)
        else
            label ← getLabel(getL, v, f)
            pty ← getPriority(getP, v, f)
            switch ds do
                case QUEUE do
                    enqueue(pDS, (label, f.edge))
                case STACK do
                    pDS ← push(pDS, (label, f.edge))
                case PQ do
                    pq.insert((label, f.edge, v.id))
                case SET do
                    pDS.add(f.dstID)
                case LIST do
                    pDS.append(f.dstID)
    return pDS
```

```
def getRandomNeighbor(p : addr | None) →
 (int | None, addr | None) :
    if !p then
        return None, None
    dstID ← None, edge ← None
    v ← get(p)
    if v.outDegree > 0 then
        rem ←$ [v.outDegree] − 1        // uniformly
          sample a leaf between 0 and out degree - 1
        path ← empty-list
        for level ← 0 to v.height − 1 do
            exp ← v.height − level − 1
            if exp < 0 then
                exp ← 0
            div ← bf^exp
            idx ← rem/div
            path.append(idx)
            rem ← rem % div
        idx ← path[0]
        p′ ← v.outChildren[idx]
        for i ← 1 to height − 1 do
            idx ← path[i]
            f ← get(p′)
            p′ ← f.children[idx]
        f ← get(p′)
        edge ← f.edge
        dstID ← f.dstID
    return dstID, edge
```

Figure 9: Additional emulating graph functions (2).

7 Experimental Evaluation

In this section, we present an empirical assessment of our OSAM$^+$ improvements. All of our evaluations and benchmarks are performed on a server with AMD Ryzen Threadripper PRO 7995WX CPU with 96 physical cores and 768 GB of RAM, running Ubuntu 22.04. We build emulating graphs following the steps outlined in Section 6. The graph is static after building. We construct the emulating graph and run only one algorithm.

Notation. We use n to refer to the number of nodes in the graph and d to refer to the average degree. We use bs to refer to the block size used in Path OSAM$^+$ and bf to refer to the branching factor of fanout trees that occurs when a node has more edges than can fit in bs.

Graphs. We present results from Erdős–Rényi graphs and datasets from the Stanford Large Network Dataset Collection (SNAP). We begin with discussion on Erdős–Rényi graphs to show the scaling of all variants with respect to $n, d,$ `bs` and then discuss differences in real datasets.

We consider average degree $d \in \{20, 100\}$ to represent different network densities. The number of nodes ranges over $n \in \{2^2, ..., 2^{20}\}$ for $d = 20$ and $n \in \{2^2, ..., 2^{18}\}$ for $d = 100$. The memory size N depends on n, d, the distribution of edges, and the oblivious mechanism used, so we measure this empirically.

Since the Erdős–Rényi graphs are undirected, one edge in the original graph results in two directed edges in the emulating graph. The random generation can also result in a disconnected graph. If this occurs, edges are randomly inserted between disconnected components, so the actual average degree occasionally exceeds

$d+1$. Note that the additional reference comes from the pointer kept in the oblivious AVL tree discussed in §6.1.

Graph algorithms We evaluate each trial over four algorithmic phases: Build, Random Walk, PageRank, and Contact Discovery. We set the walk length l to 50 and damping factor df to 0.9. Algorithms are run 50 times on random entry points. Performance numbers are the average of all 50 runs.

Graph priming As discussed in Section 2.2, while the amortized cost of a splay tree operation is $O(\log d)$ SAM/SAM$^+$ operations, the actual cost could be $O(d)$. After building the graph, but before running any of the algorithms, we *prime* the graph by conducting random walks on random entry points to rebalance the underlying splay trees. We perform $.10n$ random walks of walk length 50. The accesses made during priming are considered a part of the Build phase.

Cryptographic backend. We test and compare our implementations of oblivious graph using different pointer options as shown in Figures 10 and 11. Each of the shorthands that is used in our graphs denotes a type of pointer as described in the following:

- OSAM$^+$: OSAM$^+$-based pointers augmented with move-style splay tree;
- OSAM: The original OSAM pointers using [AHR24]'s queue-based techniques;
- OSAM w/ Move: The OSAM pointers using queue-based techniques with move, but no multi-write handling;
- ORAM: The traditional pointer system using recursive Path ORAM.

Pointers with moves enjoy the restricted smart copies highlighted in Section 2.3. We consider two different block sizes: $\texttt{bs} \in \{64, 4096\}$ bytes to represent typical storage in cache lines and hard disk drives. This parameter is used to set the branching factor, `bf`, of the resulting graphs. For the all pointer implementations, we set $\texttt{bf} = \texttt{bs}/8 - 2$ to give space to store an object identifier and some associated data in 16 bytes. The remainder of space is used to identify edges. This assumes that ORAM/OSAM/OSAM$^+$ addresses are of size 8 bytes. Note this impacts the branching factor of the edge trees.

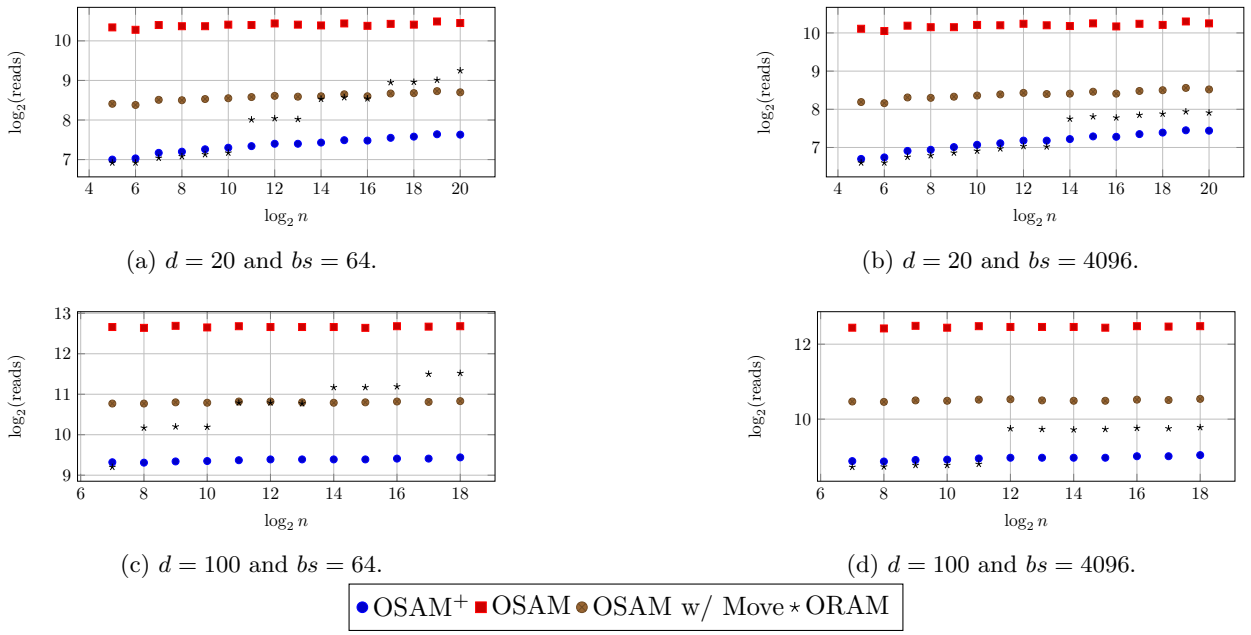
For the recursive Path ORAM implementation, we assume that client storage $\texttt{cs} = 2^{12}$ bytes (beyond ORAM caches) can be kept locally by the client. This allows the client to cut off some levels of recursion. Particularly, when $\texttt{bs} = 64$ we assume they can cut off four levels of recursion of size 8+64+512+4096 = 4680 bytes. When $\texttt{bs} = 4096$ we assume they can cut off one level of recursion of size 512. A second level of recursion for this size is 2^{18}.

7.1 Experimental Results - Erdős–Rényi

We focus on the number of ORAM reads, which serves as a proxy for the number of roundtrips made to the server.

We first consider Contact Discovery in Figure 10. The figures display the average number of reads across the 50 trials. Focusing on Figure 10a, at smaller values of n, the pointer using recursive ORAM is most efficient. This is because all levels of recursion can be stored locally and there is a single ORAM read per logical memory read. Increases in the number of recursive steps are the jumps in the figures, with the first occurring after $n = 2^{10}$. For both OSAM variants and OSAM$^+$, note the nearly constant number of reads as n increases. This is because d is a constant in each figure. Some variation is due to the oblivious map lookups in the algorithms, which scale with $O(\log n)$. We recall that OSAM and OSAM$^+$ use different data structures under the hood ([AHR24]'s queue-based system and our splay tree). The slight decrease with increasing n for OSAM variants appears to be due to poor amortization on queue accesses. At $n = 2^{20}$, our pointer handling achieves improvements of 4× and 1.5× over the recursive pointer for Contact Discovery and PageRank (and 8× and 12× over the original OSAM) with $bs = 64, d = 20$.

Block size. As shown in Figures 10b and 10d, both OSAM variants and OSAM$^+$ perform better for $\texttt{bs} = 64$ than $\texttt{bs} = 4096$. If $\texttt{bs} = 64$, we allow the client 4680 bytes of local storage, which is enough to cut off at least 4 levels of recursion. However, we see that for the tested degrees when $bs = 4096$, most storage of each node is empty. This is because it can store the 16 bytes of node data and all 20 or 100 addresses of pointers in



(a) $d = 20$ and $bs = 64$.

(b) $d = 20$ and $bs = 4096$.

(c) $d = 100$ and $bs = 64$.

(d) $d = 100$ and $bs = 4096$.

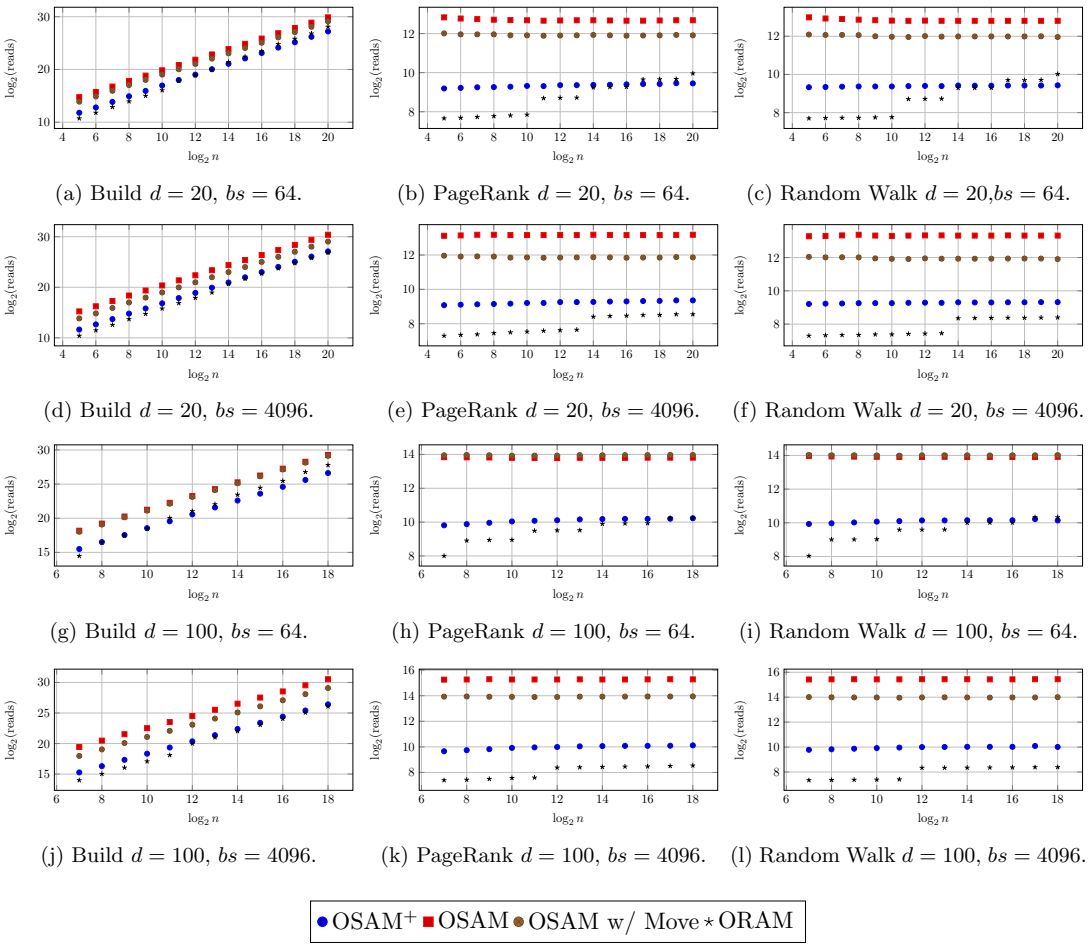
Figure 10: Contact Discovery performance.

only 816 bytes. The same is true for the recursive ORAM, but the recursive storage of position maps is able to fill these blocks and use storage more effectively. Looking ahead to our cryptographic implementation, $bs = 4096$ increases bandwidth blow by a factor 32. Contact Discovery is the only application where for all $(\texttt{bs}, d)$ configurations, OSAM$^+$ at $n = 2^{20}$ is already more efficient than recursive Path ORAM.

Average degree. Recursive ORAM must store every object in the graph, so increasing d does increase the size of the memory. Recall that for OSAM$^+$ there are two trees: the edge tree and the splay tree underlying the smart pointers. The edge tree has a $bf = 6$ when $bs = 64$ and $bf = 510$ when $bs = 4096$. There is no splay tree for the recursive ORAM implementation. As d increases, a larger fraction of reads are splay tree traversals, hurting the relative performance of OSAM$^+$ and OSAM variants. However, larger d also means that recursive ORAM requires more recursive depth for each access it makes. These two factors compete. In general, we see the relative performance between OSAM$^+$ and recursive Path ORAM is better for $d = 100$ compared to $d = 20$, see Figure 10a compared to Figure 10c. Recall that each pointer read for OSAM$^+$ requires $O(\log d)$ ORAM accesses while a recursive ORAM read requires $O(\log(N))$, where the size N is proportional to nd and makes the number of accesses scale as $O(\log n + \log d)$.

Impact of Move. Move's value depends on the algorithm being used. We use Contact Discovery as an example. In Contact Discovery without move, copies are made of the entire edge tree for vertex v_1 which are then deleted before accessing v_2. However, all neighbors of v_1 are smart copied and kept for comparison with the neighbors of v_2. This increases the number of references to the common neighbors of v_2 when these neighbors are being accessed. When move is enabled, these neighbors are not accessed on the server and instead directly retrieved from the cache. Enabling move results in a nearly 4× improvement. On the other hand, PageRank (Figure 11) creates many fewer copies of objects as it only retrieves a single neighbor, accesses the id, and then deletes the copy. As a result we see a smaller improvement of less than 2× for PageRank.

Comparison to [AHR24]. Path OSAM$^+$ is consistently 8× better than [AHR24]. This improvement increases to 16× on some PageRank and random walk configurations. The improvement over [AHR24] increases for graphs of higher degree. Recall d determines the average number of references to each graph node. Our approach appears to gain a small additional advantage over [AHR24] for larger bs. We attribute this trend to the fact that a smaller fraction of operations traverse fanout trees; these tree are not impacted by our pointer sharing optimizations (since fanout nodes have in-degree of 1), and hence a larger fraction of operations do benefit from our optimizations.



(a) Build $d = 20$, $bs = 64$. (b) PageRank $d = 20$, $bs = 64$. (c) Random Walk $d = 20$,$bs = 64$.

(d) Build $d = 20$, $bs = 4096$. (e) PageRank $d = 20$, $bs = 4096$. (f) Random Walk $d = 20$, $bs = 4096$.

(g) Build $d = 100$, $bs = 64$. (h) PageRank $d = 100$, $bs = 64$. (i) Random Walk $d = 100$, $bs = 64$.

(j) Build $d = 100$, $bs = 4096$. (k) PageRank $d = 100$, $bs = 4096$. (l) Random Walk $d = 100$, $bs = 4096$.

Figure 11: Read complexity results across Build, PageRank, and Random Walk for all (d, bs) configurations. Note $l = 50$ and $df = 0.9$ where applicable.

Build, PageRank, and Random Walks Here we display all the results we generated during the experiments described in Section 7. Figure 11 summarizes results for Build, PageRank, and Random Walks.

Figures 11a, 11d, 11g, and 11j show the number of reads during Build for $\{d = 20, bs = 64\}$, $\{d = 20, bs = 4096\}$, $\{d = 100, bs = 64\}$, and $\{d = 100, bs = 4096\}$ respectively. We see the linear increase in the number of reads as we increase n during Build.

Figures 11b, 11e, 11h, and 11k reflect the experiment results for PageRank and Figures 11c, 11f, 11i, and 11l show results for Random Walks experiments. Due to their algorithmic similarities, the results of Random Walk closely match those of PageRank. We see that for $\{d = 20, bs = 64\}$ there exist some n that multi-write outperforms recursive. It should be noted that such a threshold exists for other settings of d and bs as the rate of increase in the number of reads is higher in recursive pointer compared to the multi-write pointer.

Regarding the aforementioned experiments, specifically PageRank and Random Walk, OSAM$^+$ and the OSAM variants perform better for larger degrees and lower block sizes which is consistent with the performance of Contact Discovery.

7.2 Experimental Results - Datasets

Table 1 reports experiments on several graph datasets from SNAP. These datasets exhibit significant structural variation including sparsity, skewed degree distributions, directed edges, and sink nodes. These properties influence the performance of Random Walk (RW), PageRank (PR), and Contact Discovery (CD).

Dataset	bs	Impl.	Build	CD	RW	PR
email-EU-core Directed $\log n = 10.0$ $d = 25$	64	OSAM$^+$	16.9	7.8	8.8	9.5
		ORAM	16.9	8.5	8.1	8.8
		w/ Move	19.2	9.0	12.2	13.0
		OSAM	19.5	10.8	12.5	13.2
	4096	OSAM$^+$	16.7	7.3	8.7	9.4
		ORAM	15.7	7.2	6.8	7.6
		w/ Move	19.1	8.7	12.1	12.9
		OSAM	20.6	10.6	13.7	14.3
p2p-Gnutella04 Directed $\log n = 13.4$ $d = 3.7$	64	OSAM$^+$	18.6	6.4	5.3	9.4
		ORAM	18.8	6.8	5.1	9.2
		w/ Move	19.1	7.0	6.1	10.4
		OSAM	19.6	8.3	7.1	11.4
	4096	OSAM$^+$	18.5	6.3	5.3	9.3
		ORAM	18.4	6.1	4.9	8.8
		w/ Move	19.0	6.8	6.1	10.3
		OSAM	19.7	8.1	7.0	11.3
ego-Gplus Directed $\log n = 16.7$ $d = 127$	64	OSAM$^+$	25.1	10.2	9.2	12.6
		ORAM	26.7	12.3	6.7	10.1
		w/ Move	27.2	11.4	12.7	16.6
		OSAM	27.2	13.2	12.1	15.9
	4096	OSAM$^+$	25.0	9.9	9.2	12.6
		ORAM	25.2	10.6	5.7	9.0
		w/ Move	27.2	11.1	12.7	16.6
		OSAM	27.9	13.0	13.3	16.8
roadNet-PA Undirected $\log n = 20.1$ $d = 1.4$	64	OSAM$^+$	26.4	6.6	8.5	8.7
		ORAM	26.9	7.0	9.3	9.3
		w/ Move	27.3	7.0	9.9	9.9
		OSAM	28.4	7.9	11.3	11.2
	4096	OSAM$^+$	26.4	6.6	8.5	8.7
		ORAM	26.4	6.7	8.4	8.6
		w/ Move	27.3	7.0	9.9	9.9
		OSAM	28.4	7.9	11.3	11.2
com-Youtube Undirected $\log n = 20.1$ $d = 2.6$	64	OSAM$^+$	27.1	7.0	9.6	9.5
		ORAM	27.5	8.1	9.8	9.7
	4096	OSAM$^+$	27.0	6.8	9.4	9.4
		ORAM	26.5	7.2	8.5	8.6
twitch-gamers Undirected $\log n = 17.4$ $d = 40$	64	OSAM$^+$	26.0	9.0	10.2	10.1
		ORAM	27.0	11.1	10.5	10.3
	4096	OSAM$^+$	25.8	8.7	10.1	10.0
		ORAM	25.3	9.5	8.5	8.6
Higgs-Twitter Directed $\log n = 18.8$ $d = 33$	64	OSAM$^+$	26.6	8.1	10.0	10.0
		ORAM	27.5	9.9	10.3	10.1
	4096	OSAM$^+$	26.4	7.7	9.9	9.8
		ORAM	25.9	8.3	8.4	8.6

Table 1: Number of reads averaged over 50 trials for Build, Contact Discovery, Random Walk, and PageRank. For com-Youtube, twitch-gamers, Higgs-Twitter, OSAM and OSAM w/ Move experiments exhausted memory in our environment.

Overall, the real-world datasets broadly follow the same trends observed in Erdős–Rényi graphs with comparable values of n and average degree d, though sparsity and graph directionality can produce deviations. The real-world datasets with results similar to our Erdős–Rényi testing include `email-EU-core`,

log	$bs = 64$		$bs = 4096$	
Memory Size	Comp	BW	Comp	BW
19	270	9.7	1500	310
20	290	10	1600	330
21	320	11	1700	340
22	340	11	1700	360
23	360	12	1800	380
24	370	12	1800	390
25	400	13	–	–
26	420	13	–	–
27	430	14	–	–
28	440	14	–	–

Table 2: Cryptographic backend performance metrics per operation for read and write. Comp is in microseconds and bandwidth is in kilobytes. Omitted size for BS 4096 was due to out of running out of memory.

`twitch-gamers`, and `Higgs-Twitter`.

Recall that RW terminates whenever a sink node is reached. This means directed graphs with nodes with out-degree 0 can produce substantially lower RW costs than PR, since PR continues traversals probabilistically while RW halts early. In the undirected datasets, nodes have an out-degree of at least 1, causing RW and PR to exhibit very similar behavior without risk of early termination.

The `p2p-Gnutella04` and `roadNet-PA` datasets have much lower degree than any tested Erdős–Rényi graph. The lower d density reduces the benefit of OSAM$^+$ in comparison to all other pointers as expected for CD. Due to having small d and being directed, the `p2p-Gnutella04` dataset has many sink nodes that terminate RW early. Across pointer types and block sizes, the RW numbers for Erdős–Rényi are 5–80× larger than real results.

The is behavior carries over to the `ego-Gplus` dataset. Excluding ORAM, the RW measurements on Erdős–Rényi graphs are roughly 2–4× higher than the results for `ego-Gplus` because the graph is directed and contains sink nodes. This effect is especially pronounced for ORAM, where RW costs on Erdős–Rényi graphs are approximately 6× or 12× higher depending on block size. In contrast, PR measurements for OSAM$^+$, OSAM, and OSAM w/ Move on Erdős–Rényi are roughly 0.15–0.35× smaller than that of `ego-Gplus`.

`Higgs-Twitter` is the closest match to a tested Erdős–Rényi graph with $n = 2^{19}$ and $d = 20$. Build costs are nearly identical between the real and synthetic settings. CD costs are similar for $bs = 4096$ but moderately larger for real data for $bs = 64$. Unlike Erdős–Rényi graphs, the real-world datasets exhibit highly skewed degree distributions with hub nodes. While Erdős–Rényi graphs concentrate most vertices near the average degree d, real datasets contain a small number of very high-degree vertices. These hubs increase variance in traversal costs for OSAM$^+$ because accesses to shared nodes induce deeper or more frequently accessed splay trees. RW and PR measurements are generally comparable to, or slightly larger than, those observed on Erdős–Rényi graphs.

Memory exhaustion for OSAM and OSAM w/ Move occurred on several real-world datasets despite testing on Erdős–Rényi graphs with a larger $n \cdot d$. We attribute this to a larger variation in degree distribution leading to larger queues trees in OSAM, highlighting the benefit of our splay pointers.

Cryptographic efficiency. Table 2 shows the cost of a single read/write in microseconds and the bandwidth in KB. We make three observations:

1. The BW depends heavily on the block size,
2. Computation time has a smaller dependence on block size, and
3. The dominant cost between latency, computation time, and bandwidth is application dependent.

All our experiments were done with the entire oblivious structure in main memory. For our hardware, sizes beyond those listed would require use of hard disk. As a reminder, even for our most efficient local algorithms Path OSAM$^+$ requires roughly 320 reads for a random walk moving computation time to the millisecond scale and bandwidth to the megabyte scale.

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A Counterexample to GKW18's Proof

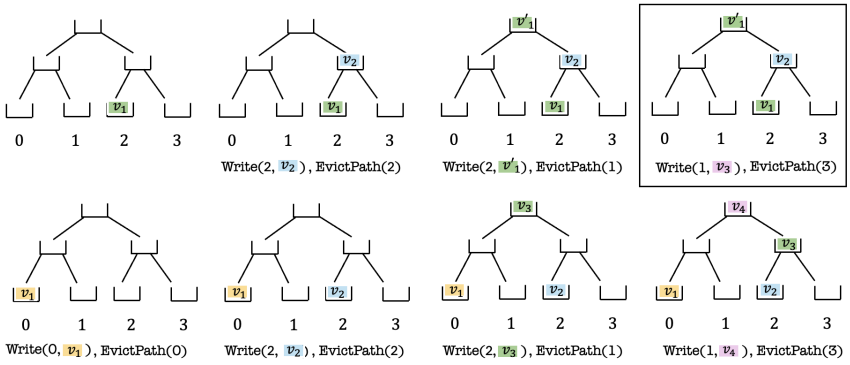

Figure 12: The top row depicts a sequence of writes that includes a repeated write (v_1 and v_1') and results in a value (v_3) being stashed. The bottom row depicts the same sequence, except that the older repeated write (v_1) is replaced with a write to a uniformly random leaf (we also rename v_1' to v_3 and v_3 to v_4). The bottom sequence *does not result* in a value being stashed.

[GKW18] claims to prove the following proposition, which is adjusted from their Lemma 3.1 to our syntax:

Proposition 1. *Consider a modification to Path OSAM (Figure 3) such that on operation* `Write`(i, v)*, all prior writes to address i are removed from the OSAM tree (without regard for obliviousness). This modification does not affect the size of the stash, regardless of the sequence of memory accesses.*

In other words, the following two strategies are claimed to be equivalent from the perspective of stash size:

- Use Path OSAM's lazy strategy of allowing repeated writes to accumulate in the tree, then delete them lazily on eviction.
- When two values are written to the same address, aggressively (and insecurely) remove the older write from the tree.

In Figure 12, we present a counter-example that demonstrates that the above two strategies are *not equivalent* even when the older write is replaced with a write to a uniformly random leaf (as in the original Path ORAM construction that disallows multiple writes to the same address). We show the stash size is *different* in both cases: when the latter write is not removed the stash size is 1; but when it is removed, the stash is empty.

B Inverted Pointer Tree

B.1 The Splay Operation

The `splay` operation (see Figure 4) is a recursively defined operation that balances the path being traversed when a pointer is dereferenced. `splay` takes as input the address p of an node n on this path, and addresses a and b of n's children. `splay` balances the path from n to the root by performing two types of rotations (depicted in Figure 13) based on two cases.

Case 1 - n has no grand parent: This is the case when n's parent is the root. The client performs a *zig* rotation. This causes a to hold the address of the root, and links b with n's sibling.

Case 2 - n has a grand parent: The client performs a *zig-zag* rotation. This balances the path by linking n's sibling with the sibling of n's parent. The client continues to balance the path by recursively calling `splay` on n's grand-parent.

The `splay` operations halts in the base case when the root is reached, and returns a newly allocated (un-written) address of the root, along with the value at the root.

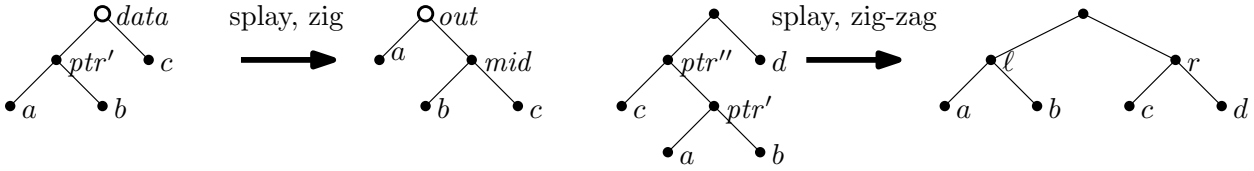

Figure 13: zig and zig-zag rotations performed by `splay`.

B.2 Analysis

Theorem 3. *Each pointer tree operation implemented by Figure 4 has an amortized cost of $O(\log n)$ SAM$^+$ operations, where n is the number of leaves of the pointer tree.*

Proof. We prove that the amortized cost of each pointer analysis is $O(\log n)$ using the potential method. We use the same potential function used by splay trees. Let $\texttt{size}(x)$ denote the size of a subtree rooted at node x, and define $\texttt{rank}(x) = \log(\texttt{size}(x))$. The potential of the splay tree T is *thrice* the sum of the ranks of all of its nodes.

When using the potential method, the amortized cost of an operation on T its actual cost, summed up with the difference in the potential of T. We now analyze the cost of each pointer operation.

The `new` operation incurs a single SAM$^+$ operation, and adds a single node to T, which increases its potential by 0. Thus, the amortized cost of `new` is $O(1)$.

Recall that the `copy` operation extends a path in the tree T by creating a new leaf. This incurs a constant number of SAM$^+$ operations. To calculate the change in potential, suppose that, before the operation, this path had ℓ nodes of sizes $n_1, \ldots, n_\ell$, from the root, to the leaf. Suppose that the new tree that we obtain is T'. The change in potential of is:

$$\begin{aligned}\Phi(T') - \Phi(T) &= 3 \cdot (\log(n_1 + 1) - \log(n_1) + \ldots \\ &\quad + \log(n_\ell + 1) - \log(n_\ell)) \\ &= 3 \cdot \log\left(\frac{n_1 + 1}{n_1} \cdot \frac{n_2 + 1}{n_2} \cdot \cdots \cdot \frac{n_\ell + 1}{n_\ell}\right)\end{aligned}$$

Since $n_1 > n_2 > \cdots > n_\ell$, $n_1 \geq n_2 + 1, n_2 \geq n_3 + 1, \ldots, n_{\ell-1} \geq n_\ell + 1$. This allows us to simplify the above expression.

$$\begin{aligned}\Phi(T') - \Phi(T) &\leq 3 \cdot \log\left(\frac{n_1+1}{n_1} \cdot \frac{n_1}{n_2} \cdot \frac{n_2}{n_3} \cdot \ldots \cdot \frac{n_{\ell-1}}{n_\ell}\right) \\ &\leq 3 \cdot \log\left(\frac{n_1+1}{n_\ell}\right) \leq 3 \cdot \log(n_1+1)\end{aligned}$$

Hence, the amortized cost of `copy` is $O(\log n)$. The `delete` operation performs the *opposite* of `copy` by truncating the path to a leaf. This incurs $O(1)$ SAM$^+$ operations, and only causes a decrease in the potential, and so, the amortized cost of `delete` is $O(1)$. To analyze the cost of `get` and `put`, we show that the amortized cost of `splay` is $O(\log n)$. Since `get` and `put` call `splay` (through `move`), this bounds the cost of `get` and `put`.

To bound the cost of `splay`, consider a zig-zag operation performed on a node x with parent y and grand-parent z. Let us use x', y' and z' to denote the positions of these nodes after the operation. We show that the amortized cost of the zig-zag operation is $2 \cdot (\texttt{rank}(x') - \texttt{rank}(x))$. Since `splay` repeatedly performs the zig-zag operation until the root, the amortized cost of `splay` is $2 \cdot (\texttt{rank}(x') - \texttt{rank}(x) + \texttt{rank}(x'') - \texttt{rank}(x') + \cdots + \texttt{rank}(\texttt{root})) = 2 \cdot (\texttt{rank}(\texttt{root}) - \texttt{rank}(x))$. By the definition of `rank`, this is $O(\log n)$.

It remains to show that the amortized cost of the zig-zag operation is $2 \cdot (\texttt{rank}(x') - \texttt{rank}(x))$. The amortized cost c of this operation is

$$\begin{aligned}c = 6 &+ \texttt{rank}(x') - \texttt{rank}(x) + \texttt{rank}(y') - \texttt{rank}(y) \\ &+ \texttt{rank}(z') - \texttt{rank}(z)\end{aligned}$$

Note that, by how zig-zag is performed, $\texttt{rank}(x') = \texttt{rank}(z)$ and that $\texttt{rank}(x) < \texttt{rank}(y)$. Hence,

$$\begin{aligned}c &= 6 - \texttt{rank}(x) + \texttt{rank}(y') - \texttt{rank}(y) + \texttt{rank}(z') \\ &\leq 6 - 2 \cdot \texttt{rank}(x) + \texttt{rank}(y') + \texttt{rank}(z')\end{aligned}$$

Now, consider the term $2 \cdot \texttt{rank}(x') - \texttt{rank}(y') - \texttt{rank}(z')$. This is

$$3 \cdot \left(\log \frac{\texttt{size}(x')}{\texttt{size}(y')} + \log \frac{\texttt{size}(x')}{\texttt{size}(z')}\right)$$

By how zig-zag is performed, $\texttt{size}(x') \geq \texttt{size}(y') + \texttt{size}(z')$. The smallest value of the above term occurs when $\texttt{size}(y') = \texttt{size}(z') = \texttt{size}(x')/2$. The above term is, thus, at least 6. Hence,

$$\begin{aligned}c &\leq 6 - 2 \cdot \texttt{rank}(x) + \texttt{rank}(y') + \texttt{rank}(z') \\ &\leq 2 \cdot \texttt{rank}(x') - \texttt{rank}(y') - \texttt{rank}(z') \\ &\quad - 2 \cdot \texttt{rank}(x) + \texttt{rank}(y') + \texttt{rank}(z') \\ &\leq 2 \cdot (\texttt{rank}(x') - \texttt{rank}(x))\end{aligned}$$

□

C Oblivious Data Structures

This section provides an overview of modified or new oblivious data structures utilized to implement oblivious graphs. Stacks and priority queues are implemented as explained in [AHR24].

C.1 Queues

Address queues. The queues described in Section 2.1 are referred to as *address queues* in [AHR24]. They are managed through two separate addresses head and tail. We fetch the next entry by reading head and add items by writing to tail. Address queues are used to (1) enable reference sharing for the original queue-based pointer and (2) store pointers or vertex information during traversal algorithms like BFS. The original `dequeue` procedure always reads head. If head was written to during `enqueue`, this gives the next queued object and a head′ pointing to the following entry. Otherwise, head equals tail, which is an unwritten address that yields None when read. Detecting None indicates the queue is empty, as there are no further addresses to read.

Our changes. We create a new queue object that tracks head and tail together. During each `dequeue`, we compare head to tail to determine if the queue is empty. When tail equals head, `dequeue` outputs None without reading anything. Since tail is always an allocated but not written address, a matching head signifies there are no more entries. As stated in 7.1, we use the number of reads as a proxy for the number of roundtrips. Every time we exhaust a queue, this change saves one read that is made by address queues. To leave the queue-based pointer intact, we only use the new queue in graph traversal. The implementation can be seen in Figure 14.

```
struct queue :
    head : address
    tail : address

def initQueue() → queue :
    q ← queue
    q.head ← Alloc()
    q.tail ← q.head
    return q

def enqueue(q : queue, v : userT) :
    tail′ ← Alloc()
    Write(q.tail, {v, tail′})
    q.tail ← tail′

def dequeue(q : queue) → userT :
    if q.head == q.tail then
        q.head ← None
        q.tail ← None
        return None
    else
        v, head′ ← Read(q.head)
        q.head ← head′
        return v
```

Figure 14: Queue methods.

C.2 AVLtrees

The oblivious maps we mention in Section 6.1 are oblivious AVLtrees built on OSAM$^+$ and not outlined by [AHR24]. Each node is written to a SAM$^+$ address and stores a key, value, and left and right child attributes (addresses, heights, and balances). A root address must be locally maintained to access the AVLtree. This implementation supports three key algorithms in $O(\log n)$: `search`, `insert`, and `delete`. Each procedure follows the general framework of (1) read down the tree until some stopping point (finding the target node or a leaf to insert a new node), (2) perform some procedure-specific steps at the bottom, and (3) write the tree back up to the root. By maintaining child metadata (heights and balances) at each node, we can perform rotations without reading any nodes not encountered during the initial downward traversal. These procedures are too complex to neatly summarize in pseudocode, so please see the Python implementation for further details.

D Oblivious Graph Background

Chase and Kamara [CK10] introduced the notion of graph encryption as a form of structured encryption. Later, Meng et al. [MKNK15] presented *GRECS* to support approximate shortest-distance queries

through three different schemes with trade-off between communication complexity, computation complexity, and amount of leakage. Ghosh, Kamara, and Tamassia [GKT21] presented a graph encryption scheme (GKT) for shortest path queries using SP-matrices, where they support unbounded recursive algorithms non-interactively. Falzon and Paterson [FP22] presented a query recovery attack against GKT.

In another line of work [ZDB+17, CDP+24], trusted hardware enclaves are combined with oblivious graph algorithms to ensure that leakage is minimized. OPAQUE [ZDB+17] combines Intel SGX with a modified Spark SQL architecture where they move query planning to the client side. However, when using this design on graph databases, it suffers from leaking some information about the topology of the graph. GraphOS [CDP+24], on the other hand, because of the use of a doubly oblivious map, only leaks the size of the graph, the type of query and its response size.

Compass [ZPZP24] introduces a graph-based semantic search scheme from the Hierarchical Navigable Small World (HNSW) index. The focus of Compass is the systems where multiple clients store their documents on a single server and later each client queries from their own set of documents; so file sharing is not considered in this scheme.

Classical HNSW, generates a multi-layer graph, begins the greedy search from an entry point in the first layer, and continues the search over each layer. The search complexity of HNSW is poly-logarithmic. One way to protect search privacy in such indexes is applying ORAM. However, due to the high number of data accesses in HNSW where each access itself may contain a large number of neighbors for each node, applying an ORAM on its own results in high bandwidth and roundtrips.

Compass addresses this issue by utilizing two other tools and combining them with a Ring ORAM which is tailored for Graph-Traversal. These three techniques are described below. Note that the first two techniques use ORAM as a black-box.

1. In order to reduce bandwidth, they use *Directional Neighbor Filtering*. This method only returns those subsets of neighbors that are closer to the query point. The size of the filter is denoted by efn.

    (a) To implement this, they store a "directional hint" along with each node, which helps identifying the top efn closest neighbors.

2. In order to reduce the number of roundtrips, they use *Speculative Neighbor Prefetch*, so that in each ORAM roundtrip, extra nodes that are likely to be visited in the future are also retrieved.

    (a) In order to batch multiple ORAM accesses, one must know which nodes are more likely to be needed. Compass uses a greedy search algorithm for this purpose to reduce the number of roundtrips by a factor of $efspec$.

3. In order to optimize graph traversal, Compass modifies how graph nodes are stored and accessed in ORAM. Considering a graph which consists of coordinates and neighbor list, each graph node and its neighbor list are stored in the same ORAM block. To reduce roundtrips, using the above technique, parallel requests are performed.

    (a) To reduce the search latency, Compass introduces "multi-hop lazy eviction", where they delay the writeback from the stash until the end of the query.

Compass is built on top of HNSW, but its method can also be used on other graph-based ANN index structures.

Security. Using ORAM protects the access pattern. To guarantee integrity, Compass constructs a Merkle tree on top of the ORAM tree.