γ_{11}: Unequal Triangles and Adapted References

THE MACADAMIA 0050

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Abstract. This paper settles the ninth of the ten *macadamias* of the series on the Hermite constant in dimension 11: the perfect form $P =$ `0050:024468269` of rank 9, of determinant $8192/9$ at minimum 4. The root paper reduces γ_{11} to chains $L \supset M \supset N$ of lattices of ranks 11, 10, 9, each with minimum at least 4; we prove that every admissible chain whose rank-9 member is admissibly marked by P has $\det L \geq 972$. No argument at rank 10 can do this, because an actual lattice of rank 10 with determinant $3328/3 < 4 \cdot 972^{9/11}$ lies over P. The proof is computer-assisted, with explicit finite checks. Its decisive step is elementary: the three additive dual vectors of a section covector and the first normal usually have different lengths, and keeping the three lengths in the circle obstruction, instead of their maximum, gives a determinant cap that sharpens every subsequent contact and spectral bound. Adapted reference forms turn the contact conditions into exact rational tests, a common slack is retained throughout, and four computational results reduce the problem to a five-parameter root, exhausted by a finite cover of 6 993 terminal regions replayed in exact arithmetic.

Keywords: Hermite constant · Lattice packings · Perfect forms · Voronoi domain · Additive dual vectors · Exact certificates · Verified computation

1 Introduction

This paper belongs to a series of seven papers, made public concurrently, whose aim is to establish the Hermite constants γ_{10}, γ_{11} and γ_{12}: the root paper [3] fixes the conventions, proves γ_{10} and derives γ_{12} from γ_{11}; the root paper also reduces γ_{11} to chains $L \supset M \supset N$ of ranks 11, 10 and 9 (its Proposition 15). The γ_{11} paper [4] selects the 4 487 813 rank-nine forms, excludes all but ten by exact certificates, and settles `0052`. The five companions settle the other nine forms, including the present one.

This paper settles the ninth macadamia, the perfect form $P =$ `0050:024468269` of [2], of determinant $8192/9$ at minimum 4 with 65 pairs of minimal vectors (Appendix A). An unnamed lattice of rank 10 with 180 minimal vectors, determinant $3328/3 = 1109.33\ldots$ and dual minimum $32/39$ lies over it [3, Table 4], and $3328/3 < d_{\mathrm{cut}} = 4 \cdot 972^{9/11} = 1113.04\ldots$; so no argument at rank 10 can exclude the form, and the statement to prove concerns the lattice of rank 11:

Theorem 1. *Assume Computational results 14–17. Every chain $L \supset M \supset N$ of Definition 2 whose rank-9 member N has a Gram matrix admissibly marked by $P =$ `0050:024468269` satisfies* $\det L \geq 972$.

The idea. The decisive improvement in this case is elementary. A section covector of N^* has two nearest lifts to M^*, and with the first normal (the shortest vector of M^*, of squared length u) they form an *additive triangle* of three dual vectors summing to zero; the second gluing (the way L is glued over M) assigns to the three a circle distance each, and the three distances have sum at most one. Three additive dual vectors usually have different lengths. Keeping those three lengths in the circle obstruction, instead of their maximum, gives a stronger determinant cap, $e < 972\,u^2 h_\triangle(r)$ with $h_\triangle(r) = (3-2r)^2/(8(1-2r))$ in place of the inherited $4/3$ (Lemma 26); the cap then improves the phase bounds and the relative spectral bounds used by every subsequent contact test. The second ingredient, the *adapted references*, replaces the actual matrix by rational reference forms chosen leaf by leaf, against which the contact conditions become exact rational tests. A third feature, less visible but essential for the exactness of the final cover, is that one *common slack* $J \geq 0$ of the marking is kept as an actual variable throughout the necessary inequalities and the final cover.

What the proof consists of. Section 2 states the admissible-chain conditions and recalls the common tools. The chain is then written in canonical coordinates adapted to P with eight *normal energies* η_i, eight *phases* ε_i, an axial phase t, a coupling budget V and the slack J (Section 4). A cofactor argument caps every normal energy by a finite cover (Section 5); the nearest-phase centre is shown to be the *good* one of its two classes (Section 6); phase caps, an axial-phase exclusion and a middle-height band confine the outer heights (Sections 7–9); the unequal-triangle lemma is proved with its full endpoint argument (Section 10); determinant, contact, flag-normal, Schur and primal-trace tests that retain the common slack are then stated as exact rational box tests (Sections 11–15); the height reduction (Section 16) confines a counterexample to a five-parameter root, which a final cover of $13\,985$ nodes and $6\,993$ terminal regions, replayed in exact arithmetic, exhausts (Section 17). The computational dependencies are separated into four results (Section 3), so that a completed final cover cannot conceal an unverified outer range. Section 18 says what is verified and where, and what the theorem rests on.

2 The local setting and common tools

2.1 Notation and admissible chains

We use the common coordinates of [4, Appendix D]. The determinant of a lattice is the determinant of its Gram matrix, and its minimum is the least nonzero squared length. Its dual is $L^* = \{y \in \operatorname{span} L : y \cdot x \in \mathbb{Z} \text{ for all } x \in L\}$. A sublattice $M \subset L$ is *primitive* if $M = L \cap \operatorname{span} M$; a primitive vector generates a primitive rank-one sublattice. Write $Q[x] = x^{\mathsf{T}} Q x$, with T denoting transpose, and $\operatorname{tr} A$ for the sum of the diagonal entries. For symmetric matrices, $A \succeq 0$ means $A[x] \geq 0$ for every x; $A \succ 0$

means $A[x] > 0$ for $x \neq 0$, and $A \preceq B$ means $B - A \succeq 0$. The nested sections form a flag of ranks 11, 10 and 9; its notation is

$$\begin{gathered} e = \det N, \qquad d = \det M = \frac{e}{u}, \qquad \Delta = \det L = \frac{e}{uv}, \\ u = \min M^{*}, \qquad v = \min L^{*}. \end{gathered} \tag{1}$$

The cutoffs are the positive numbers $d_{\mathrm{cut}}^{11} = 4^{11}972^{9}$ and $e_{\mathrm{cut}}^{55} = 2^{246}3^{197}$.

Definition 2 (admissible residual chain). *A chain $L \supset M \supset N$ is admissible for P if L has rank eleven and minimum at least four, $M = L \cap n_{11}^{\perp}$ and $N = M \cap n_{10}^{\perp}$ for shortest nonzero vectors $n_{11} \in L^{*}$ and $n_{10} \in M^{*}$, its rank-nine Gram matrix is marked by P as defined below, and*

$$\begin{gathered} 768 \le d < d_{\mathrm{cut}} = 4\,972^{9/11}, \\ e < e_{\mathrm{cut}} = \Big(\frac{4096}{3}\Big)^{1/10} d_{\mathrm{cut}}^{9/10} < 1137. \end{gathered} \tag{2}$$

Shortest dual vectors are primitive, so the determinant identities above follow by an adapted basis. Rounding a lift of n_{10} across the last cut gives $v \le u + v/4$, hence $v \le 4u/3$ and $\Delta \ge 3d/(4u)$. The global reduction [3, Proposition 15] supplies such a chain from a hypothetical counterexample.

Definition 3 (marking). *Let Q be the Gram matrix of N in an integral basis, and $X = Q^{-1}$. A perfect form P of minimum four marks Q if $\min Q \ge 4$ and there are weights satisfying*

$$\begin{gathered} X = \sum_{c \in \mathcal{C}(P)} w_c\, cc^{\mathsf{T}}, \qquad w_c \ge 0, \qquad \sum_{c} w_c \le \frac{9}{4}, \qquad w_c \le \frac{1}{4}, \\ \operatorname{tr}(PX) \le 9, \qquad \det P \le e. \end{gathered} \tag{3}$$

Here $\mathcal{C}(P)$ contains one vector from each opposite pair of minimum vectors of P.

Perfect means that the matrices cc^{T} span the symmetric matrices. A marking is transported by an integral change of basis. Its existence for arbitrary Q is proved in [3, Proposition 7]; here the marker is fixed to the printed form.

Gluing coordinates. Vectors $\beta, \alpha \in \mathbb{R}^{9}$ and a scalar $\vartheta \in \mathbb{R}$ give the dual squared lengths at $(z, k, k') \in \mathbb{Z}^{9} \times \mathbb{Z}^{2}$ as

$$X[z] + u(\beta^{\mathsf{T}} z + k)^{2} + v(\alpha^{\mathsf{T}} z + \vartheta k + k')^{2}, \tag{4}$$

and the primal squared lengths as

$$Q[x - k\beta - k'(\alpha - \vartheta\beta)] + \frac{(k - \vartheta k')^{2}}{u} + \frac{k'^{2}}{v}. \tag{5}$$

Write $\|t\| = \min_{j \in \mathbb{Z}} |t - j|$. The first shortest-normal condition implies

$$X[z] + u\,\|\beta^{\mathsf{T}} z\|^{2} \ge u \qquad (0 \neq z \in \mathbb{Z}^{9}). \tag{6}$$

The second condition applies to every nonzero vector of M^{*} and every lift to L^{*}.

Physical coordinates. Choose C_P with $P = C_P^{\mathsf{T}} C_P$ and write

$$Q = C_P^{\mathsf{T}} S C_P, \qquad R = S^{-1}.$$

The columns of C_P realise the reference lattice in Euclidean space. Thus S measures squared lengths in these fixed coordinates, with reference metric I, and R is its inverse: $\det S = e/\det P$, $\operatorname{tr} R = \operatorname{tr}(PX) \le 9$; integral vectors c, z become $C_P c, C_P^{-\mathsf{T}} z$. We write $|x| = \|x\|$ for Euclidean length and $\|E\|_F^2 = \operatorname{tr}(E^{\mathsf{T}} E)$ for the squared Frobenius norm, and $\rho_G^2 = \sup_x \min_{a \in \Lambda} G[x-a]$ for the squared covering radius of the reference lattice Λ under discussion, with $x \in \operatorname{span} \Lambda$. Eigenvalues are listed in decreasing order. The notation $I_n, e_i, \mathbf{1}$ denotes the identity, coordinate vectors and all-one vector; Sym_n denotes the symmetric $n \times n$ matrices. Schur complements, cofactors and adjugates have the conventions and block identities of [4, Appendix D.2]. Exact LDL^{T} factorisation and bounded short-vector enumeration are described in [4, Section 5.1]. In the canonical coordinates below, $a = u^{-1} - 1$, $\mathcal{N}$ is the normal-energy budget, and $\widetilde{Q}, \widetilde{X}$ are the section Gram matrix and its inverse.

2.2 Statements of the common toolbox

The arguments of this paper invoke the following statements, common to the whole series. Their proofs appear in the common toolbox [4, Appendix D], including the primitive-extension, cell-averaging and directional inequalities. Positive parts are $(x)_+ = \max(x, 0)$; for a box coordinate, x_-, x_+ denote its lower and upper endpoints; the root lattice $D_n = \{z \in \mathbb{Z}^n : \sum_i z_i \in 2\mathbb{Z}\}$ has roots $e_i \pm e_j$. See also [9] for the matrix inequalities.

Lemma 4 (circle triangle; [4, Lemma 22]). *If $t_0 + t_1 + t_2 = 0$ in $\mathbb{R}/\mathbb{Z}$, then $\sum_i \|t_i\| \le 1$. Consequently, if $y_0 + y_1 + y_2 = 0$ are nonzero vectors of M^* of squared lengths b_i, then the squared length v of the last normal satisfies*

$$\sum_{i=0}^{2} \sqrt{\max\left\{0, 1 - \frac{b_i}{v}\right\}} \le 1. \tag{7}$$

In particular $v \le (9/8) \max_i b_i$.

Lemma 5 (inverse-trace eigenvalue bound; [4, Lemma 27]). *Let $B \succ 0$ have size $n \ge 2$, $\operatorname{tr} B^{-1} \le n$, and $\det B < \rho$. Define*

$$F_n(t) = t\Big(\frac{n-1}{n - \frac{1}{t}}\Big)^{n-1} \qquad \Big(t > \frac{1}{n}\Big).$$

Every eigenvalue λ of B satisfies $F_n(\lambda) < \rho$. In particular, if $H > 1$ and $F_n(H) \ge \rho$, then $B \prec HI$.

It is applied in Section 12 with $n = 9$ and $\rho = e/(4 \det R_{\mathrm{aux}})$ to a rational reference R_{aux}: the test $H(8/(9 - 1/H))^8 \ge e/(4 \det R_{\mathrm{aux}})$ proves $\widetilde{Q} \preceq H R_{\mathrm{aux}}$.

Lemma 6 (a shared D_n determinant bound; [4, Lemma 32]). *Let E be symmetric of size $n \geq 7$, $E_{ii} + E_{jj} \geq 2|E_{ij}|$ for $i \neq j$, and $0 \leq s = \operatorname{tr} E \leq 1$. If $1 + s - (n-1)s^2/4 > 0$, then*

$$I + E \succ 0, \qquad \det(I + E) \geq 1 + s - \frac{n-1}{4}s^2.$$

In particular this applies for $n = 7, 8$ with $s \leq 1$, and for $n = 10$ with $s \leq 3/10$.

Lemma 7 (determinant from trace and inverse trace; [4, Lemma 34]). *Let $n \geq 3$, $B \succ 0$, $\operatorname{tr} B = \sigma$, and $\operatorname{tr} B^{-1} = U$. Then $\sigma U \geq n^2$. If equality holds, $B = (\sigma/n)I$. Otherwise define*

$$t = \frac{U\sigma - n(n-2) + \sqrt{(U\sigma - n^2)(U\sigma - (n-2)^2)}}{2U}, \qquad \ell = \frac{\sigma - t}{n-1}.$$

Then $\det B \geq t\ell^{n-1}$. This lower bound increases with σ at fixed U, and decreases with U at fixed σ.

For interval use with $\sigma \geq \sigma_0$ and $U \leq U_1$, replace σ_0 by $\max(\sigma_0, n^2/U_1)$ and apply the lemma at (σ_0, U_1); a rational upper enclosure for the square root gives an upper enclosure for t, and at fixed σ, $t[(\sigma - t)/(n-1)]^{n-1}$ decreases for $t \geq \sigma/n$, so this gives a rational *lower* determinant bound.

Lemma 8 (a shared trace–inverse logarithm bound; [4, Lemma 38]). *Let $0 < \ell \leq r$, $H > 0$, and let $a_{\log} = 1/(\ell + r)$, $b_{\log} = \ell r/(\ell + r)$. If*

$$q_{\log} \leq \min\left\{\log \ell - a_{\log}\ell + \frac{b_{\log}}{\ell},\ \log H - a_{\log}H + \frac{b_{\log}}{H}\right\},$$

then $\log x \geq a_{\log}x - b_{\log}/x + q_{\log}$ for $0 < x \leq H$. In particular,

$$x - 1 \leq 3\log x + 2\left(\frac{1}{x} - 1\right) \quad \left(0 < x \leq \frac{14}{5}\right),$$
$$\log(1 + x) \geq \frac{17}{40}x + \frac{23}{40}\frac{x}{1 + x} \quad \left(-1 < x \leq \frac{9}{16}\right).$$

Lemma 9 (concentrating bounded mass; [4, Lemma 40]). *Let $0 \leq x_i \leq r$, $\sum_i x_i \leq m$, and $r > 0$. If f is convex and nondecreasing on $[0, r]$, with $f(0) = 0$, then*

$$\sum_i f(x_i) \leq jf(r) + f(m - jr), \qquad j = \lfloor \frac{m}{r} \rfloor.$$

In particular, define $\mathcal{C}_r(m) = jr^2 + (m - jr)^2$; then $\sum_i x_i^2 \leq \mathcal{C}_r(m)$. For zero capacity use $\mathcal{C}_0(m) = 0$.

Lemma 10 (rational linear certificates; [4, Lemma 44]). *Suppose a vector z satisfies $Az \leq b$ and coordinate bounds $\ell \leq z \leq h$, all rational. For any rational $\lambda \geq 0$, let $a = A^{\mathsf{T}}\lambda$. If*

$$\sum_j \min(a_j l_j, a_j h_j) > \lambda^{\mathsf{T}} b,$$

there is no such z. More generally, for a linear objective $q^{\mathsf{T}}z$ and $r = q - A^{\mathsf{T}}\lambda$,

$$q^{\mathsf{T}}z \le \lambda^{\mathsf{T}}b + \sum_j \max(r_j l_j, r_j h_j).$$

Every finite cover of this paper obeys the following rule, enforced with this lemma: the root contains every counterexample, each terminal witness excludes its whole region, and the regions exhaust the root; a contraction uses a necessary bound; a split covers its parent by two closed children; logarithms are enclosed by the rational series $\log r = 2\sum_{j<m} z^{2j+1}/(2j+1) + \mathcal{R}_m$, $z = (r-1)/(r+1)$, $|\mathcal{R}_m| \le 2|z|^{2m+1}/((2m+1)(1-z^2))$; square roots by squaring rational endpoints; and a polynomial on $[0,1]$ is nonnegative if all its Bernstein coefficients are.

Lemma 11 (a primitive extension; [4, Lemma 41]). *If a rank-eleven lattice of minimum at least four contains a primitive rank-ten lattice M, then*

$$\Delta \ge d\big(4 - \rho^2(M)\big),$$

where $\rho^2(M)$ is the squared covering radius of M.

Lemma 12 (averaging in a cell; [4, Lemma 42]). *Let a_i be points, $\pi_i \ge 0$, $\sum_i \pi_i = 1$, and $\sum_i \pi_i a_i = x$. For every positive metric G,*

$$\min_i G[a_i - x] \le \sum_i \pi_i G[a_i] - G[x].$$

A Euclidean Delaunay cell is the convex hull of the lattice points on an empty sphere; the cells tile space [1,10], and this paper uses a fixed reference tiling only as a partition, with probability distributions on its vertices.

Fact 13 (the directional bound; [4, Lemma 30]). *For D_n with $n \ge 3$, let E be symmetric with $E_{ii} + E_{jj} \ge 2|E_{ij}|$ for $i \ne j$ and $T_E = \operatorname{tr} E$; for a vector p set $a_p = \max |p_i|$, $b_p = \min |p_i|$ and $S_p = \sum |p_i|$. Then*

$$p^{\mathsf{T}}Ep \le T_E S_p \max\Big(a_p, \frac{S_p - 2b_p}{n-2}\Big). \tag{8}$$

It is used in Lemma 20.

3 The theorem and its four computational results

The computational dependencies in this longer case are separated into four results, so that a completed final cover cannot conceal an unverified outer range. They are stated here in the terms of the sections that follow, where every quantity they mention is defined; their verification obligations are those of Section 18.

Computational result 14 (the phase and centre covers for `0050`). In the coordinates of Section 4, the cofactor covers prove the capacities (13) and the stated two-coordinate refinement on $5/6 < u \le 12/13$. For a bad centre, the boundary-region certificates exclude $12/13 \le u \le 17/18$ and, below $12/13$, the regions $V \le 1/100$ or $t \le 1/8$. Lemma 19 excludes $u \ge 17/18$ directly. On the rest the certificates give the containing root stated in Section 6. The 1 249-tree forest, with 24 622 334 terminal regions, covers that entire root. Every terminal witness contradicts a necessary budget, determinant or legal contact inequality, treating both half-spinor parities when applicable. Every graft is at exactly the pending parent box, and the final frontier is empty.

Computational result 15 (the reference cells for `0050`). A complete cell certificate for $\sqrt{2}D_{10}^{+}$ proves, for every minimum-four metric with $0 \le s_G = \operatorname{tr} G - 10 \le 1/8$,

$$\rho_G^2 \le \max\left(\frac{9}{4} + \frac{9s_G}{2},\ \frac{5}{2} + \frac{5s_G}{2}\right).$$

It consists of a fundamental-domain reference partition and barycentric contact identities whose nonnegative coefficients bound every point of every cell. The implication of such identities is Lemma 12; the required coverage and coefficient checks are specified in Section 18.

Computational result 16 (the height and collar covers for `0050`). For good centres, the $u \ge 12/13$ cover and the 323 437-leaf strip cover exclude $u \ge 459/500$. The 146 291-leaf lower cover excludes $419/500 \le u \le 181/200$. On $181/200 \le u \le 459/500$, the 4 257-leaf cover excludes $\mathcal{N} \ge 31/100$ and the 15-box collar excludes $\mathcal{N} \le 1/200$. Every leaf is justified by the determinant, contact, triangle, trace or repair inequalities printed in Sections 10–16. The formal-path convexity and endpoint comparisons used for repair are checked on the whole strip where they are invoked. The assertions include all stated boundaries and all remaining phase and common-slack values. It also includes the cover excluding $t \ge 1/3$ and the five-coordinate repair cover of Section 9. The four axial-phase certificates of Input 24 and the fourteen radius records of Input 27 are included in this computational result.

Computational result 17 (the final cover for `0050`). On the five-parameter root of Section 17, the five resumed journals give a complete tree of 13 985 nodes and 6 993 terminal regions. Each contraction is a necessary inequality printed in Sections 4–17, rounded outward. Every terminal witness contradicts one of the seven stated inequality types. Conditional phase subdivisions cover their whole parent intervals, and a resumed root is exactly a preceding pending box. The completed frontier is empty.

The form itself enters through its printed matrix (Appendix A): the printed A_P is positive definite, has minimum $s_P = 12$, and its 65 minimum-vector pairs span the 45-dimensional space of symmetric matrices; we re-checked the determinant, the minimum and the number of minimal vectors in exact arithmetic (Appendix A).

4 Coordinates, phases and one common slack

Use the coordinate lattice

$$N_{\mathrm{ref}} = (D_8 \times 2\mathbb{Z}) \cup \big((h_8 - e_1, 1) + (D_8 \times 2\mathbb{Z})\big), \qquad h_8 = \frac{1}{2}\mathbf{1},$$

whose coordinate covolume is two. The printed marker (Appendix A), in these coordinates, is

$$\widetilde{P} = \begin{pmatrix} 2I_8 & -\frac{\mathbf{1}}{6} \\ -\mathbf{1}^{\mathsf{T}}_{6} & 1 \end{pmatrix}.$$

For an explicit change to the printed basis, define $\zeta = (x_1 + \cdots + x_7 - 3x_8)/2$ and $C_{\mathrm{can}}x = y$ by

$$y_1 = -x_8 + \frac{x_9}{2}, \qquad y_j = \zeta - x_{9-j} \quad (2 \le j \le 8), \qquad y_9 = \zeta - x_8.$$

Direct multiplication gives $C_{\mathrm{can}}^{\mathsf{T}} P C_{\mathrm{can}} = \widetilde{P}$ and $\det C_{\mathrm{can}} = 1/2$. The displayed generators of N_{ref} have integral images, whose covolume is one; hence $C_{\mathrm{can}} N_{\mathrm{ref}} = \mathbb{Z}^9$. Its 65 contact pairs are the 56 horizontal D_8 pairs, $2e_9$, and $(h_8 - e_i, 1)$ for $1 \le i \le 8$. These four assertions (the change of basis, its determinant, the integrality of the images, and the value 4 of $\widetilde{P}$ on the 65 listed contacts) were re-checked in exact arithmetic. To distinguish these coordinates from the integral basis of Section 2.1, let

$$\widetilde{X} = \widetilde{Q}^{-1} = \begin{pmatrix} B & b \\ b^{\mathsf{T}} & c \end{pmatrix}, \qquad T = \operatorname{tr} B, \qquad e = 4 \det \widetilde{Q}, \qquad Q_0 = \operatorname{diag}(2I_8, 1).$$

The factor four is the square of the coordinate covolume two. Equivalently $\widetilde{Q} = C_{\mathrm{can}}^{\mathsf{T}} Q C_{\mathrm{can}}$. Set $a = u^{-1} - 1$. After a simultaneous coordinate sign change, $b_i \ge 0$, $\sum b_i = 3uV$, $b_i \le uV/2$, and $\widetilde{Q}_{99} \ge 1$. Apply this coordinate change to the actual matrix and its contact vectors together. For the eight integral normal covectors $n_i = (e_i, -1/2)$ in this coordinate lattice, define

$$\widetilde{X}[n_i] = u\left(\frac{3}{4} + \eta_i\right), \qquad \mathcal{N} = \sum_{i=1}^{8} \eta_i.$$

There is one actual scalar $J \ge 0$ such that

$$\begin{aligned} T &= 6 - 2u - \frac{u(\mathcal{N} + V)}{3} - \frac{2J}{3}, \\ c &= 4u - 3 + \frac{u(2\mathcal{N} + 5V)}{3} + \frac{J}{3}, \\ 12(1-u) - 2u(\mathcal{N} + V) - J &\ge 0. \end{aligned} \tag{9}$$

In particular $\mathcal{N}, V \ge 0$ and $\mathcal{N} + V \le 6a$. All uses of these equations concern the same J.

Derivation of the slack equations. Before changing signs, give the eight extra contact tensors weights $\lambda_i \ge 0$ and let $\tau_\lambda = \sum_i \lambda_i$. Their coupling entries are $\tau_\lambda/2 - \lambda_i$. Hence $3uV = \sum_i |b_i| \ge 3\tau_\lambda$ and $|b_i| \le \tau_\lambda/2 \le uV/2$. The axis coefficient gives $c - \tau_\lambda \le 1$, since $\widetilde{Q}_{99} \ge 1$. Marking reads $2T + c - \tau_\lambda \le 9$, so define $J = 9 - 2T - c + uV \ge 0$. Summing the definitions of η_i gives $u\mathcal{N} = T + 2c - 3uV - 6u$. Solving these two equations gives (9); its last left side equals $3(1 - c + uV) \ge 0$. In particular $u \le 1$. Now (3), $d < d_{\mathrm{cut}} < 1114$, $v \le 4u/3$, and $\Delta < 972$ give

$$u > \frac{16\sqrt{2}}{27}, \qquad \frac{8192}{9} \le e < \min(1114u,\ 1296u^2). \tag{10}$$

Phases. Choose nearest half-integer phases for the favoured covectors n_i and a nearest integer phase for e_9. After simultaneous negation, write their displacement as

$$\delta_\beta = \left(\varepsilon + \left(\frac{t}{2}\right)\mathbf{1},\ t\right), \quad 0 \le t \le \frac{1}{2}, \quad A_\varepsilon = \sum \varepsilon_i^2, \quad Z = \sum \varepsilon_i, \quad p = A_\varepsilon + tZ.$$

The normal inequalities become

$$\begin{gathered} \eta_i \ge |\varepsilon_i| - \varepsilon_i^2, \qquad |\varepsilon_i| \le \frac{1}{2}, \\ \mathcal{N} + 2t(1-t) \le 9a, \qquad 3V + \mathcal{N} + p \ge 4t(1-t), \\ 3V + \frac{\mathcal{N} + A_\varepsilon - (1-2t)Z}{2} \ge 4t(1-t). \end{gathered} \tag{11}$$

The first line of (11) comes from the sixteen covectors $(e_i, \pm\frac{1}{2})$: their energies sum to $2\mathcal{N} + 6V$ by (9), the two phases of each pair differ by t, and $f(x) = x - x^2$ satisfies $f(p) + f(q) \ge f(d)$ whenever $d \le p + q$ on $[0, \frac{1}{2}]$; hence $8t(1-t) \le 2\mathcal{N} + 6V$, and $\mathcal{N} + V \le 6a$ gives $\mathcal{N} + 2t(1-t) \le \frac{3}{2}(\mathcal{N} + V) \le 9a$. The other two lines are the normal inequalities at $(e_i, \pm\frac{1}{2})$ with the actual phases. The normal inequalities concern these actual phases. The nearest-phase centre has two classes modulo N_{ref}: $\beta_g = (h_8, 0)$ and $\beta_b = e_9$, before the sign change. Indeed its ninth coordinate is an integer; reducing it modulo two leaves the parity of the horizontal integer coordinates as the only choice. We call the image of β_g *good* and the other class *bad*. We track the two classes and their half-spinor parities through the coordinate change.

Table 1. How the reductions for `0050` fit together. Here u is the first dual minimum and $\mathcal{N}$ the normal-energy budget. Every row retains all remaining phase and common-slack values.

stage	argument and scope	finite input
Centre choice	Section 6 excludes the bad centre over the entire admissible domain	Computational result 14
Extension bound	The common D_{10}^{+} trace criterion supplies the inverse-trace gap	Reference cells, Computational result 15
Outer heights	Section 16 excludes $u \geq 459/500$ and the lower-height band	Computational result 16
Normal budget	On $181/200 \leq u \leq 459/500$, exclude $\mathcal{N} \leq 1/200$ and $\mathcal{N} \geq 31/100$	Collar and large-budget covers, Computational result 16
Final root	Section 17 covers every surviving parameter tuple	13 985 nodes, 6 993 terminals; Computational result 17

5 A common cofactor argument controls every phase

The following calculation explains the uniform cap used throughout the proof. Suppose $\eta_i \geq h_\eta \geq 0$, and let $y = uV$. The budgets imply

$$0 \leq y \leq 6(1-u) - uh_\eta, \qquad T \leq \frac{9 + y - c}{2},$$
$$c_{\min} = 4u - 3 + \frac{5y}{3} + \frac{2uh_\eta}{3} \leq c \leq 1 + y.$$

Deleting row i from $\widetilde{X}$, taking the Schur complement in c, and applying Cauchy–Schwarz to the other seven couplings gives

$$\operatorname{cof}_{ii} \widetilde{X} \leq c \left[\frac{T - B_{ii} - \frac{(3y - b_i)^2}{7c}}{7} \right]^7.$$

Here $B_{ii} - b_i = u(3/4 + \eta_i) - c/4$. The bracket decreases with b_i, since $y/c \leq 1/2$ implies its derivative is at most $-4/7$. Thus, for a necessary cap $e < E$,

$$\widetilde{Q}_{ii} < U_i := \frac{Ec}{4} \left[\frac{\frac{9}{2} + \frac{y}{2} - u\left(\frac{3}{4} + h_\eta\right) - \frac{c}{4} - \frac{9y^2}{7c}}{7} \right]_+^7. \tag{12}$$

The positive part extends the bound outside the physically possible domain; a physical positive cofactor cannot have a nonpositive enclosing bracket.

The primal pair contacts give $\widetilde{Q}_{ii}+\widetilde{Q}_{jj}\ge 4$. Therefore, for the horizontal primal block H_Q,

$$\operatorname{tr} H_Q \ge \max(16, 28-6U_i), \qquad \det H_Q = \frac{ec}{4} < \frac{Ec}{4},$$
$$\operatorname{tr} H_Q^{-1} \le W := \frac{9+y-c}{2} - \frac{9y^2}{8c}.$$

The determinant minimum of Lemma 7, evaluated at trace $\max(16, 28-6U_i, 64/W)$ and inverse trace W, can contradict this cap. This is a three-variable rational test in (u, y, c). Its containing interval for c is parametrised by $c = c_{\min} + \xi(1+y-c_{\min})$, $0\le\xi\le 1$. The case $y=0$ is included.
On $5/6 < u \le 12/13$, at $h_\eta = 15/64$ the complete certificate has 3 978 481 nodes, 1 989 241 leaves, and no pending box. Thus

$$\eta_i < \frac{15}{64}, \qquad |\varepsilon_i| < \frac{3}{8} \qquad (1\le i\le 8). \tag{13}$$

The cover is part of Computational result 14.

Lemma 18 (the second normal-energy capacity). *For $5/6 < u \le 1$, at most one of the eight normal energies satisfies $\eta_i \ge 55611/250000$. Consequently at most one phase satisfies $|\varepsilon_i| \ge 167/500$. This statement requires no interval cover.*

Proof. Let

$$a_* := \frac{3}{4} + \frac{55611}{250000}, \qquad u_* := \frac{557}{648}, \qquad \ell := \frac{3711}{10000}, \qquad r := \frac{3713}{10000}.$$

Suppose $\eta_i \ge 55611/250000$. The preceding budgets give $0\le y\le 1$ and $0 < c \le 1+y$. The two normal flags give $e < E := \min(1114u, 1296u^2)$. We bound the right side of (12) in three elementary steps.

First set $A = (9+y)/2 - a_*u$ and $B = 9y^2/7$. Since $a_*u \le 1$ and $y\le 1$, we have $A \ge 2(1+y)$. For $0 < c \le 1+y$,

$$\frac{d}{dc}\Big\{c\Big(A - \frac{c}{4} - \frac{B}{c}\Big)^7\Big\} = \Big(A - \frac{c}{4} - \frac{B}{c}\Big)^6\Big(A - 2c + \frac{6B}{c}\Big) \ge 0.$$

The same endpoint bound holds with the positive part: if the bracket is nonpositive its positive part vanishes. Thus, with

$$s_0(y) := \frac{17}{4} + \frac{y}{4} - \frac{9y^2}{7(1+y)}, \qquad c\Big[A - \frac{c}{4} - \frac{B}{c}\Big]_+^7 \le (1+y)[s_0(y) - a_*u]^7.$$

The endpoint bracket is positive, since $27/7 \le s_0(y) \le 9/2$ and $a_*u \le 1$. For the lower bound, multiply by $28(1+y)$ and use $28(1+y)(s_0(y)-27/7) = (1-y)(29y+11) \ge 0$; the upper bound follows by dropping the nonnegative subtracted term.

Second, the two caps agree at u_*, because $1296u_* = 1114$. For every $s\in[27/7, 9/2]$, the function $u^2(s-a_*u)^7$ increases on $[0, u_*]$, whereas $u(s-a_*u)^7$

decreases on $[u_*, 1]$. Indeed their derivatives have respectively the signs of $2s - 9a_*u$ and $s - 8a_*u$, and the exact rational inequalities

$$9a_*u_* \le \frac{54}{7}, \qquad 8a_*u_* \ge \frac{9}{2}$$

supply those signs. It follows that $E[s_0(y) - a_*u]^7 \le 1114u_*[s_0(y) - a_*u_*]^7$.

Finally put $s_* := 17/4 - a_*u_*$ and $g(y) := s_* + y/4 - 9y^2/(7(1+y))$. Here $s_* \ge 2$, so $g > 0$ on $[0, 1]$. Direct differentiation gives

$$g'(y) = \frac{36 - 29(1+y)^2}{28(1+y)^2}, \qquad \frac{d}{dy}\big\{(1+y)g(y)^7\big\} = \frac{g(y)^6 P(y)}{7(1+y)},$$
$$P(y) = 7s_* + \frac{49}{4} + \Big(7s_* - \frac{399}{4}\Big)y - 58y^2.$$

The polynomial P decreases on $[0, 1]$. The following are exact rational inequalities, verified by substitution and clearing positive denominators:

$$P(\ell) \ge 0, \qquad P(r) \le 0, \qquad 29(1+\ell)^2 \ge 36, \qquad \frac{1114u_*}{8 \cdot 7^7}(1+r)g(\ell)^7 < 1.$$

Hence $(1+y)g(y)^7$ increases up to ℓ, decreases after r, and on $[\ell, r]$ is at most $(1 + r)g(\ell)^7$, because g decreases there. Combining all three bounds with (12) gives

$$\frac{\widetilde{Q}_{ii}}{2} < \frac{1114u_*}{8 \cdot 7^7}(1+r)g(\ell)^7 < 1.$$

Two such coordinates would contradict $\widetilde{Q}_{ii} + \widetilde{Q}_{jj} \ge 4$. Finally $x - x^2$ increases on $[0, 1/2]$ and $(167/500)(1 - 167/500) = 55611/250000$, so the phase assertion follows from (11). All the numerical tests displayed in this proof are fixed rational calculations; no covering data are used. □

These are capacities for the actual normal energies, including any unused normal slack. The mass bound used below is Lemma 9:

$$\sum x_i^2 \le \mathcal{C}_r(m) \quad \text{when } 0 \le x_i \le r,\ \sum x_i \le m. \tag{14}$$

The same mass transfer applies to any convex increasing function of the normal energies, using one capacity $15/64$ and seven capacities $55611/250000$.

6 From a gluing centre to a uniform extension bound

At the good centre, the rank-ten reference is $\sqrt{2}D_{10}^+$, where $D_{10}^+ = D_{10} \cup (D_{10} + \frac{1}{2}\mathbf{1})$. The reference coset determines the basis; the actual first extension is represented by a metric G on it. This metric has minimum four and satisfies

$$\det G = \frac{e}{1024u} < \frac{557}{512}, \qquad \operatorname{tr} G = \operatorname{tr}(Q_0^{-1}\widetilde{Q}) + \widetilde{Q}[\delta_\beta] + \frac{1}{u}. \tag{15}$$

Legal first-layer section components have $\widetilde{Q}$-value at least $4 - 1/u$. The integer components include $(\varepsilon + (t/2)\mathbf{1} - e_i,\ t - 1)$. For one parity, the half-spinor components include $x_\sigma = (\varepsilon + (t-\sigma)\mathbf{1}/2,\ t)$, $\sigma = \pm 1$; for the other parity they include $x_\sigma + \sigma e_i$. At a bad centre, the legal axial components are $\delta_\beta \pm e_9$; the horizontal half-spinors again have one of the two parities just described. Averaging the axial pair gives

$$4 - \frac{1}{u} \le \widetilde{Q}[\delta_\beta] + \widetilde{Q}_{99}. \tag{16}$$

The following direct argument handles the outer heights.

Lemma 19. *For $u \ge 17/18$, the nearest-phase centre is good.*

Proof. The slack equations give $\operatorname{tr}(\widetilde{X}\widetilde{P}) = 2T + c - uV = 9 - J \le 9$, and $\det \widetilde{P} = 2048/9$. The reference spectral comparison of Lemma 5, with $e < 1114u$ and $u \le 1$, gives

$$\widetilde{Q} \preceq H_{\mathrm{ref}}\widetilde{P}, \qquad H_{\mathrm{ref}} = 2; \qquad H_{\mathrm{ref}} = \frac{9}{5} \text{ if } u \le \frac{19}{20}.$$

The two required endpoint inequalities are the fixed rational checks

$$1114 \le \frac{8192}{9}\,2\Big(\frac{8}{9 - \frac{1}{2}}\Big)^8, \qquad \frac{10583}{10} \le \frac{8192}{9}\,\frac{9}{5}\Big(\frac{8}{9 - \frac{5}{9}}\Big)^8.$$

Indeed an eigenvalue beyond H_{ref} forces the relative determinant to be at least $H_{\mathrm{ref}}[8/(9 - 1/H_{\mathrm{ref}})]^8$, whereas $\det(\widetilde{P}^{-1}\widetilde{Q}) = 9e/8192$ is smaller. The axial contact coefficient gives $\widetilde{Q}_{99}(c - uV) \le 1$: before changing signs its coefficient is $c - \tau_\lambda$, with $\tau_\lambda \le uV$, and coefficient times contact value is at most one. Consequently

$$\widetilde{Q}_{99} \le \frac{1}{4u - 3 + \big(\frac{2u}{3}\big)(\mathcal{N} + V)} \tag{17}$$

whenever the denominator is positive.

Put $f(x) = x - x^2$ on $[0, 1/2]$ and $L_\varepsilon = \sum_i |\varepsilon_i|$. The normal-energy budget gives

$$L_\varepsilon \le \frac{s}{2} + r, \qquad A_\varepsilon \le \frac{s}{4} + r^2, \qquad \mathcal{N} = \frac{s}{4} + r - r^2, \tag{18}$$

for $s = 0$ on $[0, 1/4]$ and $s = 1$ on $[1/4, 1/2]$. Here $0 \le r \le 1/2$. To justify this mass bound, apply mass transfer to the energies $f(|\varepsilon_i|) \in [0, 1/4]$. The increasing inverse of f, and its square, are convex. For two energies with a fixed sum, either objective is convex as one energy varies, so moving to an endpoint does not decrease it. Repeating leaves at most one unsaturated energy and proves (18). Increasing the total energy to its allowed upper bound is harmless.

Only endpoint bounds for these inverse functions are needed. For $0 \le \theta \le 1$,

$$f((1-\theta)x_0 + \theta x_1) - (1-\theta)f(x_0) - \theta f(x_1) = \theta(1-\theta)(x_0 - x_1)^2 \ge 0.$$

Since f is increasing, this identity bounds its inverse by the chord through any upper endpoint enclosures. It also proves the convexity used in the mass transfer. Squaring

preserves convexity and monotonicity on the nonnegative interval. The displacement comparison is

$$\widetilde{Q}[\delta_\beta] \le H_{\mathrm{ref}}\Big\{2\Big(\frac{s}{4}+r^2\Big)+\frac{5}{3}t\Big(\frac{s}{2}+r\Big)+\frac{11}{3}t^2\Big\}. \tag{19}$$

For fixed $s \ge 0$ its right side is convex and increasing in $(r,t) \ge 0$: the quadratic coefficient matrix has positive diagonal entries and determinant $239/36 > 0$. Thus the inverse-energy chords and two endpoint substitutions bound a whole segment, even when r increases and t decreases.

First let $u \ge 19/20$. Then $\mathcal{N} \le 6/19$ and $\mathcal{N} + 2f(t) \le 9/19$. On the two mass intervals the endpoint triples (s,r,t) can be bounded respectively by

$$\Big(0,0,\frac{193}{500}\Big),\ \Big(0,\frac{1}{2},\frac{129}{1000}\Big); \qquad \Big(1,0,\frac{129}{1000}\Big),\ \Big(1,\frac{71}{1000},\frac{87}{1000}\Big).$$

For example each residual bound $\bar{r}$ satisfies $f(\bar{r}) \ge \mathcal{N} - s/4$ at its endpoint, and each axis bound $\bar{t}$ satisfies $2f(\bar{t}) \ge 9/19 - \mathcal{N}$. These are fixed rational inequalities; their affine interpolations supply the whole-interval bounds just proved. All four substitutions in (19), with $H_{\mathrm{ref}} = 2$, are at most $3/2$. Equation (17) gives $\widetilde{Q}_{99} \le 5/4$. Hence the right side of (16) is at most $11/4 < 56/19 \le 4 - 1/u$, a contradiction.

Now let $17/18 \le u \le 19/20$. We have $\mathcal{N} \le 6/17$ and $\mathcal{N} + 2f(t) \le 9/17$. If $t \le 1/3$ and $\mathcal{N} \le 13/153$, then $r \le 47/500$, so (19) on this rectangle is at most $3/2$. On the remaining two mass intervals, $[13/153, 1/4]$ and $[1/4, 6/17]$, use the endpoint triples

$$\Big(0,\frac{47}{500},\frac{1}{3}\Big),\ \Big(0,\frac{1}{2},\frac{17}{100}\Big); \qquad \Big(1,0,\frac{17}{100}\Big),\ \Big(1,\frac{3}{25},\frac{1}{10}\Big).$$

The same chord argument gives the bound $3/2$ throughout. Equation (17) gives $\widetilde{Q}_{99} \le 9/7$, and $3/2 + 9/7 = 39/14 < 50/17 \le 4 - 1/u$.

Finally let $1/3 \le t \le 1/2$. Then $f(t) \ge 2/9$, so $\mathcal{N} \le 13/153$ and (18) has $s = 0$, $r \le 47/500$. In particular $\mathcal{N} + p \le (1+t)r \le (47/500)(1+t)$. The concave quadratic $4f(t) - (47/500)(1+t)$ has endpoint values $859/1125$ and $859/1000$, both greater than $3/4$. The paired phase budget in (11) therefore gives $V \ge 1/4$. Now (17) gives $\widetilde{Q}_{99} \le 108/101$, and the corner $(r,t) = (47/500, 1/2)$ in (19) gives $\widetilde{Q}[\delta_\beta] \le 183/100$. Since $183/100 + 108/101 < 50/17 \le 4 - 1/u$, the bad-centre average is again impossible. This includes every height and split boundary. □

The bounds (17)–(19), established inside this proof, are reused in Sections 7–9.

The remaining bad-centre domain. For the remaining bad-centre domain, the finite boundary certificates exclude $12/13 \le u \le 17/18$, $V \le 1/100$, and $t \le 1/8$. For the rest, $J < 1/8$ follows from AM–GM on the determinant with the cap $e < 1114 \cdot 12/13$ of (10) (the shear and the dispersion of the coupling coordinates are retained in the same budget), and $A_\varepsilon < 3/5$, $|Z| < 9/5$ from $\mathcal{N} \le 6/5$, the capacity (13), Lemma 18 (at most one $\eta_i \ge 55611/250000$) and Lemma 9 applied to $|\varepsilon_i| \le (1 - \sqrt{1-4\eta_i})/2$;

these give the containing root

$$(u,\mathcal{N},V,J)\in\left[\frac{5}{6},\frac{12}{13}\right]\times\left[0,\frac{6}{5}\right]\times\left[\frac{1}{100},\frac{6}{5}\right]\times\left[0,\frac{1}{8}\right],$$
$$(t,A_\varepsilon,Z)\in\left[\frac{1}{8},\frac{1}{2}\right]\times\left[0,\frac{3}{5}\right]\times\left[-\frac{9}{5},\frac{9}{5}\right].$$

Each terminal box contradicts a necessary budget, determinant, axial contact, or *both* half-spinor parity classes. The reference and contact inequalities proving these implications are given in Sections 12–15. The complete forest has 1 249 component trees and 24 622 334 terminal regions. Every pending leaf of one component is replaced by a tree rooted at that exact box; the assembled frontier is empty. This proves good centre for every remaining first gluing, including both canonical coupling signs (Computational result 14).
Here is the common rank-ten result used afterwards.

Lemma 20 ([4, Lemma 43]). *Assume Computational result 15. For a minimum-four metric G on $\sqrt{2}D_{10}^{+}$ with $\det G<557/512$, either $\operatorname{tr}G\le 103/10$ or $\operatorname{tr}G^{-1}\le 151/15$ implies that every minimum-four primitive extension has determinant at least 1216.*

Computational result 15 supplies exactly the cell hypothesis of the common trace criterion.
Every remaining counterexample consequently has

$$D_G:=\operatorname{tr}G^{-1}-10>\frac{1}{15},\qquad D_G\le u(V-a+2p+5t^2). \tag{20}$$

The second inequality follows by block inversion in the same good-centre coordinates.

7 Uniform phase caps at the outer heights

The next two bounds use the same actual good-centre coordinates as (11). They require no finite covering certificate.

Lemma 21. *Every good-centre counterexample satisfies*

$$u\ge\frac{19}{20}\implies|\varepsilon_i|<\frac{21}{50},\qquad u\ge\frac{17}{18}\implies|\varepsilon_i|<\frac{23}{50}\qquad(1\le i\le 8).$$

No preliminary upper bound on t other than $t\le 1/2$ is required.

Proof. All four integer components

$$\left(\varepsilon+\left(\frac{t}{2}\right)\mathbf{1}-\sigma e_i,\ t-\tau\right),\qquad\sigma,\tau\in\{-1,1\},$$

are legal first-layer vectors. Indeed, before the sign change, $(h_8+\sigma e_i,1)-(h_8-e_1,1)=(\sigma e_i+e_1,0)$ belongs to $D_8\times 2\mathbb{Z}$ for either sign, and subtracting $2e_9$ changes the axial

sign. The coordinate sign change only permutes these four choices. Their section values are at least $4 - 1/u$.

Fix i, put $x = |\varepsilon_i|$, and suppose $x \geq \rho > 0$. Shift ε_i by its sign, leaving every other error fixed; write the resulting vector as ε' and put $\delta' = (\varepsilon' + (t/2)\mathbf{1}, t)$. Averaging the two legal axial signs cancels their cross terms and gives

$$\widetilde{Q}[\delta'] \geq 4 - \frac{1}{u} - \widetilde{Q}_{99}. \tag{21}$$

The shifted error has absolute value $1 - x \leq 1 - \rho$. Since $f(x) = x - x^2$ increases on $[0, 1/2]$, the other seven errors have total energy at most $\mathcal{N} - f(\rho)$. If this number is in $[0, 1/4]$, define $r \in [0, 1/2]$ by $f(r) = \mathcal{N} - f(\rho)$. The unfilled mass bound in (18) gives

$$\sum_j |\varepsilon'_j| \leq 1 - \rho + r, \qquad \sum_j (\varepsilon'_j)^2 \leq (1 - \rho)^2 + r^2.$$

Thus $\widetilde{Q} \preceq H\widetilde{P}$ implies

$$\widetilde{Q}[\delta'] \leq H\mathcal{P}_\rho(r, t), \qquad \mathcal{P}_\rho(r, t) = 2\big((1 - \rho)^2 + r^2\big) + \frac{5}{3}t(1 - \rho + r) + \frac{11}{3}t^2. \tag{22}$$

The coefficients $5/3$ and $11/3$ follow by expanding $\widetilde{P}[(\varepsilon' + (t/2)\mathbf{1}, t)]$. This quadratic is convex and increasing for $r, t \geq 0$: its quadratic matrix has positive diagonal and determinant $239/36$. The inverse-energy chord identity in the proof of Lemma 19 therefore bounds it by its two endpoint values when $f(r) = \mathcal{N} - f(\rho)$ and $f(t) \leq (9a_{\max} - \mathcal{N})/2$. The following rational data give $\mathcal{P}_\rho \leq 7/8$ throughout each displayed interval:

ρ	$a_{\max}$	interval for $\mathcal{N}$	(r_0, t_0)	(r_1, t_1)
$\frac{21}{50}$	$\frac{1}{19}$	$\left[\frac{609}{2500}, \frac{6}{19}\right]$	$\left(0, \frac{133}{1000}\right)$	$\left(\frac{79}{1000}, \frac{87}{1000}\right)$
$\frac{23}{50}$	$\frac{1}{17}$	$\left[\frac{621}{2500}, \frac{6}{17}\right]$	$\left(0, \frac{17}{100}\right)$	$\left(\frac{3}{25}, \frac{1}{10}\right)$

Specifically, at each endpoint ν the data satisfy $f(r_j) \geq \nu - f(\rho)$, $2f(t_j) \geq 9a_{\max} - \nu$, and $\mathcal{P}_\rho(r_j, t_j) \leq 7/8$. All arguments lie in $[0, 1/2]$; each check is a rational inequality. The affine energy budgets are interpolated first, then the increasing convex quadratic is applied. This justifies the whole interval even though the two energy budgets have opposite slopes.

For $u \geq 19/20$, we have $a \leq 1/19$ and $\mathcal{N} \leq 6/19$. If $u \geq 97/100$, then $\mathcal{N} \leq 18/97 < 609/2500 = f(21/50)$, which already excludes $x \geq 21/50$. Otherwise $e < 1114(97/100) = 54029/50$ and the same spectral comparison used in Lemma 19 gives $\widetilde{Q} \preceq (19/10)\widetilde{P}$. Its fixed rational endpoint check is

$$\frac{54029}{50} \leq \frac{8192}{9}\,\frac{19}{10}\Big(\frac{8}{9 - \frac{10}{19}}\Big)^8.$$

Equations (17) and (21) give $\widetilde{Q}[\delta'] \geq 129/76$. The first row of the table instead gives $\widetilde{Q}[\delta'] \leq (19/10)(7/8) = 133/80 < 129/76$. This proves the first cap.

It remains to consider $17/18 \le u \le 19/20$. Here $a \le 1/17$, $\mathcal{N} \le 6/17$, and $\widetilde{Q} \preceq (9/5)\widetilde{P}$. The second row of the table bounds the shifted value by $(9/5)(7/8) = 63/40$, whereas (17) and (21) bound it below by $4 - 18/17 - 9/7 = 197/119 > 63/40$. For larger u, the first cap is stronger. This proves the second cap. □

Lemma 22 (middle-height endpoint bound). *If $u \ge 17/18$ and $t \le 1/3$, then $E = D_G + J \le 65/81$.*

Proof. Put $f(x) = x - x^2$, $\rho = 23/50$, and $F_0 = f(\rho) = 621/2500$. Lemma 21 and capacity filling give a chart $s \in \{0, 1\}$, $0 \le r \le \rho$, such that

$$\mathcal{N} = sF_0 + f(r), \qquad A_\varepsilon \le S = s\rho^2 + r^2, \qquad |Z| \le R = s\rho + r.$$

Also $\mathcal{N} \le 6/17$ and $\mathcal{N} + 2f(t) \le 9/17$. Set $P_s(r, t) = 2S + 2tR + 5t^2$. We claim $P_s(r, t) - \mathcal{N} \le 5/9$. On $[0, 1/4]$, the inverse g of $f|_{[0,1/2]}$ is increasing and convex. On each of the following three intervals, replace r by $g(\mathcal{N} - sF_0)$ and t by the indicated bound:

s	interval for $\mathcal{N}$	bound for t
0	$\left[0, \frac{13}{153}\right]$	$\frac{1}{3}$
0	$\left[\frac{13}{153}, F_0\right]$	$g\left(\frac{\frac{9}{17} - \mathcal{N}}{2}\right)$
1	$\left[F_0, \frac{6}{17}\right]$	$g\left(\frac{\frac{9}{17} - \mathcal{N}}{2}\right)$

The quadratic P_s is convex and increasing in both nonnegative coordinates: its quadratic part is $2r^2 + 2rt + 5t^2$, with positive definite Hessian, and its linear coefficients are nonnegative. Consequently each resulting function of $\mathcal{N}$, after subtracting $\mathcal{N}$, is convex. Its maximum is at an endpoint. The rational upper bounds needed there are

$\mathcal{N}$	s	r bound	t bound
0	0	0	$\frac{1}{3}$
$\frac{13}{153}$	0	$\frac{47}{500}$	$\frac{1}{3}$
F_0	0	$\frac{23}{50}$	$\frac{17}{100}$
F_0	1	0	$\frac{17}{100}$
$\frac{6}{17}$	1	$\frac{3}{25}$	$\frac{1}{10}$

For each row, applying f to its rational bound verifies that it bounds the required inverse value; direct substitution gives $P_s - \mathcal{N} \le 5/9$. This proves the claim. Finally $V + \mathcal{N} \le 6a$ and $p \le S + tR$ give

$$E = u(V - a + 2p + 5t^2) \le u(5a - \mathcal{N} + P_s) \le 5(1 - u) + \frac{5u}{9} \le \frac{65}{81}.$$

□

Lemma 23. *If $n \ge 0$ and $\mathcal{N} \ge n$, then $u \le 6/(6 + n)$. If $0 \le s \le t \le 1/2$, then $u \le 9/(9 + 2s(1 - s))$. In particular,*

$$\mathcal{N} \ge \frac{53}{200} \implies u \le \frac{1200}{1253} < \frac{24}{25}, \qquad t \ge \frac{2}{7} \implies u \le \frac{441}{461} < \frac{24}{25}.$$

Proof. The normal budget gives $n \le \mathcal{N} \le 6a$. The phase budget and monotonicity of f give $2s(1-s) \le 2t(1-t) \le 9a$. Multiplying by $u > 0$ and using $ua = 1-u$ proves the two bounds. The stated specialisations follow by rational substitution. □

Thus a height cover treating either of these two branches can use $u < 24/25$ directly from its own branch condition.

8 Excluding large axial phases at the outer heights

We next prove $t < 2/7$ whenever $u \ge 19/20$. This step needs only four finite polynomial certificates, which we specify before using them. Let

$$\overline{w}(R) = \frac{6}{19} - R + R^2, \qquad \underline{w}(R,t) = \frac{(1-t)(4t-R)}{3},$$
$$\kappa(R,w) = \frac{5w}{3} - \frac{2\overline{w}(R)}{3}, \qquad \mathcal{V}(R,t,w) = \frac{3w^2}{2} - \frac{w(1-t)(t-R)}{2},$$
$$\mathcal{F}(R,t,w) = \frac{25}{24}\kappa(R,w) - \frac{73}{20}\mathcal{V}(R,t,w) + \frac{73}{33}\Big(\kappa(R,w)^2 - w\kappa(R,w) + \frac{73}{160}w^2\Big).$$

Here w is a polynomial argument; its actual value will be the coupling budget V. Define the rational rectangle

$$\mathcal{R} = \left[0, \frac{71}{1000}\right] \times \left[\frac{2}{7}, \frac{39}{100}\right].$$

Input 24 (four axial-phase polynomial certificates). The certificate gives the margin $\delta = 1/1000$ and a nonnegative tensor Bernstein expansion on $\mathcal{R}$ for each of the four polynomials

$$\kappa(R,\underline{w}), \qquad \kappa(R,\overline{w}), \qquad \mathcal{F}(R,t,\underline{w}) - \delta, \qquad \mathcal{F}(R,t,\overline{w}) - \delta. \tag{23}$$

In this formula $\underline{w} = \underline{w}(R,t)$ and $\overline{w} = \overline{w}(R)$. Explicitly, set $x = 1000R/71$ and $y = 700(t-2/7)/73$. For each polynomial P in (23), the certificate consists of two nonnegative integers n, m and rational coefficients $c_{ij} \ge 0$ such that

$$P(R,t) = \sum_{i=0}^{n}\sum_{j=0}^{m} c_{ij} x^i (1-x)^{n-i} y^j (1-y)^{m-j}. \tag{24}$$

The checker expands both sides over $\mathbb{Q}$ and compares every coefficient. Binomial factors, if used, are included in c_{ij}. These four finite identities and their coefficient signs are the complete computational input for this step.

These coefficients are recorded in `physical_followup_2026-10-20/lower_height_phase_balance_verified.json`, under `positive_certificates`. The rows `phase_kappa`, `budget_kappa`, `phase_margin`, `budget_margin` correspond, respectively, to the four polynomials in (23). The stored degrees are $(2,2), (2,0), (4,4), (4,2)$. The two margin polynomials have minimum Bernstein coefficients $37767101/4633704306$ and $503/409640$; thus $\delta = 1/1000$ is permitted. Subtract δ from each of their normalised Bernstein coefficients and absorb the binomial factors to obtain (24). The companion `lower_height_phase_balance_replayed.json` records the independent rational replay of all 52 coefficients. This input is part of Computational result 16.

Proposition 25. *Under Input 24, every good-centre counterexample with $u \geq 19/20$ satisfies $t < 2/7$.*

Proof. Suppose $t \geq 2/7$. The phase budget gives

$$a \leq \frac{1}{19}, \qquad \mathcal{N} \leq \frac{9}{19} - 2\left(\frac{2}{7}\right)\left(\frac{5}{7}\right) = \frac{61}{931}.$$

There is therefore a unique $R \in [0, 71/1000]$ with $R - R^2 = \mathcal{N}$: the function $f(R) = R - R^2$ increases there, and $f(71/1000) > 61/931$. Also $t \leq 39/100$, since $2f(39/100) > 9/19$. The unfilled case of (18) gives

$$L_\varepsilon \leq R, \qquad A_\varepsilon \leq R^2, \qquad \mathcal{N} + A_\varepsilon \leq R. \tag{25}$$

All comparisons here are fixed rational inequalities.

Write $\beta_i = b_i/u$, so $0 \leq \beta_i \leq V/2$ and $\sum_i \beta_i = 3V$, and put $\nu = \sum_i \beta_i^2$. Since $t + \varepsilon_i \geq 2/7 - R > 0$, the opposite normal inequality is

$$t(1-t) + (1-2t)\varepsilon_i - \varepsilon_i^2 \leq \eta_i + 2\beta_i. \tag{26}$$

Summing it and using (25) and $Z \geq -R$ yields

$$6V \geq 8t(1-t) + (1-2t)Z - (\mathcal{N} + A_\varepsilon) \geq 2(1-t)(4t - R).$$

Together with $\mathcal{N} + V \leq 6a$, this places the actual coupling in

$$\underline{w}(R,t) \leq V \leq \overline{w}(R). \tag{27}$$

The same normals give a sharper square bound than the individual caps alone. Set $\ell_i = V/2 - \beta_i$; then $0 \leq \ell_i \leq V/2$ and $\sum_i \ell_i = V$. Multiplying (26) by ℓ_i and summing, use

$$\sum_i \ell_i \varepsilon_i \geq -\frac{VR}{2}, \qquad \sum_i \ell_i(\eta_i + \varepsilon_i^2) \leq \frac{VR}{2}.$$

It follows that

$$2\sum_i \ell_i \beta_i \geq V(1-t)(t-R).$$

Since $\sum_i \ell_i \beta_i = 3V^2/2 - \nu$, this proves

$$\nu \leq \mathcal{V}(R,t,V). \tag{28}$$

This weighted argument uses the same eight couplings and the same phases as the unweighted normal budget.

The actual directional Schur inequality. Let $q = \widetilde{Q}_{99} \geq 1$ and $s_b = b^{\mathsf{T}} B^{-1} b$. Block inversion gives

$$c - s_b = \frac{1}{q}, \qquad \widetilde{Q}[(b,z)] = s_b + q(s_b - z)^2 \quad (z \in \mathbb{R}). \tag{29}$$

By Lemma 23, the hypothetical branch itself forces $u < 24/25$. Consequently $e < 26736/25$. The common reference

$$\widetilde{P} = 2\begin{pmatrix} I_8 & -\frac{\mathbf{1}}{12} \\ -\mathbf{1}^{\intercal}_{12} & \frac{1}{2} \end{pmatrix}$$

has determinant $2048/9$ and $\operatorname{tr}(\widetilde{X}\widetilde{P}) = 9 - J \le 9$. The reference eigenvalue bound applies with $M = 73/40$, since the following is an exact rational check:

$$\frac{26736}{25} \le \frac{8192}{9}\,\frac{73}{40}\Big(\frac{8}{9 - \frac{40}{73}}\Big)^{8}.$$

Thus $\widetilde{Q} \preceq M\widetilde{P}$. In the same direction as (29), the reference value is

$$\widetilde{P}[(b, z)] = 2u^2\nu - uVz + z^2.$$

Replacing q by its lower bound one and choosing $z = (MuV - 2s_b)/(2(M-1))$ in the resulting quadratic inequality gives

$$s_b + \frac{73}{33}\Big(s_b^2 - uVs_b + \frac{73}{160}(uV)^2\Big) \le \frac{73}{20}u^2\nu. \tag{30}$$

Indeed subtraction of the right side before this choice leaves a concave quadratic in z; the displayed value is its maximiser. No division by $M - q$ or assumption about its sign is used.

The first identity in (29) gives $s_b \ge c - 1$. By (9),

$$c - 1 - u\kappa(R, V) = \frac{80}{19}u - 4 + \frac{J}{3} \ge 0.$$

Hence $s_b \ge u\kappa(R, V)$. Also the common budget gives $uV \le 6(1-u) \le 3/10$.

The finite endpoint reduction. The four identities in Input 24 imply their polynomials are nonnegative on the entire closed rectangle, since $0 \le x, y \le 1$. The function $\kappa(R, w)$ is affine in w; the coefficient of w^2 in $\mathcal{F}(R, t, w)$ is exactly $-95411/47520$. Thus, for $w = (1-\theta)\underline{w} + \theta\overline{w}$ and $0 \le \theta \le 1$,

$$\kappa(R, w) = (1-\theta)\kappa(R, \underline{w}) + \theta\kappa(R, \overline{w}) \ge 0,$$
$$\mathcal{F}(R, t, w) = (1-\theta)\mathcal{F}(R, t, \underline{w}) + \theta\mathcal{F}(R, t, \overline{w}) + \frac{95411}{47520}\theta(1-\theta)(\overline{w} - \underline{w})^2 \ge \delta > 0.$$

If the coupling interval degenerates, its single endpoint already gives the same conclusions. Apply them at $w = V$ using (27).

Finally, replacing s_b by $u\kappa(R, V)$ decreases the left side of (30). Its difference is

$$(s_b - u\kappa)\Big[1 + \frac{73}{33}(s_b + u\kappa - uV)\Big] \ge 0,$$

where $\kappa = \kappa(R, V)$: the bracket is at least $1 - 73/110 > 0$ and both $s_b, u\kappa$ are nonnegative. Use (28), divide by $u^2 > 0$, and use $1/u \ge 25/24$ and $\kappa \ge 0$. The result is $\mathcal{F}(R, t, V) \le 0$, contradicting the strict lower bound δ. Every matrix inequality was applied at the actual slack J; only scalar polynomial endpoints were interpolated. □

9 Projected determinants and the middle height band

The following argument supplies both the test excluding $t \geq 1/3$ and one family of terminal tests for $17/18 \leq u \leq 19/20$. Throughout this section, H denotes the horizontal 8×8 principal block of $\widetilde{Q}$; B still denotes the horizontal block of $\widetilde{Q}^{-1}$. Let

$$\beta = \frac{b}{u}, \qquad \nu_* = |\beta|^2, \qquad \kappa = \frac{5V + 2\mathcal{N}}{3} - 4a, \qquad n = \frac{\mathbf{1}}{\sqrt{8}}.$$

The tests below are used only when $\kappa > 0$. The normal budget gives $\kappa \leq V$; hence $V > 0$, so $\nu_* > 0$ and $v_b = \beta/\sqrt{\nu_*}$ is a unit vector. The identities for c and the axial contact give $c \geq 1 + u\kappa$ and $q = \widetilde{Q}_{99} \geq 1$. Block inversion gives

$$H[v_b] = \frac{c(cq-1)}{u^2\nu_*}, \qquad B = H^{-1} + \frac{u^2\nu_*}{c} v_b v_b^{\mathsf{T}}, \qquad (v_b \cdot n)^2 = \frac{9V^2}{8\nu_*}. \tag{31}$$

Consequently, for any proved upper bound $\nu \geq \nu_*$,

$$H[v_b] \geq \frac{\kappa(1+a+\kappa)}{\nu}, \qquad 1 - (v_b \cdot n)^2 \leq 1 - \frac{9V^2}{8\nu}. \tag{32}$$

These statements concern the actual common slack J.

Loss when changing the projected direction. For $u \leq 19/20$ the reference comparison gives

$$\frac{5}{4} I \leq H < \frac{18}{5} I. \tag{33}$$

The upper comparison is the relative spectral bound $9/5$ already used above. The lower comparison follows by applying inverse-trace AM–GM to a hypothetical relative eigenvalue below $5/8$: the required rational comparison is

$$\frac{10583}{40} \, \frac{8}{5} \Big(\frac{37}{40}\Big)^8 \leq \frac{2048}{9}.$$

The determinant bound and inverse trace at most 9 then give the relative lower bound $5/8$, hence the horizontal bound $5/4$. The horizontal root contacts $H[e_i - e_j] \geq 4$ imply, on summing over all 28 unordered pairs,

$$\operatorname{tr} H - H[n] \geq 14. \tag{34}$$

Write $\Delta_H = 18/5 - 5/4 = 47/20$ for this spectral gap. Two entirely rational certificates for $\operatorname{tr} H - H[v_b] \geq K$ are available. Choose lower bounds $v \leq V$ and $C \leq H[v_b]$, an upper bound $\nu \geq \nu_*$, and rational $z \geq 0$ with $1 - 9v^2/(8\nu) \leq z^2$, where $v \geq 0$ and $\nu > 0$. The ordinary certificate is

$$K \leq 14 - \Delta_H z. \tag{35}$$

Indeed, the difference of the two unit rank-one projectors has rank at most two, with possibly nonzero eigenvalues $\pm\sqrt{1-(v_b\cdot n)^2}$. Subtracting $(5/4)I$ from H, then using (33), bounds $H[v_b]-H[n]$ by $\Delta_H z$. The refined certificate chooses also $g\ge 0$ and checks

$$C\ge\frac{97}{40},\qquad \left(\frac{18}{5}-C\right)\left(C-\frac{5}{4}\right)\le g^2,\qquad K\le 14-\Delta_H z^2-2zg. \tag{36}$$

If $C\ge 18/5$ there is already a contradiction. Otherwise let $q_b=H[v_b]$. The positive semidefinite matrix $(H-(5/4)I)((18/5)I-H)$ gives $|(H-q_bI)v_b|^2\le(18/5-q_b)(q_b-5/4)\le g^2$; the last quadratic decreases above its midpoint $97/40$. Decomposing n into its component along v_b and its orthogonal component now gives $H[v_b]-H[n]\le\Delta_H z^2+2zg$. This proves both certificates, including coincident directions.

The projected determinant inequality. Set $r_b=1/B^{-1}[v_b]>0$. A second Schur-complement identity gives

$$r_b\left(c-\frac{1}{q}\right)=u^2\nu_*,\qquad r_b\le\frac{u\nu}{\kappa},\qquad r_b\le T=\operatorname{tr}B.$$

We claim that any $K>0$ proved by the preceding tests yields

$$\det B\le\frac{7r_b}{K}\Big(\frac{T-r_b}{7}\Big)^6. \tag{37}$$

Choose an orthonormal basis whose last vector is v_b, and write $B=\left(\begin{smallmatrix}D & g_b\\ g_b^{\mathsf T} & d\end{smallmatrix}\right)$. Let $\gamma=g_b^{\mathsf T}D^{-1}g_b\ge 0$ and $s_D=\operatorname{tr}D$. Then $r_b=d-\gamma$, $T-r_b=s_D+\gamma$, and $\det B=r_b\det D$. The rank-one identity in (31) says that H^{-1} has the same block D and column g_b. Since $H\le(18/5)I$, its inverse is at least $(5/18)I$, so $D^{-1}\le(18/5)I$. The bordered inverse formula therefore gives

$$K\le\operatorname{tr}H-H[v_b]\le\operatorname{tr}D^{-1}+\left(\frac{18}{5}\right)^2\gamma.$$

For the seven positive eigenvalues of D, the elementary symmetric sum of order six and AM–GM give

$$\det D\,\operatorname{tr}D^{-1}\le 7\Big(\frac{s_D}{7}\Big)^6,\qquad \det D\le\Big(\frac{s_D}{7}\Big)^7.$$

The first inequality can also be proved by pair averaging: with the sum of two eigenvalues fixed, the symmetric sum has the form $A+B(x+y)+Cxy$ with $C\ge 0$, so replacing x,y by their average cannot decrease it. Repeatedly average the largest and smallest entries; their squared deviations from the common mean decrease, and the entries converge to that mean. Continuity gives the stated bound at the equal seventuple. Also $T\le 6-2u\le 13/3$, whence $(18/5)^2T^2<294$. The binomial expansion, using $s_D,\gamma\ge 0$, gives

$$7\Big(\frac{s_D}{7}\Big)^6+\left(\frac{18}{5}\right)^2\gamma\Big(\frac{s_D}{7}\Big)^7\le 7\Big(\frac{s_D+\gamma}{7}\Big)^6:$$

the first nonconstant term on the right suffices, since its comparison with the extra term is exactly $(18/5)^2 s_D^2 \le 294$. Multiplication by r_b/K proves (37). Define the rational expression

$$\mathcal{P}(T,R) = 7r\Big(\frac{T-r}{7}\Big)^6, \qquad r = \min\Big\{R, \frac{T}{7}\Big\}.$$

For $0 \le r \le T$, the function $r(T-r)^6$ increases up to $T/7$ and then decreases. Thus upper bounds $T \le \overline{T}$ and $r_b \le \overline{R}$ give $\det B \le \mathcal{P}(\overline{T},\overline{R})/K$. The cofactor identity $\det B = q/\det \widetilde{Q}$ also gives

$$4 < 1114u \det B. \tag{38}$$

These are the upper and lower bounds compared by a projected leaf.

Keeping the height normalisation. The sharper endpoint test keeps the factor u in (38). Suppose finite box bounds give

$$0 \le n_0 \le \mathcal{N}, \quad 0 \le v \le V, \quad 0 \le a \le \overline{a} \le \frac{1}{10}, \quad n_0 + v \le \frac{3}{5}, \quad \frac{5v+2n_0}{3} \le 1.$$

Let $d_0 = (5v+2n_0)/3$, $\kappa_0 = d_0 - 4\overline{a} > 0$, $u_0 = 1/(1+\overline{a})$, and

$$C_0 = \frac{\kappa_0(1+\overline{a}+\kappa_0)}{\nu}, \qquad T_0 = 4 + \frac{u_0(6\overline{a}-n_0-v)}{3}, \qquad R_0 = \frac{u_0\nu}{\kappa_0}.$$

Then $H[v_b] \ge C_0$. In fact, on putting $\delta = \overline{a} - a \ge 0$, the difference between the two numerators in (32) is at least $\delta(4 + 4\overline{a} + 7\kappa_0 + 12\delta) \ge 0$. Moreover

$$u \det B \le \frac{u_0 \mathcal{P}(T_0,R_0)}{K}. \tag{39}$$

Here is the endpoint proof, including the clipping in $\mathcal{P}$. Write $L_0 = n_0 + v$, $h(x) = 4 + 6x - L_0/3$ and

$$z(x) = \min\Big\{\frac{\nu}{(d_0-4x)h(x)}, \frac{1}{7}\Big\}.$$

After substituting $u = 1/(1+a)$, the relevant upper bound for $u\mathcal{P}(T,R)$ factors as

$$\frac{h(a)^7}{(1+a)^8}\, 7z(a)\Big(\frac{1-z(a)}{7}\Big)^6.$$

Both factors increase on $0 \le a \le \overline{a}$. For the first, the derivative has the sign of $42(1+a) - 8h(a) = 10 - 6a + 8L_0/3 > 0$. For the second, use

$$(d_0-4a)h(a) - (d_0-4\overline{a})h(\overline{a}) = (\overline{a}-a)\Big[16 - 6d_0 - \frac{4L_0}{3} + 24(a+\overline{a})\Big] \ge 0$$

and the increasing function $z(1-z)^6$ on $[0,1/7]$. This proves (39). A spectral leaf checks $C_0 \ge 18/5$. A projected leaf instead supplies (35) or (36), with $C \le C_0$, and checks

$$K > 0, \qquad 1114u_0\mathcal{P}(T_0,R_0) \le 4K. \tag{40}$$

Either leaf contradicts the actual matrix inequalities.

A finite coupling-variance certificate. Let $f(x) = x - x^2$ and

$$w_i = \frac{\eta_i + \varepsilon_i^2 - (1-2t)\varepsilon_i}{2}, \quad W_2 = \sum_i w_i = \frac{\mathcal{N} + A_\varepsilon - (1-2t)Z}{2}, \quad \ell_i = \frac{V}{2} - \beta_i.$$

The opposite-normal contacts give

$$w_i \geq 0, \quad 0 \leq \ell_i \leq \frac{V}{2}, \quad \sum_i \ell_i = V, \quad \ell_i \leq \frac{V - f(t)}{2} + w_i, \quad \nu_* = V^2 + \sum_i \ell_i^2.$$

Suppose $V \leq M$, $W_2 \leq W$, and $(V - f(t))/2 \leq b_0$. Every sum of k entries of ℓ is at most $\min\{kM/2, kb_0 + W, M\}$. It is enough to supply eight rational numbers $y_1 \geq \cdots \geq y_8 \geq 0$ satisfying, for $0 \leq k \leq 8$,

$$\sum_{i=1}^{k} y_i \geq \min\left\{\frac{kM}{2},\ kb_0 + W,\ M\right\}, \qquad M^2 + \sum_{i=1}^{8} y_i^2 \leq \nu. \tag{41}$$

This proves $\nu_* \leq \nu$. Indeed, order ℓ decreasingly and apply summation by parts to $\sum_i (y_i - \ell_i)(y_i + \ell_i)$: all prefix sums of the first factor are nonnegative, and the second factor is nonnegative and decreasing. No eigenvector or phase enumeration is part of this finite certificate.

The cover excluding $t \geq 1/3$. Suppose $u \geq 17/18$ and $t \geq 1/3$. Then $a \leq 1/17$ and the phase budget gives $\mathcal{N} \leq 13/153$. There is a unique $R \in [0, 47/500]$ with $f(R) = \mathcal{N}$. The phase-capacity argument used in (18) gives

$$\sum_i |\varepsilon_i| \leq R, \qquad A_\varepsilon \leq R^2, \qquad W_2 \leq (1-t)R.$$

The opposite budget $4f(t) \leq 3V + W_2$ gives $V \geq 1/4$, because $4 \cdot \frac{2}{9} - \frac{2}{3} \cdot \frac{47}{500} > \frac{3}{4}$; the normal budget gives $V \leq 6/17$. Thus (R, t, V) belongs to the closed rational root box

$$\left[0, \frac{47}{500}\right] \times \left[\frac{1}{3}, \frac{1}{2}\right] \times \left[\frac{1}{4}, \frac{6}{17}\right].$$

A finite binary subdivision of this box has the following permitted terminal tests. On a leaf let $n_0 = f(R_-)$, $f_0 = f(t_-)$, $W = (1 - t_-)R_+$, $M = V_+$, and $b_0 = (M - f_0)/2$. A leaf is empty if any of

$$n_0 + 2f_0 > \frac{9}{17}, \qquad n_0 + V_- > \frac{6}{17}, \qquad 3V_+ + W < 4f_0$$

holds. Otherwise it supplies (41) and a spectral or projected leaf as above, with $v = V_-$ and $\overline{a} = 1/17$. All endpoint signs and inequalities are rational checks. Induction on the subdivision tree proves that no point of the root box can occur. Conditional on this finite cover, every counterexample with $u \geq 17/18$ therefore has

$$t < \frac{1}{3}. \tag{42}$$

This cover is part of Computational result 16.

The five-coordinate repair cover. For $17/18 \le u \le 19/20$, use (42) and the uniform phase cap $|\varepsilon_i| < \rho = 23/50$. Since $\mathcal{N} \le 6/17 < 2f(\rho)$, capacity filling gives one of two charts, indexed by $s \in \{0,1\}$:

$$\mathcal{N} = sf(\rho) + f(r), \qquad A_\varepsilon \le S = s\rho^2 + r^2, \qquad \sum_i |\varepsilon_i| \le R = s\rho + r.$$

Here $0 \le r \le \rho$ for $s = 0$ and $0 \le r \le 1/8$ for $s = 1$; the latter follows from $6/17 - f(\rho) < f(1/8)$. The actual point (a, r, t, p, V) lies in

$$\left[\frac{1}{19}, \frac{1}{17}\right] \times [0, r_s] \times \left[0, \frac{1}{3}\right] \times \left[-\frac{2}{17}, \frac{9}{20}\right] \times \left[0, \frac{6}{17}\right]. \tag{43}$$

For the bounds on p, use $p \ge -t\mathcal{N}$, $p \ge -2t^2$, and $p \le S + tR$, together with $R \le 117/200$ and $S \le 9089/40000$. The square completion proving $p \ge -2t^2$ is $\sum_i(\varepsilon_i + t/2)^2 = p + 2t^2$.

Here are the extra moment bounds used after subdivision. Let $n_0 = sf(\rho) + f(r_-)$, $N_+ = \min\{6a_+, sf(\rho) + f(r_+)\}$, $R_+ = \min\{3/5, s\rho + r_+\}$ and $S_+ = s\rho^2 + r_+^2$. Choose rational $q_+, L_+ \ge 0$ with $q_+^2 \ge 2p_+ + 4t_+^2$ and $L_+^2 \ge 8\overline{A}$, where

$$\overline{A} = \min\Big\{S_+,\ \frac{p_+ + t_+N_+}{1 - t_+},\ p_+ + 4t_+^2 + 2t_+q_+\Big\}, \qquad \overline{L} = \min\{R_+, N_+ + \overline{A}, L_+\}.$$

These bound A_ε and $\sum|\varepsilon_i|$ respectively. The second square bound uses $p \ge (1 - t)A_\varepsilon - t\mathcal{N}$; the third solves $p \ge A_\varepsilon - t\sqrt{8A_\varepsilon}$. For the latter, squaring after moving terms, or completing the square in $\sqrt{A_\varepsilon}$, gives $A_\varepsilon \le p + 4t^2 + 2t\sqrt{2p + 4t^2}$. The mass bounds use $\sum|\varepsilon_i| \le \mathcal{N} + A_\varepsilon$ and Cauchy–Schwarz. An upper bound for W_2 is the positive part of the minimum of

$$(1 - t_-)R_+, \qquad \frac{N_+ + \overline{A} + (1 - 2t_-)\overline{L}}{2},$$

and, if $t_- > 0$, the two-endpoint maximum of

$$\frac{N_+}{2} + \frac{(1 - t)\overline{A} - (1 - 2t)p_-}{2t}, \qquad t \in \{t_-, t_+\}.$$

The last bound follows on substituting $Z = (p - A_\varepsilon)/t$; its dependence on t is affine in $1/t$, so these two endpoints suffice. It can feed the same variance certificate and projected leaves already proved. For repair leaves, additionally let $M_\phi = \rho^2 + t_+\rho$. Then

$$\sum_i(\varepsilon_i^2 + t\varepsilon_i)_+ \le \max\Big\{0, \min\Big\{2M_\phi,\ S_+ + t_+R_+,\ \frac{\overline{A} + t_+\overline{L} + p_+}{2}\Big\}\Big\}.$$

The first two bounds are phase-capacity filling; the third follows from $\sum x_i^+ = (\sum x_i + \sum|x_i|)/2$ with $x_i = \varepsilon_i^2 + t\varepsilon_i$.

The finite five-dimensional forest starts from the two boxes (43). Its leaves are a polynomial contradiction, one of the projected tests, or the repair test proved in Section 16. A

coordinate contraction must be justified by a finite polynomial identity before its narrowed child box is used. For precision, the permitted nonnegative analytic polynomials are box bounds and

$$\begin{gathered} 6a-\mathcal{N}-V, \qquad 9a-\mathcal{N}-2f(t), \qquad p+t\mathcal{N}, \qquad p+2t^2, \\ S+tR-p, \qquad 3V+\mathcal{N}+p-4f(t), \qquad \mathcal{N}+p, \\ E-\frac{3}{80}, \qquad \frac{65}{81}-E, \qquad \frac{4}{3}-c_E, \qquad \frac{37}{9}-T_E, \\ 3V+\overline{W}_2-4f(t), \qquad \overline{A}, \end{gathered}$$

where $E = u(V - a + 2p + 5t^2)$, $c_E = 4u - 3 + u(2\mathcal{N} + 5V)/3$, and $T_E = 6 - 2u - u(\mathcal{N} + V)/3$. The endpoint cap $E \le 65/81$ was proved in Lemma 22; the lower bound $E \ge 3/80$ follows from $E = D_G + J \ge D_G > 1/15$ in (20); the bounds on c_E and T_E follow from the normal budget and $u \ge 17/18$. Multiply these rational expressions by the positive denominator $1 + a$ when needed. Derivations may add and multiply previously nonnegative polynomials, use squares and nonnegative rational constants, or use a finite Bernstein expansion with nonnegative coefficients. Every asserted identity is a rational coefficient check. Thus the induction on the forest uses only the actual matrix inequalities proved here and the repair inequalities of Section 16; it requires no unrecorded assertion about all points in a box. This cover is part of Computational result 16.

10 The unequal-triangle lemma, with its full endpoint proof

Lemma 26 (unequal triangles). *For $0 \le \mathcal{N} \le 10/9$, let*

$$r_{\mathcal{N}} = \frac{1-\sqrt{1-\frac{\mathcal{N}}{2}}}{2}, \qquad h_{\triangle}(r) = \frac{(3-2r)^2}{8(1-2r)}.$$

Then

$$\frac{v}{u} \le h_{\triangle}(r_{\mathcal{N}}), \qquad e < 972u^2 h_{\triangle}(r_{\mathcal{N}})$$

in a counterexample. Above $\mathcal{N} = 10/9$, the valid replacement is $4/3$.

Proof. Some $\eta_i \le q = \mathcal{N}/8$. Set $x = |\varepsilon_i|$. By (11) alone, $0 \le x \le r_{\mathcal{N}} \le 1/6$ and $q = r_{\mathcal{N}} - r_{\mathcal{N}}^2$. The first normal and the two nearest lifts of n_i form an additive triangle. Dividing their squared lengths by u gives

$$B_0 = 1, \qquad B_- = 1 + \eta_i - x + x^2, \qquad B_+ = 1 + \eta_i + x + x^2.$$

The circle lemma (Lemma 4) says that for $k = v/u$, $\sum_{j=0,-,+} \sqrt{(1 - B_j/k)_+} \le 1$. Replacing η_i by q weakens this condition. Let $r = r_{\mathcal{N}}$, $k_0 = h_{\triangle}(r)$. Since $k_0 - (1+2r) = (1-6r)^2/[8(1-2r)] \ge 0$, both lift radicands are nonnegative on $0 \le x \le r$. Set $T_x = k_0 - 1 - q - x^2$. The sum of the two lift square roots, multiplied by $\sqrt{k_0}$, is $F(x) = \sqrt{T_x + x} + \sqrt{T_x - x}$. For $0 \le x \le r$ we have $T_x \ge T_r \ge r$, and hence

$$F(x)^2 = 2T_x + 2\sqrt{T_x^2 - x^2} \ge 2T_r + 2\sqrt{T_r^2 - r^2} = F(r)^2.$$

Since $F(x), F(r) \geq 0$, this gives $F(x) \geq F(r)$, including the endpoints. At $x = r$ the three length bounds are $1, 1, 1 + 2r$, and the corresponding distances sum to

$$2\,\frac{1+2r}{3-2r} + \frac{1-6r}{3-2r} = 1.$$

For $k > k_0$ the sum increases strictly, giving a contradiction. The cap is continuous and increasing on $[0, 1/6]$: for $0 \leq r < s \leq 1/6$,

$$h_{\triangle}(s) - h_{\triangle}(r) = \frac{(s-r)(6+4r+4s(1-2r))}{8(1-2r)(1-2s)} > 0.$$

At $r = 1/6$ the bound is $4/3$, agreeing with the inherited bound. □

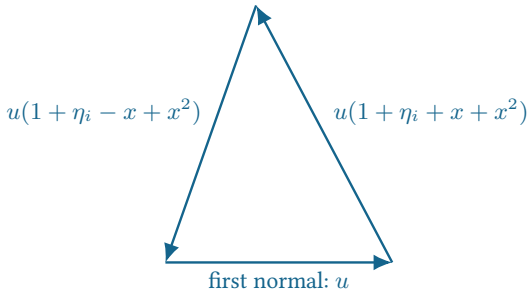


Fig. 1. The labels are squared lengths; the three vectors are drawn head to tail, so that their sum is zero. Additivity forces the three second-gluing circle distances to have sum at most one. Keeping the three different lengths strengthens the flag uniformly over all allocations of the eight first-gluing phases.

Retaining the squared phase mass. On the phase-cap domain (13), the same argument also retains the squared phase mass. For $\mathcal{N} \leq 6/5$, define $g(q) = (1 - \sqrt{1-4q})/2$ and

$$r_* = \min\Big\{\frac{1}{6},\ g\Big(\frac{\mathcal{N}}{8}\Big),\ \frac{\mathcal{N}+3A_\varepsilon}{8+2\mathcal{N}+2A_\varepsilon},\ \frac{3\mathcal{N}-5A_\varepsilon}{24-8\mathcal{N}-8A_\varepsilon}\Big\}, \tag{44}$$

omitting $g(\mathcal{N}/8)$ if $\mathcal{N} > 10/9$. All denominators are positive, since $A_\varepsilon \leq 3\mathcal{N}/5$ follows from $|\varepsilon_i| \leq 3/8$. We prove $v/u \leq h_{\triangle}(r_*)$ directly, including equality at every radius endpoint.

Proof of the refined cap. Suppose instead that $k := v/u > h_{\triangle}(r)$ for some $0 \leq r \leq 1/6$. The inherited bound $k \leq 4/3 = h_{\triangle}(1/6)$ forces $r < 1/6$. Each triangle requires

$$\eta_i \geq f_r(x_i) := \max\Big(x_i - x_i^2,\ \frac{r-(3-2r)x_i^2}{1-2r}\Big).$$

In fact, the second inequality is strict for every i. For $x_i > r$, the first term exceeds the second by $(x_i - r)(1 + 2x_i)/(1 - 2r) > 0$. For $x_i \le r$, equality in the second term makes the two lift square roots at $h_\triangle(r)$ sum to $\sqrt{h_\triangle(r)} - \sqrt{h_\triangle(r) - 1}$, whose square is $(1 - 2r)/2$. Increasing k makes the circle sum strictly larger than one; decreasing η_i can only increase that sum. Thus equality and all smaller values violate the circle inequality. Summing the strict second inequalities gives

$$r < \frac{\mathcal{N} + 3A_\varepsilon}{8 + 2\mathcal{N} + 2A_\varepsilon}.$$

Let $R_* = 3/8$. Concavity of $\sqrt{s} - s$ as a function of s on $[r^2, R_*^2]$ gives

$$f_r(x) \ge r - r^2 + \Big(\frac{1}{R_* + r} - 1\Big)(x^2 - r^2).$$

For $x \le r$, the decreasing second branch also lies above this line. Consequently the inequality for η_i is strict whenever $x_i < R_*$: use the strict second-branch bound for $x_i \le r$, and strict concavity for $r < x_i < R_*$. At least one coordinate is unsaturated, since otherwise $\sum_i \eta_i \ge 8(R_* - R_*^2) = 15/8 > 6/5 \ge \mathcal{N}$. The summed chord inequality is therefore strict and yields

$$r < \frac{R_*\mathcal{N} - (1 - R_*)A_\varepsilon}{8R_* - \mathcal{N} - A_\varepsilon} = \frac{3\mathcal{N} - 5A_\varepsilon}{24 - 8\mathcal{N} - 8A_\varepsilon}.$$

If $\mathcal{N} \le 10/9$, the preceding phase-budget cap and monotonicity of $h_\triangle$ also force $r < g(\mathcal{N}/8)$. Apply these conclusions at $r = r_*$. They make r_* strictly smaller than every term defining its minimum in (44), a contradiction. □

11 Determinant tests that retain the common slack

Let $T_* = 6 - 2u - u(\mathcal{N} + V)/3$ and $c_* = 4u - 3 + u(2\mathcal{N} + 5V)/3$. The Schur complement and Cauchy–Schwarz give

$$\det \widetilde{X} \le c \left[\frac{T - \frac{9u^2V^2}{8c}}{8}\right]^8 \le c_* \left[\frac{T_* - \frac{9u^2V^2}{8c_*}}{8}\right]^8.$$

For the second inequality decrease J to zero in (9). Equivalently increase the nonnegative budget Ω for which $c' = -u, T' = 2u$. Writing $\nu_0 = 9u^2V^2/8$ and $S_* = T - \nu_0/c$, the logarithmic derivative has the sign of $16c - T - 7\nu_0/c$. Using $uV/c \le 1/2$ gives the positive lower bound $66u - 54 + (1107/48)uV > 1$ on $u > 5/6$. Thus the comparison is valid for every actual J. Since $e \det \widetilde{X} = 4$, a strict upper bound below four after multiplication by any valid cap E excludes the box.

There is a complementary axis test. For any positive matrix B and nonzero b, let $\nu = |b|^2$, $k = b^{\mathsf{T}}B^{-1}b$, $z = \nu/k$. In an orthonormal basis beginning with $b/|b|$, the Schur complement gives $\det B \le z[(T - z)/7]^7$. Here $k = c - 1/\widetilde{Q}_{99} \ge c - 1$, while $\nu \le 3u^2V^2/2$. For $c > 1$ one may therefore take

$$z = \min\Big(\frac{T}{8},\ \frac{3u^2V^2}{2(c - 1)}\Big).$$

The axis contact requires $e \det B \geq 4$. The zero-coupling case with $c > 1$ is already impossible; otherwise ordinary AM–GM covers it.

Coupling concentration from unused phase slack. The axis test becomes stronger when all but one coupling loss have a smaller cap. We prove that reduction using the actual phases, with no change to J. Let

$$\beta_i = \frac{b_i}{u}, \qquad \ell_i = \frac{V}{2} - \beta_i, \qquad \xi_i = \eta_i - |\varepsilon_i| + \varepsilon_i^2, \qquad \Xi = \sum_i \xi_i.$$

Here $\xi_i \geq 0$ is unused normal energy; Ξ is distinct from the determinant cap E. The cone bounds give

$$0 \leq \ell_i \leq \frac{V}{2}, \qquad \sum_i \ell_i = V, \qquad \sum_i \beta_i^2 = V^2 + \sum_i \ell_i^2. \tag{45}$$

Define the actual positive and negative phase masses and their normal-energy upper bounds by

$$L_+ = \sum_i (\varepsilon_i)_+, \quad L_- = \sum_i (-\varepsilon_i)_+, \quad P_+ = \frac{\mathcal{N} + A_\varepsilon + Z}{2}, \quad P_- = \frac{\mathcal{N} + A_\varepsilon - Z}{2}.$$

Since $\Xi = \mathcal{N} + A_\varepsilon - L_+ - L_-$ and $Z = L_+ - L_-$,

$$L_+ = P_+ - \frac{\Xi}{2}, \qquad L_- = P_- - \frac{\Xi}{2}, \qquad P_+, P_- \geq 0. \tag{46}$$

The opposite normal gives $|t + \varepsilon_i| - (t + \varepsilon_i)^2 \leq \eta_i + 2\beta_i$. Replacing its absolute value by $t + \varepsilon_i$ yields

$$2\ell_i - V + t - t^2 \leq \xi_i + 2t(\varepsilon_i)_+ + 2(1-t)(-\varepsilon_i)_+. \tag{47}$$

This argument applies even when $t + \varepsilon_i < 0$.
Fix a proposed loss cap $k > 0$. If two distinct losses exceed k, let $m \in \{0, 1, 2\}$ count their nonnegative phases and let $d = 2k - V + t - t^2$. Sum (47) separately over these two sign groups. Each group's unused energy is at most Ξ, so (46) gives

$$md \leq (1-t)\Xi + 2tP_+, \qquad (2-m)d \leq t\Xi + 2(1-t)P_-. \tag{48}$$

An empty group contributes the harmless inequality $0 \leq \Xi$ before substitution. Thus all three sign counts are included, including zero phases, which may be assigned to the nonnegative group.
Here are the complete finite checks for turning this observation into a box exclusion. Subscripts $-, +$ now denote the lower and upper rational endpoints for $\mathcal{N}, t, p, V, A_\varepsilon$ in the current box. Also retain established phase bounds $-r_- \leq \varepsilon_i \leq r_+$ with rational $r_-, r_+ \geq 0$. These bounds concern the actual point and are inherited from the necessary phase refinements; the unrefined choice $r_- = r_+ = 1/2$ is always valid. Require

$$0 < t_- \leq t_+ \leq \frac{1}{2}, \qquad 0 < k \leq \frac{V_+}{2}.$$

Choose rational $\overline{P}_+, \overline{P}_- \geq 0$ and check, at both $s = t_-$ and $s = t_+$,

$$\frac{\mathcal{N}_+ + A_- + \frac{p_+ - A_-}{s}}{2} \leq \overline{P}_+, \qquad \frac{\mathcal{N}_+ + A_+ + \frac{A_+ - p_-}{s}}{2} \leq \overline{P}_-. \tag{49}$$

These four checks imply $P_\pm \leq \overline{P}_\pm$. Indeed $Z = (p - A_\varepsilon)/t$; the positive expression decreases with A_ε and the negative one increases with it. After the indicated endpoint substitutions each expression is affine in $1/t$, whose extrema occur at the two endpoints. Set

$$\begin{gathered} d_0 = 2k - V_+ + t_- - t_-^2, \\ \Delta_m = \max\Big\{0,\ \frac{m d_0 - 2t_+\overline{P}_+}{1 - t_-},\ \frac{(2-m)d_0 - 2(1-t_-)\overline{P}_-}{t_+}\Big\} \quad (m = 0, 1, 2). \end{gathered} \tag{50}$$

All denominators are positive. Since $t - t^2$ increases on $[0, 1/2]$, we have $d \geq d_0$. Bounding the right sides of (48) outward gives $\Xi \geq \Delta_m$, hence

$$L_+ \leq \overline{P}_+ - \frac{\Delta_m}{2} =: M_{+,m}, \qquad L_- \leq \overline{P}_- - \frac{\Delta_m}{2} =: M_{-,m}.$$

For each of the three values of m, require either $M_{+,m} < 0$, or $M_{-,m} < 0$, or the following two finite capped-square witnesses satisfying

$$0 \leq s_\pm \leq r_\pm, \qquad M_{\pm,m} \leq j_\pm r_\pm + s_\pm, \qquad j_\pm r_\pm^2 + s_\pm^2 \leq Q_\pm, \qquad Q_+ + Q_- < A_-, \tag{51}$$

for each sign an integer $j_\pm \geq 0$ and rational $s_\pm, Q_\pm$. The first two alternatives contradict $L_\pm \geq 0$. In the last, Lemma 9, applied to the positive and negative parts, gives $A_\varepsilon \leq Q_+ + Q_- < A_-$, also impossible. A zero cap causes no difficulty: all entries of that sign then vanish. Consequently at most one loss ℓ_i exceeds k.

To finish the harmonic test, supply a rational Q_ℓ bounding $\sum_i \ell_i^2$. There are two allowed finite witnesses:

$$V_+ \geq 0, \qquad V_+^2 \leq Q_\ell,$$

or an integer $j \geq 0$ and rational s with

$$0 \leq s \leq k \leq \frac{V_+}{2}, \qquad V_+ \leq \frac{V_+}{2} + jk + s, \qquad \left(\frac{V_+}{2}\right)^2 + jk^2 + s^2 \leq Q_\ell. \tag{52}$$

The first uses $\sum_i \ell_i^2 \leq (\sum_i \ell_i)^2 \leq V_+^2$. For the second, the exceptional coordinate has cap $V_+/2$, all others have cap k, and their total mass is at most V_+. The same mass-transfer proof as Lemma 9 fills the large cap first: if necessary swap a largest coordinate into that position, then transfer mass to it, and finally fill the smaller caps. Each transfer toward the larger coordinate increases the sum of squares. Padding with zero coordinates and unused mass up to $V_+/2 + jk + s$ proves the bound in (52). Thus

$$|b|^2 = u^2 \sum_i \beta_i^2 \leq u^2 (V_+^2 + Q_\ell). \tag{53}$$

Finally take rational height bounds $0 < u_- \le u \le U$. For a box in a, these follow from the finite checks $1 + a_- > 0$, $u_- \le (1 + a_+)^{-1}$ and $(1 + a_-)^{-1} \le U$. Take nonnegative $\mathcal{N}_-, V_-$ and rational c_0, T_0 satisfying

$$1 < c_0 \le 4u_- - 3 + \frac{u_-(2\mathcal{N}_- + 5V_-)}{3}, \qquad 6 - 2u_- - \frac{u_-(\mathcal{N}_- + V_-)}{3} \le T_0. \tag{54}$$

Monotonicity in these nonnegative variables and (9) give $c \ge c_0$ and $T \le T_0$ for the actual $J \ge 0$. In particular $b \ne 0$, since its Schur energy is at least $c - 1 > 0$. The actual reciprocal Schur quantity in the axis test therefore satisfies

$$z = \frac{|b|^2}{b^{\mathsf{T}} B^{-1} b} \le \frac{U^2(V_+^2 + Q_\ell)}{c_0 - 1} =: z_0.$$

For fixed T_0, the function $z[(T_0 - z)/7]^7$ increases up to $T_0/8$ and decreases thereafter. It also increases with T_0 on its nonnegative domain. Therefore the single rational check

$$1114U\, r\Big(\frac{T_0 - r}{7}\Big)^7 \le 4, \qquad r = \min\Big\{z_0, \frac{T_0}{8}\Big\}, \tag{55}$$

contradicts $e \det B \ge 4$ and $e < 1114u \le 1114U$. Equations (49)–(55), with one of the stated alternatives for each sign count, are the entire finite witness. The reduction uses actual normal energies, actual signed masses and actual inverse blocks throughout.

Affine anchored versions. Affine anchored versions of both tests avoid losing correlations. For example, y^2/c has the tangent lower bound $2y_0 y/c_0 - y_0^2 c/c_0^2$, whose difference is $(c_0 y - y_0 c)^2/(c_0^2 c)$. Weighted AM–GM then bounds $u^k c[(T - 9y^2/(8c))/8]^8$ by an explicit positive power of an affine expression, for $k = 1, 2$. For the axis test use positive anchors u_0, z_0, B_0 with $B_0 \ge 7z_0$. Its AM–GM expression is bounded by

$$\frac{ku}{u_0} + \left(\frac{7}{B_0}\right)T + \left(\frac{1}{z_0} - \frac{7}{B_0}\right)\frac{3u^2V^2}{2(c-1)}.$$

This is convex in each of $u, \mathcal{N}, V, J$ separately on a box with $c > 1$, because the nonaffine term is a quadratic perspective of affine functions. Its maximum is therefore at a corner. This proves why the corner checks are bounds on whole boxes.

12 Adapted references and exact contact bounds

Let $R_{\mathrm{aux}} > 0$ be any rational reference with $\operatorname{tr}(R_{\mathrm{aux}}\widetilde{X}) \le 9$, and set $Y = R_{\mathrm{aux}}^{-1/2}\widetilde{Q}R_{\mathrm{aux}}^{-1/2}$. Then $\operatorname{tr} Y^{-1} \le 9$ and $\det Y < E/(4 \det R_{\mathrm{aux}})$. By Lemma 5, any rational $H > 1$ satisfying

$$H\Big(\frac{8}{9 - \frac{1}{H}}\Big)^8 \ge \frac{E}{4 \det R_{\mathrm{aux}}}$$

proves $\widetilde{Q} \preceq H R_{\mathrm{aux}}$. A legal component x with $H R_{\mathrm{aux}}[x] < 4 - 1/u_-$ then contradicts the minimum. The reference is any rational positive definite matrix satisfying the displayed conditions.

The selected references used in the final cover are

$$R_{\mathrm{aux}} = a_0 B_i, \qquad B_i = \begin{pmatrix} I_8 & -d_0\mathbf{1} \\ -d_0\mathbf{1}^{\mathsf{T}} & q_0 \end{pmatrix} - k_0 n_i n_i^{\mathsf{T}},$$

$$q_0 \le 2, \qquad \Delta_0 = q_0 - 8d_0^2 > 0, \qquad \Delta_k = \Delta_0 - k_0\left\{\Delta_0 + \left(d_0 - \frac{1}{2}\right)^2\right\} > 0.$$

Conjugating by the unmodified reference shows that its single changed eigenvalue is Δ_k/Δ_0; hence $B_i > 0$ and $\det B_i = \Delta_k$. Negative k_0 is allowed: it adds a positive rank-one matrix. Equation (9) and $q_0 \le 2$ give

$$\mathrm{tr}(B_i\widetilde{X}) \le T_* - 6d_0uV + q_0c_* - k_0u\left(\frac{3}{4} + \eta_i\right).$$

For $k_0 \ge 0$, a selected phase lower bound $h\varepsilon_i \ge \ell$ ($h = \pm 1$) permits $\eta_i \ge m_\ell - m_\ell^2$ with $m_\ell = \max(0, \ell)$. For $k_0 < 0$, use $\eta_i \le \min(\mathcal{N}, 15/64)$ instead. The remaining expression is separately affine in $u, \mathcal{N}, V$. Its corner upper bound $M_0 > 0$ permits $a_0 = 9/M_0$.

The selected coordinate exists by signed mass bounds. If $A_\varepsilon \ge A_-$, $\sum|\varepsilon_i| \le L_\varepsilon$, and $Z \in [Z_-, Z_+]$, positive and negative absolute masses are at most $P_+ = (L_\varepsilon + Z_+)/2$ and $M_+ = (L_\varepsilon - Z_-)/2$. With respective magnitude caps r_+, r_-, the positive squared mass is at least $A_- - \mathcal{C}_{r_-}(M_+)$. If this exceeds $\mathcal{C}_\ell(P_+)$, some positive coordinate exceeds ℓ. The negative version is identical. Averaging also gives $\max\varepsilon_i \ge Z/8 \ge Z_-/8$ and $\max(-\varepsilon_i) \ge -Z/8 \ge -Z_+/8$. These tests justify every finite bisection used to choose a signed coordinate interval. The selected index may vary with the metric; the inequality is independent of its label.

For the half-spinor bounds, write $w = h\varepsilon_i$ and

$$C_\sigma(t, Z) = p + (-\sigma - 2d_0t)Z + 2(t - \sigma)^2 - 8d_0t(t - \sigma) + q_0t^2.$$

Direct expansion yields

$$B_i[x_\sigma] = C_\sigma - k_0\left(w - \frac{h\sigma}{2}\right)^2,$$

$$B_i[x_\sigma + \sigma e_i] = C_\sigma + \sigma(1 - 2d_0)t + 2h\sigma w - k_0\left(w + \frac{h\sigma}{2}\right)^2.$$

The t, Z expression is affine in Z and convex in t, because $2 - 8d_0 + q_0 > 0$. The w expressions are quadratics. Their maxima are checked at the interval endpoints and, for a concave quadratic, its interior vertex. Thus all contact bounds are rational box inequalities. A half-spinor terminal record must contain a contradiction for *each* parity; the two witnesses may use different references. The integer-contact formulas follow by the same expansion of $B_i[(\varepsilon + (t/2)\mathbf{1} - e_i,\ t - 1)]$.

13 Flag-normal bounds with the common slack retained

Write $f(x) = x - x^2$. The flag test improves a bound on a selected normal energy or on a selected positive phase. Its endpoint comparison must retain the actual J: both the horizontal inverse trace and the inverse-axis entry depend on that same slack. Write H for the horizontal principal block of $\widetilde{Q}$; B remains the horizontal block of $\widetilde{Q}^{-1}$ and $T = \operatorname{tr} B$. Consider a rational box for $(u, \mathcal{N}, V, t, Z, p, A_\varepsilon, \sum |\varepsilon_i|)$. Write lower and upper endpoints with subscripts $-$ and $+$ and set

$$\begin{aligned} s_- &= u_- V_-, & s_+ &= u_+ V_+, \\ C_- &= 4u_- - 3 + \frac{2u_-\mathcal{N}_- + 5s_-}{3}, & C &= 4u_+ - 3 + \frac{2u_+\mathcal{N}_+ + 5s_+}{3}, \\ T_+ &= 6 - 2u_- - \frac{u_-(\mathcal{N}_- + V_-)}{3}, & W_0 &= T_+ - \frac{9s_-^2}{8C}. \end{aligned}$$

The finite domain guards are

$$\begin{gathered} u_- > 0, \qquad \mathcal{N}_-, V_- \geq 0, \qquad C_- > 0, \qquad 2s_+ \leq C_-, \\ T_+ - \frac{3u_-}{4} - \frac{141C_-}{14} \leq 0, \qquad \frac{W_0}{3} \leq 4\left(\frac{55}{96}\right)C_-. \end{gathered} \tag{56}$$

Ordinary interval multiplication and (9) then give

$$s_- \leq uV \leq s_+, \qquad 2uV \leq c \leq C + \frac{J}{3}, \qquad T \leq T_+ - \frac{2J}{3}, \qquad C \geq C_- > 0. \tag{57}$$

Let $\overline{e} > e$ be a rational section cap, obtained either from $\overline{e} \geq 1114u_+$ or from the unequal-triangle moment certificate, with $\overline{e} \geq 972u_+^2 h_+$ and its proved ratio bound $v/u \leq h_+$. The latter certificate uses the same retained normal and square-moment endpoints; its hypotheses and rational checks are those of the triangle argument of Section 10.

One loss function for both kinds of normal bound. For $s \geq 0$, $c > 0$ and $0 \leq w \leq s/2$, define

$$\mathcal{L}_{\mathrm{cof}}(s, c, w) = w + \frac{(3s - w)^2}{7c}.$$

When $2s \leq c$, it increases with w and has derivative between 0 and 1. It also increases with s and decreases with c in the range in question. These assertions follow directly from

$$\partial_w \mathcal{L}_{\mathrm{cof}} = 1 - \frac{2(3s - w)}{7c}, \qquad \partial_s \mathcal{L}_{\mathrm{cof}} = \frac{6(3s - w)}{7c}, \qquad \partial_c \mathcal{L}_{\mathrm{cof}} = -\frac{(3s - w)^2}{7c^2}.$$

In particular if $s_- \leq s$, $c \leq C'$ and $0 \leq w_- \leq w$, with $w_- \leq s_-/2$ and $2s \leq c$, then $\mathcal{L}_{\mathrm{cof}}(s_-, C', w_-) \leq \mathcal{L}_{\mathrm{cof}}(s, c, w)$.
The normal-energy test assumes $\eta_i \geq h = f(r)$ for some $r \in (0, 1/2)$ and uses $w_- = 0$. It follows immediately that

$$u_- h + \mathcal{L}_{\mathrm{cof}}\left(s_-, C + \frac{J}{3}, w_-\right) \leq u\eta_i + \mathcal{L}_{\mathrm{cof}}(uV, c, b_i). \tag{58}$$

The positive-phase test assumes $\varepsilon_i \geq r$ and uses

$$w_- = \min\Big\{ \frac{s_-}{2},\ \frac{u_-}{2} \max\big(0, \min\{t_-(1-t_--2r),\ t_+(1-t_+-2r)\}\big) \Big\}. \tag{59}$$

It yields exactly the same loss inequality, with $h = f(r)$. Here is a proof that includes excess normal energy. For a nonnegative phase x, put $w(x) = \min\{uV/2, [ut((1-t)/2 - x)]_+\}$. The two actual normal inequalities are $f(x) \leq \eta_i$ and $f(t+x) \leq \eta_i + 2b_i/u$. If $b_i \geq w(x)$, monotonicity of $\mathcal{L}_{\mathrm{cof}}$ proves

$$uf(x) + \mathcal{L}_{\mathrm{cof}}(uV, c, w(x)) \leq u\eta_i + \mathcal{L}_{\mathrm{cof}}(uV, c, b_i).$$

If $b_i < w(x)$, the opposite inequality gives $u(\eta_i - f(x)) \geq 2(w(x) - b_i)$, and the derivative bound for the loss gives $\mathcal{L}_{\mathrm{cof}}(uV, c, w(x)) - \mathcal{L}_{\mathrm{cof}}(uV, c, b_i) \leq w(x) - b_i$. These two inequalities prove the same conclusion. Furthermore the left side increases with $x \leq 1/2$. Before $x = (1-t)/2$, the decrease of $w(x)$ between x and y is at most $ut(y-x)$, which is at most $u(f(y) - f(x))$; the loss function is 1-Lipschitz in w there. After that endpoint $w(x) = 0$ and f increases. Splitting at the endpoint proves monotonicity across it as well. Thus $x \geq r$ permits replacement by r. Finally $t(1-t-2r)$ is concave in t, so its minimum on the box occurs at an endpoint. Formula (59) and the monotonicities of $\mathcal{L}_{\mathrm{cof}}$ prove (58).

Cofactor comparison along the enclosing scalar path. Delete horizontal coordinate i from $\widetilde{X}$, Schur complement its axis, and apply AM–GM to the other seven horizontal coordinates. Their coupling sum is $3uV - b_i$, so

$$H_{ii} < \frac{\overline{e}\, c}{4} \left(\frac{\left[T - B_{ii} - \frac{(3uV - b_i)^2}{7c} \right]_+}{7} \right)^7.$$

Use $B_{ii} = u(3/4 + \eta_i) + b_i - c/4$ and (58), then (57). Let $c_1 = C + J/3$. The resulting upper bound is

$$\frac{\overline{e} c_1}{4} \left(\frac{\left[T_+ - \frac{2J}{3} - u_-\left(\frac{3}{4} + h\right) + \frac{c_1}{4} - w_- - \frac{(3s_- - w_-)^2}{7c_1} \right]_+}{7} \right)^7.$$

This is at most its value at $J = 0$. For a fully algebraic verification put $A_0 = (3s_- - w_-)^2/7$ and

$$b_0 = T_+ - u_-\left(\frac{3}{4} + h\right) + \frac{C}{4} - w_- - \frac{A_0}{C},$$
$$b_1 = T_+ - \frac{2J}{3} - u_-\left(\frac{3}{4} + h\right) + \frac{c_1}{4} - w_- - \frac{A_0}{c_1}.$$

Since $2s_- \leq C$ and $0 \leq w_- \leq s_-/2$, $0 \leq A_0 \leq 9C^2/28$. Put $v_0 = 7/4 - A_0/(Cc_1) \geq 0$. Then $b_0 - b_1 = (J/3)v_0$. The cofactor guard in (56), weakened from C_- to C, gives $b_0 \leq 7Cv_0$; indeed first use $A_0/C \leq 9C/28$ to obtain $b_0 \leq 7C(7/4 - A_0/C^2)$, and

then $c_1 \ge C$. If $b_1 \le 0$ the clipped comparison is immediate. Otherwise $0 < b_1 \le b_0$ and

$$(c_1 - C)b_1 \le 7C(b_0 - b_1), \qquad b_0^7 \ge b_1^7 + 7(b_0 - b_1)b_1^6.$$

The second inequality is the binomial expansion. Together they give $c_1 b_1^7 \le C b_0^7$, as required. We have proved the finite endpoint bound

$$H_{ii} < U_0 = \frac{\overline{e}C}{4}\Big(\frac{[b_0]_+}{7}\Big)^7. \tag{60}$$

If $b_0 \le 0$, positivity of H_{ii} already contradicts this bound.
If instead $U_0 \le U$, paired horizontal contacts give

$$\begin{aligned} \operatorname{tr} H &\ge \max\{16, 28 - 6U\}, \qquad \det H < \frac{\overline{e}\big(C + \frac{J}{3}\big)}{4}, \\ \operatorname{tr} H^{-1} &\le W_J = T_+ - \frac{2J}{3} - \frac{9s_-^2}{8\big(C + \frac{J}{3}\big)}. \end{aligned} \tag{61}$$

For the first assertion, sum $H_{jj} + H_{kk} \ge 4$ over all pairs, and then over the seven pairs containing i. For the second, block inversion gives $\operatorname{tr} H^{-1} = T - |b|^2/c$ and $|b|^2 \ge (\sum b_i)^2/8 = 9(uV)^2/8$. For the last, $\det H = c \det \widetilde{Q}$. The common slack decreases the reciprocal-trace bound by at least $55J/96$:

$$W_J + \frac{55}{96}J \le W_0. \tag{62}$$

Indeed

$$\frac{9s_-^2}{8C} - \frac{9s_-^2}{8\big(C + \frac{J}{3}\big)} = \frac{3J}{8}\,\frac{s_-^2}{C\big(C + \frac{J}{3}\big)} \le \frac{3J}{32},$$

and $2/3 - 3/32 = 55/96$.

Elasticity of the comparison determinant. For a positive eight-dimensional matrix with trace at least σ and inverse trace at most $W > 0$, use the effective trace $\sigma_* = \max\{\sigma, 64/W\}$. The matrix Cauchy inequality forces its trace to be at least σ_*. There are numbers $\theta \ge \lambda > 0$ with $\theta + 7\lambda = \sigma_*$ and $1/\theta + 7/\lambda = W$. Indeed the polynomial

$$P(z) = Wz^2 - (\sigma_* W - 48)z + \sigma_*$$

is nonpositive at $\sigma_*/8$ and positive at σ_*; a root in that interval gives $\lambda = (\sigma_* - \theta)/7 > 0$. Lemma 7 gives $\det H \ge \theta\lambda^7$ for this comparison pair.
The needed dependence on W follows from the next comparison. For $r = \theta/\lambda \ge 1$ let

$$F(r) = \frac{(7r+1)^4}{r^3(r+7)^4}, \qquad G(r) = \frac{(7r+1)^8}{r^7}.$$

Direct substitution and differentiation give

$$\begin{aligned} (\theta\lambda^7)W^4 &= (\theta + 7\lambda)^4 F(r), & F'(r) &= -\frac{21(r-1)^2(7r+1)^3}{r^4(r+7)^5} \le 0, \\ (\theta\lambda^7)W^8 &= G(r), & G'(r) &= \frac{7(r-1)(7r+1)^7}{r^8} \ge 0. \end{aligned}$$

Consider two ordered comparison pairs with ordinary traces $\sigma_0 \le \sigma_1$ and reciprocal traces $W_1 \le W_0$. If $r_1 \le r_0$, the first identity and decreasing F show $D_0 W_0^4 \le D_1 W_1^4$, where $D_j = \theta_j \lambda_j^7$. If $r_1 \ge r_0$, use the second identity and increasing G: $D_0 W_0^8 \le D_1 W_1^8$, and divide after using $W_1 \le W_0$. Thus in both cases

$$D_1 \ge D_0 \left(\frac{W_0}{W_1} \right)^4 . \tag{63}$$

The argument includes $r = 1$, where the two eigenvalues coalesce, and allows the effective trace to increase as W decreases.
Use the two effective comparison pairs for W_0 and W_J in (61), with the same lower trace σ. Their ordinary traces are ordered because $\max\{\sigma, 64/W_J\} \ge \max\{\sigma, 64/W_0\}$. At an occupied box $W_J > 0$. The elasticity guard, weakened from C_- to C, and (62) imply

$$\frac{C + \frac{J}{3}}{C} \le \left(\frac{W_0}{W_J} \right)^4 .$$

For detail, let $x = W_0/W_J - 1 \ge 0$. Then $(55/96)J/W_0 \le x$ and $J/(3C) \le 4(55/96)J/W_0 \le 4x$; Bernoulli's inequality $(1+x)^4 \ge 1 + 4x$ proves the assertion. Combining it with (63) yields

$$\frac{D_0}{C} \le \frac{\det H}{C + \frac{J}{3}} . \tag{64}$$

This is the required transfer; the actual contact minima were used only at their original slack.

Rounded roots and the complete finite flag test. Choose rational U, σ, D_0 with

$$U_0 \le U, \qquad \sigma \le \max\{16, 28 - 6U\}, \qquad \overline{e} C < 4 D_0 .$$

A sufficient rational lower bound for the comparison determinant at W_0 is either

$$D_0 \le \left(\frac{8}{W_0} \right)^8 ,$$

or the following quadratic certificate:

$$\begin{aligned} &\sigma > 0, \qquad \sigma W_0 \ge 64, \qquad 2 W_0 z \ge \sigma W_0 - 48, \\ &W_0 z^2 - (\sigma W_0 - 48) z + \sigma \ge 0, \qquad D_0 \le z \big(\frac{\sigma - z}{7} \big)^7 . \end{aligned} \tag{65}$$

The inverse-only alternative follows from AM–GM applied to the eight reciprocals. In the quadratic alternative, the first two root checks place z on or beyond the larger root θ: subtracting $P(\theta) = 0$ from $P(z)$ factors the difference as $(z - \theta)[W_0(z + \theta) - (\sigma W_0 - 48)]$. The function $z((\sigma - z)/7)^7$ decreases for $z \ge \sigma/8$, since its derivative is $((\sigma - z)/7)^6 (\sigma - 8z)/7$. Hence rounding the larger root upward decreases its

determinant lower bound. A recorded square-root upper bound $\delta \geq 0$ can be checked solely by

$$\delta^2 \geq (\sigma W_0 - 64)(\sigma W_0 - 36), \qquad z = \frac{\sigma W_0 - 48 + \delta}{2W_0};$$

these imply both root inequalities in (65) after clearing $4W_0 > 0$. The repeated-root boundary is included.
Finally (64) and (61) give $D_0/C < \overline{e}/4$, contradicting $\overline{e}C < 4D_0$. Every finite flag row consists of the box guards, an already justified section cap, a radius $0 < r < 1/2$, and either the nonpositive-cofactor test or these determinant checks. With $w_- = 0$ it proves $\eta_i < f(r)$ for every i, hence $|\varepsilon_i| < r$ by monotonicity of f on $[0, 1/2]$. With (59) it proves $\varepsilon_i < r$ for every i. These conclusions can contract phase bounds in the finite contact forest; no continuous implication is left as a certificate field.

14 First-layer Schur compensation

The following estimate uses the eight integer first-layer contacts simultaneously. It applies on $459/500 \leq u \leq 17/18$ and $t > 0$. Keep the actual inverse block B and coupling b, and let

$$S = B^{-1}, \qquad q = \widetilde{Q}_{99}, \qquad z = -B^{-1}b.$$

Block inversion gives

$$\begin{aligned} \widetilde{Q}[(x,s)] &= S[x] + q(z\cdot x + s)^2, \qquad \widetilde{Q}_{\mathrm{hor}} = S + qzz^{\mathsf{T}}, \\ qc &= 1 + qS^{-1}[z], \qquad \det \widetilde{Q} = q \det S, \qquad \operatorname{tr} S^{-1} = T, \qquad q \geq 1. \end{aligned} \tag{66}$$

In particular $S > 0$ and $\det S < 1114u/4$.

Bounds that preserve the common slack. Define the scalar endpoint quantities

$$\Omega_* = \frac{2}{3}(6a - \mathcal{N} - V), \qquad d_* = 1 - u\Omega_*, \qquad c_* = 1 + u(V - \Omega_*).$$

They satisfy $c = c_* + J/3$ and $d_* > 0$: indeed $u\Omega_* \leq 4(1-u) \leq 82/250 < 1$ on this height interval. The axial coefficient bound used in (17) is $q(c - uV) \leq 1$. Therefore

$$q \leq d_*^{-1}, \qquad qc \leq 1 + quV \leq \frac{c_*}{d_*}. \tag{67}$$

The second bound controls the product directly, including its actual $J/3$ contribution. For the finite lower-strip boxes, let $u \leq u_+$ and $\Omega_* \leq \Omega_+$, with $0 \leq u_+ \leq 17/18$ and $0 \leq \Omega_+ \leq 4a_+ \leq 164/459$. Then the fixed rational comparisons

$$u_+\Omega_+ \leq \frac{1394}{4131}, \qquad \frac{1}{1 - u_+\Omega_+} \leq \frac{4131}{2737} < \frac{177}{100}$$

show that

$$\min\left\{\frac{177}{100}, \frac{1}{1 - u_+\Omega_+}\right\} = \frac{1}{1 - u_+\Omega_+}.$$

Thus this minimum preserves the reciprocal bound used in (67). If $\bar{c}_* \geq c_*$, one may consistently take $\bar{q} = (1 - u_+\Omega_+)^{-1}$ and $\bar{c}_q = \bar{c}_*/(1 - u_+\Omega_+)$, with both quantities bounding the same actual axis and axis-normal product.

The same reference comparison as in the preceding sections gives $\widetilde{Q}_{\mathrm{hor}} \preceq H_0 I_8$, with $H_0 = 177/50$. The required fixed rational check is

$$\frac{9469}{9} \leq \frac{8192}{9}\,\frac{177}{100}\Big(\frac{8}{9 - \frac{100}{177}}\Big)^8;$$

use $e < 1114u \leq 9469/9$ and the horizontal block $2I_8$ of $\widetilde{P}$. Hence $S \preceq H_0 I_8$ as well. One may improve this to $S \preceq mI_8$ using a proved bound $\operatorname{tr} S^{-1} \leq T_0$ and $4 \det S < E_S$:

$$m > 0, \qquad \frac{8}{T_0} \leq m, \qquad T_0 - \frac{1}{m} > 0, \qquad E_S \leq 4m\Big(\frac{7}{T_0 - \frac{1}{m}}\Big)^7. \tag{68}$$

These are the inverse-trace eigenvalue comparison of Lemma 5. If $mT_0 = 8$, inverse AM–GM already contradicts the strict determinant bound. Otherwise the comparison function $x[7/(T_0 - 1/x)]^7$ increases for $x \geq 8/T_0$, excluding any eigenvalue greater than m.

Let $\bar{q} \geq q$ and $\bar{c}_q \geq qc$ be outward bounds from (67). The same actual shift obeys

$$q|z|^2 \leq R_z := \min\Big\{m(\bar{c}_q - 1),\ H_0\Big(1 - \frac{1}{\bar{c}_q}\Big)\Big\}. \tag{69}$$

For completeness, set $k_z = S^{-1}[z] \geq 0$. Cauchy–Schwarz in S gives $|z|^4 \leq S[z]k_z$. Together with $S[z] \leq m|z|^2$ this yields $|z|^2 \leq mk_z$. Using instead $S[z] + q|z|^4 \leq H_0|z|^2$ gives $|z|^2(1 + qk_z) \leq H_0 k_z$. Both statements also hold when $z = 0$. Multiply by q, use $1 + qk_z = qc$, and then use the monotonicity of $y - 1$ and $1 - 1/y$ for $y \geq 1$ to obtain (69).

The eight-contact covariance. Let

$$d = \varepsilon + \frac{t}{2}\mathbf{1}, \qquad s = t - 1, \qquad n = \frac{\mathbf{1}}{\sqrt{8}}, \qquad w_c = \sqrt{8}\Big(d - \frac{1}{8}\mathbf{1}\Big),$$
$$C_d = I_8 - nn^{\mathsf{T}} + w_c w_c^{\mathsf{T}}.$$

Expansion of the eight outer products gives

$$\sum_{i=1}^{8}(d - e_i)(d - e_i)^{\mathsf{T}} = C_d, \qquad \sum_{i=1}^{8}(d - e_i) = \sqrt{8}w_c.$$

Thus the legal contacts $(d - e_i, s)$, each of value at least $4 - 1/u$, imply

$$32 - \frac{8}{u} \leq \operatorname{tr}(C_d S) + q\{C_d[z] + 2\sqrt{8}s(w_c \cdot z) + 8s^2\}. \tag{70}$$

Write $L = |w_c|^2$ and $h = w_c \cdot n$. The matrix $w_c w_c^{\mathsf{T}} - nn^{\mathsf{T}}$ has at most one positive eigenvalue: on $w_c^{\perp}$ its quadratic form is nonpositive. On any two-dimensional subspace

containing w_c and n, its trace is $L-1$ and its determinant is h^2-L, so its positive eigenvalue, with zero allowed, is

$$\mu_0 = \max\Big\{0,\ \frac{L-1+\sqrt{(L+1)^2-4h^2}}{2}\Big\}.$$

The formula includes the linearly dependent case. Consequently any $\mu \ge \mu_0$ gives

$$\operatorname{tr}(C_d S) \le \operatorname{tr} S + m\mu. \tag{71}$$

To see this, diagonalise the covariance correction. Its negative eigenvalues contribute nonpositively against $S > 0$; its at most one positive eigenvalue is at most μ, and the corresponding diagonal entry of S is at most m.

A resolvent controls the remaining square. For $\alpha > 0$, the projection $P = nn^{\mathsf{T}}$ satisfies

$$(\alpha I_8 + P)^{-1} = \alpha^{-1} I_8 - \frac{P}{\alpha(\alpha+1)}.$$

Set

$$K_0 = \frac{L}{\alpha} - \frac{h^2}{\alpha(\alpha+1)}.$$

This is nonnegative. Cauchy–Schwarz in $\alpha I_8 + P$ gives $(w_c \cdot z)^2 \le K_0\{\alpha|z|^2 + (n\cdot z)^2\}$. For $0 < K_0 < 1$, putting $X = w_c \cdot z$ and $b_0 = -\sqrt{8}s$, the one-variable completion of the square gives

$$X^2 - 2b_0 X + b_0^2 - \{\alpha|z|^2 + (n\cdot z)^2\} \le \frac{b_0^2}{1-K_0}.$$

Indeed replace the braced quantity by its lower bound X^2/K_0 and maximise the resulting concave quadratic in X. When $K_0 = 0$, Cauchy–Schwarz forces $X = 0$, giving the same result. Hence any $K_0 \le K < 1$ proves

$$C_d[z] + 2\sqrt{8}s(w_c \cdot z) + 8s^2 \le (1+\alpha)|z|^2 + \frac{8s^2}{1-K}. \tag{72}$$

Combining (69), (71) and (70), and taking $t_- \le t$, yields

$$\operatorname{tr} S \ge 32 - \frac{8}{u} - m\mu - (1+\alpha)R_z - \frac{8\bar{q}(1-t_-)^2}{1-K}. \tag{73}$$

Here $(1-t)^2 \le (1-t_-)^2$ follows from $t \le 1/2$.

Preserving the phase correlation in scalar bounds. We bound all parameters in (73) using the same phase vector. At fixed actual t, p, define

$$\begin{aligned}
\mathcal{L}(A) &= 8A + \frac{2(4t-1)(p-A)}{t} + (4t-1)^2,\\
\mathcal{H}(A) &= \frac{p-A}{t} + 4t - 1,\\
\mathcal{D}(A) &= (\mathcal{L}(A)+1)^2 - 4\mathcal{H}(A)^2,\\
\mathcal{K}_\alpha(A) &= \frac{\mathcal{L}(A)}{\alpha} - \frac{\mathcal{H}(A)^2}{\alpha(\alpha+1)}.
\end{aligned}$$

Direct expansion gives $L = \mathcal{L}(A_\varepsilon)$ and $h = \mathcal{H}(A_\varepsilon)$. Moreover $p + 2t^2 = \sum_i (\varepsilon_i + t/2)^2 \geq 0$. For $A \leq A'$,

$$\begin{aligned}
\mathcal{L}(A') - \mathcal{L}(A) &= \frac{2(A' - A)}{t}, \\
\mathcal{D}(A') - \mathcal{D}(A) &= \frac{32(A' - A)(p + 2t^2)}{t}, \\
\mathcal{K}_\alpha(A') - \mathcal{K}_\alpha(A) &= \frac{(A' - A)(2p + 8t^2 + 2\alpha t - A - A')}{\alpha(\alpha + 1)t^2}.
\end{aligned}$$

Therefore an upper phase-square bound $A_* \geq A_\varepsilon$ increases the root expression defining μ_0. It also increases K_0 provided

$$A_* \leq p + t(4t + \alpha). \tag{74}$$

The substitutions act on these scalar functions at fixed t, p; the actual signed sum remains Z.

For example, outward bounds $\bar{L} \geq \mathcal{L}(A_*)$ and $\underline{H} \leq \mathcal{H}(A_*)^2$ allow the finite rational checks

$$\begin{gathered}
\mu \geq 0, \qquad 2\mu \geq \bar{L} - 1, \qquad (2\mu - \bar{L} + 1)^2 \geq (\bar{L} + 1)^2 - 4\underline{H}, \\
\frac{\bar{L}}{\alpha} - \frac{\underline{H}}{\alpha(\alpha + 1)} \leq K < 1.
\end{gathered} \tag{75}$$

They imply the bounds on μ_0, K_0 required above. The real-root discriminant at the actual phase point is nonnegative; the displayed monotonicity and outward substitutions preserve that inequality. To evaluate a correlated square on a scalar interval $[a, b]$, its lower bound is zero if $a \leq 0 \leq b$, and otherwise the smaller of a^2, b^2. Its upper bound is always the larger of these squares. Thus a square retains its correlation throughout these checks.

With a proved inverse-trace bound $T \leq T_0$, the right side of (73) supplies a trace lower bound σ; a lower height endpoint u_- may replace u in $32 - 8/u$. The final rational determinant witness has two alternatives. Inverse AM–GM permits any $D_0 \leq (8/T_0)^8$ with $T_0 > 0$. Alternatively, Lemma 7 permits

$$\begin{gathered}
\sigma > 0, \qquad \frac{\sigma}{8} \leq x_{\mathrm{hi}} < \sigma, \qquad T_0 x_{\mathrm{hi}}^2 - (\sigma T_0 - 48)x_{\mathrm{hi}} + \sigma \geq 0, \\
D_0 \leq x_{\mathrm{hi}} \Big(\frac{\sigma - x_{\mathrm{hi}}}{7} \Big)^7.
\end{gathered} \tag{76}$$

Either alternative proves $D_0 \leq \det S$. A separately rounded lower root $x_{\mathrm{lo}} \leq x_{\mathrm{hi}}$ can be retained by taking $D_0 = x_{\mathrm{lo}}[(\sigma - x_{\mathrm{hi}})/7]^7$; the last check then follows because its powered factor is positive. Finally $E_S/4 \leq D_0$ contradicts $4 \det S < E_S$; the uniform choice is $E_S = 1114U$ for a height upper bound $u \leq U$. The numerical box tests must verify the stated phase, denominator, spectral and trace bounds. Every continuous implication has been proved for the actual matrix and its actual slack.

15 Weighted contact and primal-trace bounds

A useful common support inequality avoids eigenvector assumptions. If $L_Y I \preceq Y \preceq H_Y I$, $\operatorname{tr} Y \le S_Y$, $\operatorname{tr} Y^{-1} \le n$, and C has eigenvalues w_i, then for any $\alpha, \beta \ge 0$,

$$\operatorname{tr}(YC) \le \alpha S_Y + n\beta + \sum_i \max_{L_Y \le x \le H_Y} \left\{ (w_i - \alpha)x - \frac{\beta}{x} \right\}. \tag{77}$$

To prove it, take an eigenbasis v_i of C, apply the scalar inequality to Y by functional calculus, pair with v_i, and sum. The scalar maximum is at an endpoint or at $x = \sqrt{\beta/(\alpha - w_i)}$ when $\alpha > w_i$. At the interior point it equals $-2\sqrt{\beta(\alpha - w_i)}$. Integer square-root bounds therefore suffice. Versions retaining $\log\det Y$ replace the scalar function by $(w_i - \alpha)x - \beta/x - \gamma \log x$; its derivative has a quadratic numerator, so endpoints and rational root brackets again give a complete check. Coefficient signs determine which determinant bound may be substituted.

For the average of the eight integer components, let $z = t/2 - 1/8$, $r = t - 1$, $v_c = (\varepsilon + z\mathbf{1}, r)$ and $w_c = (\mathbf{1}/8, 0)$. Its covariance is $C = v_c v_c^{\mathsf T} + \operatorname{diag}(I_8, 0)/8 - w_c w_c^{\mathsf T}$. For an unselected reference $R_{\mathrm{aux}} = \left(\begin{smallmatrix} a_0 I & -d_0\mathbf{1} \\ -d_0\mathbf{1}^{\mathsf T} & q_0 \end{smallmatrix}\right)$, let $\kappa = a_0/8$, $P_c = R_{\mathrm{aux}}[v_c]$, $\omega = a_0(Z + 8z)/8 - d_0 r$ and $B_c = \omega^2/\kappa + r^2/(R_{\mathrm{aux}}^{-1})_{99}$. The eigenvalues of the relative covariance $R_{\mathrm{aux}}^{1/2} C R_{\mathrm{aux}}^{1/2}$ are

$$\kappa + \mu, \qquad \kappa \text{ (6 times)}, \qquad P_c - \mu, \qquad 0, \qquad \mu = \frac{P_c - \kappa + \sqrt{(P_c + \kappa)^2 - 4\kappa B_c}}{2}.$$

Here is a direct two-dimensional calculation. Put

$$\hat v = R_{\mathrm{aux}}^{1/2} v_c, \qquad \hat w = R_{\mathrm{aux}}^{1/2} w_c, \qquad \eta_c = \frac{R_{\mathrm{aux}}^{-1/2} e_9}{\sqrt{(R_{\mathrm{aux}}^{-1})_{99}}}.$$

The vectors $\eta_c, \hat w/\sqrt{\kappa}$ are orthonormal. The first eight columns of $R_{\mathrm{aux}}^{1/2}/\sqrt{a_0}$ are also orthonormal, with orthogonal complement spanned by η_c. Consequently the relative covariance is

$$\kappa \Pi_c + \hat v \hat v^{\mathsf T}, \qquad \Pi_c = I - \eta_c \eta_c^{\mathsf T} - \frac{\hat w \hat w^{\mathsf T}}{\kappa},$$

where Π_c is a rank-seven orthogonal projector. The squared projection of $\hat v$ onto its kernel is B_c: its two coordinates are $\omega/\sqrt{\kappa}$ and $r/\sqrt{(R_{\mathrm{aux}}^{-1})_{99}}$. Six eigenvalues remain κ and one remains zero. The other two are the eigenvalues of

$$\begin{pmatrix} \kappa + P_c - B_c & \sqrt{(P_c - B_c)B_c} \\ \sqrt{(P_c - B_c)B_c} & B_c \end{pmatrix}.$$

Its trace is $\kappa + P_c$ and determinant κB_c, giving the stated formula, including vanishing projections by continuity. Its characteristic polynomial at κ is $-\kappa(P_c - B_c) \le 0$, so $0 \le P_c - \mu \le \kappa$. Replacing the smaller eigenvalue by κ is therefore safe. Apply (77); an upper bound below $4 - 1/u$ contradicts the contact average.

For the primal trace an especially simple choice is $R_{\mathrm{aux}} = b_0 Q_0$, $b_0 = 9/(9 + u_+ V_+)$. Indeed $\operatorname{tr}(Q_0 \widetilde{X}) = 9 + uV - J$. Choose $H > 1$ satisfying the determinant-cap comparison above, $F_9(H) \geq E/(4 \det R_{\mathrm{aux}})$. The comparison spectrum consisting of H and eight copies of $8/(9 - 1/H)$ has inverse trace nine and determinant $F_9(H)$. Lemma 7 and its monotonicity therefore give $\operatorname{tr} Y < H + 64/(9 - 1/H)$: a trace at least this large would force $\det Y \geq F_9(H)$, contrary to the cap. Because $Q_0[\delta_\beta] = 2p + 5t^2$, (15) gives

$$\operatorname{tr} G \leq b_0 \Big[H(1 + 2p + 5t^2) + \frac{64}{9 - \frac{1}{H}} \Big] + \frac{1}{u}. \tag{78}$$

An upper bound at most $103/10$ invokes Lemma 20.
For a weighted variant, choose instead a reference of the form

$$R_{\mathrm{aux}} = \omega_0 \begin{pmatrix} I_8 & -d_0 \mathbf{1} \\ -d_0 \mathbf{1}^{\mathsf{T}} & q_0 \end{pmatrix}, \qquad \omega_0 > 0, \qquad q_0 > 8 d_0^2,$$

still satisfying $\operatorname{tr}(R_{\mathrm{aux}} \widetilde{X}) \leq 9$ and $\widetilde{Q} \preceq H R_{\mathrm{aux}}$. For $\delta_\beta = t d_\beta + (\varepsilon, 0)$ with $d_\beta = (\frac{1}{2}\mathbf{1}, 1)$, take $r_\varepsilon \geq 0$ with $r_\varepsilon^2 \geq A_\varepsilon$ throughout the box and $t_+ \geq t \geq 0$. Cauchy–Schwarz gives

$$\widetilde{Q}[\delta_\beta] \leq \widetilde{Q}[t d_\beta] + H \omega_0 \{ 2 t_+ r_\varepsilon \sqrt{2 - 8 d_0 + q_0} + r_\varepsilon^2 \}.$$

The relative covariance for $\operatorname{tr}(Q_0^{-1} \widetilde{Q}) + \widetilde{Q}[t d_\beta]$ is $R_{\mathrm{aux}}^{1/2} C_t R_{\mathrm{aux}}^{1/2}$ with $C_t = Q_0^{-1} + t^2 d_\beta d_\beta^{\mathsf{T}}$. On the horizontal subspace perpendicular to $\mathbf{1}$, its seven eigenvalues are $\omega_0/2$. On the remaining plane with orthonormal basis $(\mathbf{1}/\sqrt{8}, 0), e_9$, the reference and covariance blocks are

$$R_2 = \omega_0 \begin{pmatrix} 1 & -\sqrt{8} d_0 \\ -\sqrt{8} d_0 & q_0 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} \frac{1}{2} + 2t^2 & \sqrt{2} t^2 \\ \sqrt{2} t^2 & 1 + t^2 \end{pmatrix}.$$

Thus the last two eigenvalues have sum $\operatorname{tr}(R_2 C_2)$ and product $\det R_2 \det C_2$, namely

$$\omega_0 \Big[\frac{1}{2} + q_0 + (2 - 8 d_0 + q_0) t^2 \Big], \qquad \omega_0^2 (q_0 - 8 d_0^2) \Big(\frac{1}{2} + \frac{5 t^2}{2} \Big).$$

Applying the scalar support bound to these nine eigenvalues proves the weighted primal-trace test.

16 The height reduction and its finite obligations

The higher-height proof excludes $u \geq 459/500$. It joins the complete $u \geq 12/13$ certificate to a 646 873-node, 323 437-leaf cover of $[459/500, 12/13]$. Besides the determinant and contact tests above, its repair leaves use a dual-contact repair inequality. We prove its contact bound and describe the finite scalar checks used to apply it.

The repair inequality. Let $Y_d = G^{-1}/u$, $g_i = G_{ii} - 1$, and $r_i = (Y_d)_{ii} - \sum_{k\neq i} |(Y_d)_{ik}|$. Primal root lengths at least two give $g_i + g_k \geq 2|G_{ik}|$ for $i \neq k$. Symmetry therefore gives

$$\operatorname{tr}(GY_d) - \operatorname{tr} Y_d - \sum_i r_i g_i = \frac{1}{2} \sum_i \sum_{k\neq i} \big[|(Y_d)_{ik}|(g_i + g_k) + 2G_{ik}(Y_d)_{ik} \big] \geq 0.$$

Define $\epsilon_i^{\mathrm{rep}} = (-r_i)_+$, choose j maximising $(r_j)_+$, and let $\beta_r = (2(r_j)_+ - \sum_i (r_i)_+)_+$. Set $w_i = (r_i)_+$ for $i \neq j$ and $w_j = (r_j)_+ - \beta_r$. Then $w_i \geq 0$ and $2w_i \leq \sum_k w_k$ for every i. If $\beta_r = 0$, this follows from the choice of j; otherwise $w_j = \sum_{i\neq j} w_i$, which proves both assertions.
For any such weights, $\sum_i w_i g_i \geq 0$. Indeed $g_i + g_k \geq 0$ allows at most one negative g_k. If none is negative the assertion is immediate; otherwise

$$\sum_i w_i g_i \geq g_k \Big(2w_k - \sum_i w_i \Big) \geq 0.$$

Consequently $\sum_i r_i g_i \geq \beta_r g_j - \sum_i \epsilon_i^{\mathrm{rep}} g_i$. Using $\operatorname{tr}(GY_d) = 10/u$ and $u \operatorname{tr} Y_d = 10 + D_G$ in the first inequality proves

$$D_G \leq u \sum_i \epsilon_i^{\mathrm{rep}} (G_{ii} - 1) + u\beta_r (1 - G_{jj}).$$

Dual pair lengths at least two imply $r_i \geq 9 - 3(Y_d)_{ii} - \operatorname{tr} Y_d/2$. Summing the dual inequalities away from j also gives $\beta_r \leq [7 \operatorname{tr} Y_d - 72 - 6(Y_d)_{jj}]_+$. With $G_{jj}(Y_d)_{jj} \geq 1/u$, maximising the last term over $0 < G_{jj} < 1$ gives

$$D_G \leq u \sum_i \epsilon_i^{\mathrm{rep}} (G_{ii} - 1)_+ + \Big(\sqrt{\max(6, 70 + 7D_G - 72u)} - \sqrt{6} \Big)^2. \tag{79}$$

For this last maximisation, let $A = 70 + 7D_G - 72u$ and maximise $(A - 6/q)(1 - q)$; its interior maximum is $(\sqrt{A} - \sqrt{6})^2$ when $A > 6$, and otherwise the positive correction is zero. This proves the repair inequality, including degree imbalance.

The large normal-budget branch. We first dispose of the entire region

$$u \geq \frac{19}{20}, \qquad \mathcal{N} \geq \frac{53}{200}.$$

The argument below has only fourteen saved radius records: six vertices in each of two polygons, and two paired-phase endpoints. All other numerical comparisons displayed here are fixed rational inequalities.
Let $\rho = 21/50$ and $h_\rho = f(\rho) = 609/2500$, where $f(x) = x - x^2$ on $[0, 1/2]$. Define the increasing convex quadratic

$$\Phi_k(r, s, z) = 7r^2 + s^2 + 7rz + sz + 4z^2 + k\rho z = 7\Big(r + \frac{z}{2}\Big)^2 + \Big(s + \frac{z}{2}\Big)^2 + 2z^2 + k\rho z.$$

Here $k \in \{0, 1\}$ counts filled phase capacities, and z is a scalar argument; it is replaced by the actual axial phase t below.

Input 27 (large normal-budget radius records). For each row of the following table, the archived records give radius triples $(R,S,Z_0)\in\mathbb{Q}^3$ at the six listed vertices:

ℓ	h	b	g
$\frac{57}{200}$	$\frac{6}{19}$	$\frac{1}{12}$	$\frac{1}{1000}$
$\frac{53}{200}$	$\frac{57}{200}$	$\frac{3}{40}$	$\frac{1}{50}$

$$(n,\eta,k)=(n,0,1),\ (n,n-h_\rho,0),\ (n,b,0),\qquad n\in\{\ell,h\}.$$

Every triple must satisfy the finite rational checks

$$0\le R,S,Z_0\le\frac12,\quad f(R)\ge\eta,\quad f(S)\ge n-\eta-kh_\rho,\quad f(Z_0)\ge\frac{\frac{9}{19}-n}{2},\tag{80}$$

$$\Phi_k(R,S,Z_0)+k\rho^2-2n+6\eta\le\frac{4}{19}-g.\tag{81}$$

At each shared-edge vertex $(n,n-h_\rho,0)$ require $S=\rho$.
In addition, at $\eta\in\{3/40,1459/15200\}$ the archived records give rational $(R,S,W)\in[0,1/2]^3$ and one common rational $g_{\mathrm{p}}>0$ such that

$$f(R)\ge\eta,\qquad f(S)\ge\frac{53}{200}-\eta,\qquad f(W)\ge\frac{1459}{7600}-\eta,\tag{82}$$

$$-\frac{4}{19}-\frac{53}{100}+6\eta+S^2+\frac72(R^2+W^2)+\frac{3}{25}S+\frac12\Big(\frac{3}{25}\Big)^2\le-g_{\mathrm{p}}.\tag{83}$$

These fourteen triples and their stated rational inequalities are the complete computational input for the present branch.

The six triples in the first band are `horizontal_convex_vertices` in `physical_followup_2026-10-20/lower_height_large_budget_verified.json`. The six triples in the second band and the two paired endpoints are `small_slack_convex_vertices` and `paired_phase_convex_endpoints` in `physical_followup_2026-10-20/paired_phase_tail_repair_verified.json`. The fields `r_upper`, `s_upper`, `t_upper` give (R,S,Z_0); `other_phase_upper` gives W at the paired endpoints. The recorded paired upper bounds are $-3810558151191/38000000000000$ and $-9911092127/2375000000000$, so one may take $g_{\mathrm{p}}=1/1000$. Both records have independent replays in the certificate archive. They too belong to Computational result 16.

Proposition 28. *Under Input 27, no good-centre counterexample has $u\ge19/20$ and $\mathcal{N}\ge53/200$.*

Proof. Throughout the proof G, $\widetilde{Q}$, their phases and the slack J are the actual matrices and parameters already fixed. Label the last two coordinates of G by $+$ and $-$ so that the normalised inverse diagonals are

$$y_i:=(Y_d)_{ii}=\frac{2B_{ii}}{u}+2\Big(\varepsilon_i+\frac{t}{2}\Big)^2\ (1\le i\le8),\quad y_\pm:=(Y_d)_{\pm\pm}=\frac{\frac{c}{u}+(t\pm1)^2}{2},\tag{84}$$

$$\tau := \operatorname{tr} Y_d = \frac{10 + D_G}{u}, \quad D_G = u(V - a + 2p + 5t^2) - J, \quad G_{ii} = \frac{\widetilde{Q}_{ii}}{2}\ (1 \le i \le 8). \tag{85}$$

These follow by block inversion of the first extension in its fixed D_{10} frame: the inverse horizontal coordinates have the factor $\sqrt{2}$, and the final two coordinates are the sum and difference of the normal and extension coordinates, divided by $\sqrt{2}$. Thus their squared phase terms are exactly those in (84). Write $d_j = 3y_j + \tau/2 - 9$ for the ten diagonal defect bounds. The inequality preceding (79) says $\epsilon_j^{\mathrm{rep}} \le (d_j)_+$. Consequently either $d_j \le 0$ or $G_{jj} \le 1$ makes that coordinate's repair cost zero.

Ranges and the trace excess. The budgets and Lemma 23 give

$$a \le \frac{1}{19}, \quad \frac{53}{200} \le \mathcal{N} \le \frac{6}{19}, \quad \mathcal{N} + 2f(t) \le \frac{9}{19}, \quad t < \frac{3}{25}, \quad u < \frac{24}{25}.$$

For the phase bound, f is increasing on $[0, 1/2]$, and $53/200 + 2f(3/25) > 9/19$ excludes $t \ge 3/25$. Lemma 21 gives $|\varepsilon_i| < \rho$. Since $f(x) \ge (29/50)x$ and $x^2 \le (21/50)x$ for $0 \le x \le \rho$, summation gives

$$A_\varepsilon \le \frac{21}{29}\mathcal{N}, \qquad |Z| \le L_\varepsilon \le \frac{50}{29}\mathcal{N}. \tag{86}$$

In particular $Z \le 300/551$, and hence

$$-\mathcal{N} + 2A_\varepsilon + 2tZ + 5t^2 \le \frac{150}{551} + \frac{9}{125} < \frac{7}{20}.$$

Using $V \le 6a - \mathcal{N}$ and $J \ge 0$ in (85) proves

$$D_G \le 5(1 - u) + \frac{7}{20}u \le \frac{3}{5}. \tag{87}$$

The selected diagonal and its cofactor. Put $\beta_i = b_i/u$, so $0 \le \beta_i \le V/2$. The normal identity is $B_{ii} - b_i + c/4 = u(3/4 + \eta_i)$. Substitution of this identity and (9) into (84) gives

$$\begin{aligned} y_i &\le 1 + \frac{3}{2}a + \frac{1}{6}V - \frac{1}{3}\mathcal{N} + 2\eta_i + 2\left(\varepsilon_i + \frac{t}{2}\right)^2, \\ \tau &\le 10 + 9a + V + 2A_\varepsilon + 2tZ + 5t^2, \\ d_i &\le -\frac{4}{19} - 2\mathcal{N} + 6\eta_i + 6\varepsilon_i^2 + 6t\varepsilon_i + A_\varepsilon + tZ + 4t^2. \end{aligned} \tag{88}$$

The last line also uses $\mathcal{N} + V \le 6a$ and $a \le 1/19$.

For a second bound, let E_i be any real number with $E_i \le 2\beta_i$. The weighted complementary trace satisfies

$$\operatorname{tr}(\widetilde{X}Q_0) - 2B_{ii} \le \frac{37}{2} - u\left(\frac{21}{2} + \frac{3}{2}\mathcal{N} + 2\eta_i + E_i\right). \tag{89}$$

Indeed its left side is $2T + 3c/2 - 3u/2 - 2u\eta_i - 2b_i$; substitute (9), then use $2u(\mathcal{N} + V) + J \le 12(1 - u)$ and $E_i \le 2\beta_i$. On deleting horizontal coordinate i, the corresponding block of Q_0 has determinant $2^7 = 128$. Applying AM–GM to the eight positive

eigenvalues of this weighted inverse block gives

$$\frac{\widetilde{Q}_{ii}}{2} \le \frac{\det \widetilde{Q}}{256}\Big(\frac{\operatorname{tr}(\widetilde{X}Q_0) - 2B_{ii}}{8}\Big)^8.$$

Thus, if $K \le 21/2 + 3\mathcal{N}/2 + 2\eta_i + E_i$, the actual positive complementary trace and $\det \widetilde{Q} < 1114u/4$ give

$$\frac{\widetilde{Q}_{ii}}{2} < \frac{1114u}{1024}\Big(\frac{\frac{37}{2} - uK}{8}\Big)^8.$$

This is decreasing in $u \ge u_0 := 19/20$ wherever its bracket is positive, provided $37/2 \le 9u_0K$: differentiation has the sign of $37/2 - 9uK$. The two fixed rational substitutions

$$\begin{gathered} K_{\mathrm{u}} = \frac{21}{2} + \frac{3}{2}\,\frac{57}{200} + \frac{1}{6}, \qquad K_{\mathrm{m}} = \frac{21}{2} + \frac{3}{2}\,\frac{53}{200} + \frac{1459}{7600}, \\ 0 < K_\nu, \qquad 0 < \frac{37}{2} - u_0K_\nu, \qquad \frac{37}{2} \le 9u_0K_\nu, \\ \frac{1114u_0}{1024}\Big(\frac{\frac{37}{2} - u_0K_\nu}{8}\Big)^8 \le 1 \qquad (\nu \in \{\mathrm{u},\mathrm{m}\}) \end{gathered} \tag{90}$$

are exact rational checks. The first shows that $\mathcal{N} \ge 57/200$ and $\eta_i \ge 1/12$ imply $G_{ii} < 1$. For the second, use the opposite normal inequality to set

$$E_i = \big[|t+\varepsilon_i| - (t+\varepsilon_i)^2 - \eta_i\big]_+ \le 2\beta_i.$$

It shows that whenever $\mathcal{N} \ge 53/200$ and $G_{ii} > 1$,

$$2\eta_i + E_i < \kappa_0 := \frac{1459}{7600}. \tag{91}$$

In particular $\eta_i < 1459/15200$. The rational check $f(21511/200000) \ge 1459/15200$ gives $|\varepsilon_i| \le 21511/200000$. Together with $t < 3/25$, this ensures $|t+\varepsilon_i| < 1/2$, so every subsequent inversion of f uses its increasing branch.

Two polygons close all small selected energies. Fix i and suppose $\eta_i \le 1/4$. Choose $r \in [0, 1/2]$ with $f(r) = \eta_i$; then $|\varepsilon_i| \le r$. Apply the mass transfer proved at (18) to the other seven coordinates, now with capacity ρ. Because $0 \le \mathcal{N} - \eta_i \le 6/19 < 2h_\rho$, it gives $k \in \{0,1\}$ and $0 \le s \le \rho$ such that

$$\mathcal{N} - \eta_i = kh_\rho + f(s), \qquad A_\varepsilon \le \varepsilon_i^2 + k\rho^2 + s^2, \qquad Z \le \varepsilon_i + k\rho + s. \tag{92}$$

Using these bounds in (88) gives

$$d_i \le -\frac{4}{19} + \Phi_k(r,s,t) + k\rho^2 - 2\mathcal{N} + 6\eta_i.$$

For $\ell \le \mathcal{N} \le h$ and $0 \le \eta_i \le b$, the case $k = 1$ lies in the trapezoid $0 \le \eta_i \le \mathcal{N} - h_\rho$; the case $k = 0$ lies in $\mathcal{N} - h_\rho \le \eta_i \le b$. The tabulated constants satisfy $h_\rho < \ell < h$ and $h - h_\rho < b$. Each trapezoid is the convex hull of its four endpoint vertices: first interpolate the two affine vertical edges in $\mathcal{N}$, then interpolate between the lower and

upper edge. This gives nonnegative vertex weights summing to one and reproducing both $\mathcal{N}$ and η_i.

Here is why the finite vertex tests suffice despite the inverse energies. If $w_j \geq 0$, $\sum w_j = 1$ and the radius R_j at each vertex bounds its energy, concavity of f gives

$$f\Big(\sum_j w_j R_j\Big) \geq \sum_j w_j f(R_j).$$

Each energy budget is affine in $(\mathcal{N}, \eta_i)$. Monotonicity of f therefore bounds the actual radius by $\sum_j w_j R_j$. Do this for all three radii, then use the coordinatewise monotonicity and convexity of Φ_k, apparent from its displayed sum of squares. The affine term $k\rho^2 - 2\mathcal{N} + 6\eta_i$ is reproduced exactly. At a shared-edge vertex, replacing $(k, s) = (0, \rho)$ by $(1, 0)$ changes neither this expression nor its bound. Hence the six records per band in Input 27 suffice for both four-vertex trapezoids. They prove $d_i \leq -g < 0$ whenever $\eta_i \leq b$. In the upper band $\mathcal{N} \geq 57/200$, the cofactor bound already handles $\eta_i \geq 1/12$. Every horizontal repair cost there is zero.

Paired phases close the remaining middle-band rows. Suppose $53/200 \leq \mathcal{N} \leq 57/200$, $\eta_i \geq 3/40$ and $G_{ii} > 1$; the other alternatives have just been handled. By (91), $\eta_i < \kappa_0/2$. In (92) the filled count must be zero, since $\mathcal{N} - \eta_i \leq 57/200 - 3/40 = 21/100 < h_\rho$. Write its residual radius as s_N, so $f(s_N) = \mathcal{N} - \eta_i$ and $s_N \leq 3/10$. Choose $s_0, w \in [0, 1/2]$ with

$$f(s_0) = \frac{53}{200} - \eta_i, \qquad f(w) = \kappa_0 - \eta_i.$$

The ranges just proved ensure these energies lie in $[0, 1/4]$; moreover $s_0 \leq s_N \leq 3/10$ and $|t + \varepsilon_i| \leq w$ by (91). We may lower the normal budget to $53/200$, because

$$[-2\mathcal{N} + s_N^2 + t s_N] - \left[-2\left(\frac{53}{200}\right) + s_0^2 + t s_0\right] = (s_N - s_0)\{3(s_N + s_0) + t - 2\} \leq 0.$$

The braces are negative by $s_N, s_0 \leq 3/10$ and $t \leq 3/25$. Finally the identity

$$7\varepsilon_i^2 + 7t\varepsilon_i + 4t^2 = \frac{7}{2}\{\varepsilon_i^2 + (t + \varepsilon_i)^2\} + \frac{1}{2}t^2$$

reduces the bound for d_i to the left side of (83), with (η, R, S, W) replaced by (η_i, r, s_0, w). The three energy bounds are affine in η_i; the same inverse-energy interpolation and convexity argument, now on the interval $[3/40, 1459/15200]$, bounds this expression by its two certified endpoint values. Thus $d_i < 0$. All eight horizontal repair costs vanish in the middle band too.

The two tails. Directly from (84) and the slack identities,

$$d_\pm = 3V + \mathcal{N} + p \pm 3t + 4t^2 - 1, \qquad d_+ \leq -\frac{1}{19} - 2\mathcal{N} + A_\varepsilon + tZ + 3t + 4t^2. \quad (93)$$

The elementary half-phase bounds $A_\varepsilon \leq \mathcal{N}$ and $Z \leq 2\mathcal{N}$ give $d_- \leq 3(V + \mathcal{N}) - 1 + t(4t - 3) \leq 18/19 - 1 < 0$. So the smaller tail contributes zero. For the other tail, capacity filling with ρ has exactly one filled phase, since $h_\rho < \mathcal{N} < 2h_\rho$. Thus

$$\mathcal{N} = h_\rho + f(s), \qquad A_\varepsilon \leq \rho^2 + s^2, \qquad Z \leq \rho + s, \qquad 0 \leq s \leq \rho.$$

Define $\Psi(s,t) = s^2 + st + 4t^2 + (\rho+3)t + \rho^2 - 1/19$. Its quadratic part is $(s+t/2)^2 + 15t^2/4$; hence it is convex and increasing on the nonnegative quadrant. The following fixed rational endpoint table is sufficient:

ν	$\bar{s}$	$\bar{t}$	$\Psi(\bar{s},\bar{t}) - 2\nu$ is at most
$\frac{53}{200}$	$\frac{21879}{1000000}$	$\frac{118349}{1000000}$	$\frac{3}{50}$
$\frac{57}{200}$	$\frac{43273}{1000000}$	$\frac{21093}{200000}$	0
$\frac{6}{19}$	$\frac{3133}{40000}$	$\frac{17283}{200000}$	0

In every row $0 \le \bar{s}, \bar{t} \le 1/2$, $f(\bar{s}) \ge \nu - h_\rho$ and $f(\bar{t}) \ge (9/19 - \nu)/2$; these and the displayed last-column comparisons are the complete rational checks for this table. The preceding convex interpolation proof therefore gives $d_+ \le 0$ on $[57/200, 6/19]$, and $d_+ \le 3/50$ on $[53/200, 57/200]$.

In the middle band, $f(s) = \mathcal{N} - h_\rho \le 57/200 - h_\rho < f(1/20)$, so $s \le 1/20$, $A_\varepsilon \le 1789/10000$ and $Z \le 47/100$. The actual complementary trace obeys

$$\tau - y_+ \le 9 + \frac{23}{2}a - \frac{1}{2}\mathcal{N} + 2A_\varepsilon + 2tZ + \frac{9}{2}t^2 - t \le \frac{501}{50}.$$

The first inequality is obtained by subtracting y_+ from (85) and using the common budget in (9); every dropped slack term is nonpositive. For the second, use the displayed moment bounds, $a \le 1/19, \mathcal{N} \ge 53/200, t \le 3/25$ and $2tZ - t \le 0$. As $u < 24/25$, the complementary trace of G^{-1} is at most $(24/25)(501/50) = 6012/625$. The principal-cofactor identity and AM–GM in its nine-dimensional positive block now give the fixed rational bound

$$G_{++} \le \det G \Big(\frac{\operatorname{tr} G^{-1} - (G^{-1})_{++}}{9} \Big)^9 < \frac{557}{512} \Big(\frac{6012}{5625} \Big)^9 < 2.$$

Its repair cost is therefore at most $u(d_+)_+(G_{++} - 1)_+ \le (24/25)(3/50) = 36/625$. All other coordinates have zero cost.

Finally (87) gives $70 + 7D_G - 72u \le 29/5 < 6$, so the balance term in (79) is zero. That inequality yields $D_G \le 36/625 < 1/15$, contradicting $D_G > 1/15$. This closes both normal-budget bands with the same actual slack. □

The outer small-normal-budget branch. It remains at heights $u \ge 19/20$ to consider $\mathcal{N} \le 53/200$ and $t \le 2/7$. The uniform phase bound is $|\varepsilon_i| \le \rho = 21/50$. Capacity filling supplies $s \in \{0,1\}$ and $r \ge 0$ such that

$$\mathcal{N} = s\frac{609}{2500} + r - r^2 = R - S, \quad R = s\rho + r, \quad S = s\rho^2 + r^2, \quad A_\varepsilon \le S, \quad |Z| \le R. \tag{94}$$

The two parameter rectangles are $[0, 21/50] \times [0, 2/7]$ and $[0, 11/500] \times [0, 2/7]$, with coordinates (r,t). The second radius follows from $\mathcal{N} - 609/2500 \le 53/200 - 609/2500 < f(11/500)$. In both charts $R \le 1/2$ and $S \le 1/4$.
First we record the uniform bound

$$\frac{1}{15} < D_G \le \frac{125}{196}. \tag{95}$$

For its upper bound let $P_s(r,t) = 2(s\rho^2 + r^2) + 2t(s\rho + r) + 5t^2$. The normal and phase budgets give $D_G \le u(5a - \mathcal{N} + P_s(r,t))$ and $\mathcal{N} + 2f(t) \le 9/19$. The following three rows prove $P_s(r,t) - \mathcal{N} \le 20/49$:

s	$\mathcal{N}$ interval	(r_-, t_-)	(r_+, t_+)	upper bound for $f(t)$
0	$\left[0, \frac{61}{931}\right]$	$\left(0, \frac{2}{7}\right)$	$\left(\frac{71}{1000}, \frac{2}{7}\right)$	$\frac{10}{49}$
0	$\left[\frac{61}{931}, \frac{609}{2500}\right]$	$\left(\frac{71}{1000}, \frac{2}{7}\right)$	$\left(\frac{21}{50}, \frac{27}{200}\right)$	$\frac{9}{38} - \frac{\mathcal{N}}{2}$
1	$\left[\frac{609}{2500}, \frac{53}{200}\right]$	$\left(0, \frac{27}{200}\right)$	$\left(\frac{11}{500}, \frac{3}{25}\right)$	$\frac{9}{38} - \frac{\mathcal{N}}{2}$

The two listed radii are outward bounds at the left and right endpoints of the $\mathcal{N}$ interval, respectively. All checks are rational: the radii lie in $[0, 1/2]$, their f-values dominate the corresponding affine energy bounds, and $P_s(r_\pm, t_\pm) - \mathcal{N}_\pm \le 20/49$. To see why endpoint checks suffice, f^{-1} is increasing and convex on $[0, 1/4]$, so each actual radius is at most the affine interpolant of its two outward endpoint bounds. The polynomial P_s is increasing in nonnegative r, t and jointly convex: its quadratic part is $2r^2 + 2rt + 5t^2$. Its value at the two affine interpolants is therefore bounded by the corresponding interpolation of endpoint values. Subtracting $\mathcal{N}$ preserves this interpolation. Finally $u(5a + 20/49) = 5(1-u) + 20u/49 \le 125/196$.

Shared load bounds every active subset. For this paragraph let

$$L_i = \eta_i + \varepsilon_i^2 + t\varepsilon_i, \qquad L_* = \sum_i L_i = \mathcal{N} + p, \qquad \xi_i = \ell_i - L_i, \qquad \ell_i = \frac{V}{2} - \beta_i.$$

The phase inequalities give $L_i \ge (1-t)|\varepsilon_i| \ge 0$ and

$$\sum_i \xi_i = V - L_*, \qquad \xi_i \le \frac{V - f(t)}{2}. \tag{96}$$

For the latter, add the two inequalities for the first and opposite normal and use $t \le |\varepsilon_i| + |t + \varepsilon_i|$; expansion gives $2(\beta_i + L_i) \ge f(t)$. Thus every set I of k horizontal coordinates satisfies

$$\sum_{i \in I} \xi_i \ge -L_*, \qquad \sum_{i \in I} \xi_i \ge V - L_* - (8-k)\frac{V - f(t)}{2}.$$

The first bound uses $\ell_i \ge 0$ and the same total load L_*; the second bounds the omitted coordinates using (96). Let $z_i = u\epsilon_i^{\mathrm{rep}}$ on an active repair row. The exact diagonal identities preceding (79) give

$$z_i \le D_G - u(1 - 10a + t^2) - uL_* - 6u\xi_i.$$

Summing the two preceding residue bounds proves that its mean is at most both

$$M_2(k) = D_G - u(1 - 10a + t^2) + u\left(\frac{6}{k} - 1\right)L_*,$$

$$M_1(k) = M_2(k) + \frac{3u}{k}\{(6-k)V - (8-k)f(t)\}.$$

For $k \geq 6$ these formulas and $4f(t) \leq 3V + L_*$ give

$$M_1(k) \leq D_G - u(1 - 10a + t). \tag{97}$$

Indeed k times the difference equals $u(k-6)(4f(t) - 3V - L_*) \leq 0$. A negative upper bound on the right therefore excludes all active sets of size at least six at once. The larger tail has defect at most $u(3V + L_* + 3t + 4t^2 - 1)$. The smaller tail has zero cost: its degree is at most $3V + L_* - 3t + 4t^2 - 1 \leq 18a - 1 \leq 0$. For the middle inequality use $L_* \leq 2(1+t)\mathcal{N}$, $V \leq 6a - \mathcal{N}$, and $0 \leq t \leq 1/2$.
All these bounds admit a common two-variable enlargement. Define the following affine functions of a, with polynomial coefficients in R, S, t:

$$\begin{aligned}
F_D &= 5a - R + 3S + 2tR + 5t^2, \\
F_2(k) &= 15a - 1 + 3S + \left(\frac{6}{k} - 2\right)R + \left(1 + \frac{6}{k}\right)tR + 4t^2, \\
F_1(k) &= \left(\frac{108}{k} - 3\right)a - 1 + \left(\frac{18}{k}\right)S + \left(1 - \frac{12}{k}\right)R + \left(1 + \frac{6}{k}\right)tR \\
&\qquad + \left(3 - \frac{24}{k}\right)t + \left(1 + \frac{24}{k}\right)t^2, \\
F_+ &= 18a - 1 - 2R + 3S + tR + 3t + 4t^2, \\
F_6 &= 15a - 1 - R + 3S + 2tR + 5t^2 - t.
\end{aligned}$$

Respectively they bound D_G/u, $M_2(k)/u$, $M_1(k)/u$, the larger-tail defect divided by u, and $(D_G - u(1 - 10a + t))/u$. To verify the substitutions, use $V \leq 6a - R + S$, $p \leq S + tR$, $L_* \leq R(1+t)$ and $J \geq 0$; all coefficients used to replace V and L_* are nonnegative for $1 \leq k \leq 5$. Each function has the form $ba + c$ with $c \leq b$ on $0 \leq R \leq 1/2$, $0 \leq S \leq 1/4$, $0 \leq t \leq 1/3$. This last assertion follows by substitution for each of the five integer values of k, using $tR \leq 1/6$ and $t^2 \leq 1/9$. Hence $u(ba + c) = b + u(c - b) \leq (19/20)(b/19 + c)$. After substituting (94), the fixed-height expressions $(19/20)F(1/19, R, S, t)$ are rational quadratic polynomials in (r, t) with nonnegative coefficients of r^2 and t^2. They are therefore convex in either coordinate separately. Applying one-dimensional convex interpolation twice bounds each on a rectangle by its four corner values, including degenerate rectangles.

Degree-nine repair costs and their finite tangent tests. Write $d = D_G$ temporarily and define

$$Q_9(d, z) = \frac{557}{512}\Big(\frac{\frac{529}{60} + \frac{7d}{6} - \frac{z}{3}}{9}\Big)^9, \qquad \Psi_d(z) = z(Q_9(d, z) - 1).$$

For every active coordinate of the actual rank-ten matrix,

$$G_{ii} \leq Q_9(D_G, z_i), \qquad 0 < z_i < 2. \tag{98}$$

In fact $z_i \leq 3(G^{-1})_{ii} + \operatorname{tr}(G^{-1})/2 - 9u$, so the trace of the complementary inverse block is at most $529/60 + 7D_G/6 - z_i/3$. The cofactor identity and AM–GM on its nine

eigenvalues prove the first bound, using $\det G < 557/512$. The base is nonnegative because it bounds a positive trace. If $z_i \geq 2$, monotonicity in d and z would give $G_{ii} \leq Q_9(125/196, 2) < 1$, contrary to activity. The last strict inequality is a single rational power comparison.

Fix $z \in [0, 2]$. The function $d \mapsto Q_9(d, z)$ is convex for $d \geq 0$, and $Q_9(0, z) \leq Q_9(0, 0) < 1$. Convex interpolation between 0 and $\overline{d}$ gives, for $0 < d \leq \overline{d}$,

$$\frac{(Q_9(d, z) - 1)_+}{d} \leq \frac{(Q_9(\overline{d}, z) - 1)_+}{\overline{d}}.$$

For active rows both relevant positive parts can be removed. Thus each cost per unit excess is at most $\Psi_{\overline{d}}(z_i)/\overline{d}$. No matrix at the artificial value $\overline{d}$ is used.

The cost function is concave on $[0, 2]$ when $0 < \overline{d} \leq 125/196$. Indeed, with $b = 529/60 + 7\overline{d}/6$,

$$\Psi''_{\overline{d}}(z) = \frac{557}{512}\Big(\frac{b - \frac{z}{3}}{9}\Big)^7 \frac{-6b + 10z}{81} \leq 0.$$

For a nonempty active set of size k, Jensen's inequality bounds the sum by $k\Psi_{\overline{d}}(\overline{z})$, where $\overline{z} = k^{-1} \sum z_i$. One can prove this instance of Jensen directly: apply the tangent inequality at $\overline{z}$ to each z_i and sum; the linear terms cancel. If $\overline{z} \leq U$, it suffices to bound Ψ on $[0, \min\{2, U\}]$. A finite certificate chooses a rational t_0 in this interval and checks that the tangent

$$\Psi_{\overline{d}}(t_0) + \Psi'_{\overline{d}}(t_0)(x - t_0)$$

is at most a rational C at its two endpoints. Here

$$\Psi'_{\overline{d}}(z) = \frac{557}{512}\Big(\frac{b - \frac{z}{3}}{9}\Big)^8 \frac{b - \frac{10z}{3}}{9} - 1.$$

All values and derivatives are rational. If $U \leq 0$, the active set is empty instead, since each of its weights is positive. This is a prefix test on the mean; individual weights may occupy the whole interval $[0, 2]$.

The finite rectangle cover and absorption. For each rectangle, record four-corner upper bounds d_+, $U_1(k)$, $U_2(k)$, U_+ and U_6 for the five fixed-height expressions above, for $k = 1, \ldots, 5$. Set $\overline{d} = \min\{125/196, d_+\}$. An occupied repair rectangle has $\overline{d} \geq D_G > 1/15$. Require $U_6 < 0$, and choose nonnegative rational caps C_k, C_+ using the preceding prefix certificates with respective mean bounds $\min\{U_1(k), U_2(k)\}$ and U_+. The remaining five finite checks are

$$kC_k + C_+ \leq \frac{19}{20}\overline{d}, \qquad k = 1, \ldots, 5. \tag{99}$$

The active horizontal set has at most five elements by (97). If it has $k > 0$ elements, its cost divided by D_G is at most $kC_k/\overline{d}$; the larger-tail cost divided by D_G is at most $C_+/\overline{d}$. For an empty horizontal set, the $k = 1$ check and $C_1 \geq 0$ still bound the tail. Thus the total repair cost is at most $(19/20)D_G$ in every case. The balance term in (79) is at most $27/156800$: use $70 + 7D_G - 72u - 6 \leq 9/140$ and $(\sqrt{6 + x} - \sqrt{6})^2 \leq x^2/24$ for $x \geq 0$. It follows that $D_G/20 \leq 27/156800 < 1/300$, contradicting $D_G > 1/15$.

The two root rectangles in (94) are covered by finite closed binary subdivisions. Besides a repair leaf, the allowed terminal tests are

$$s\frac{609}{2500} + f(r_-) > \frac{53}{200}, \qquad s\frac{609}{2500} + f(r_-) + 2f(t_-) > \frac{9}{19}, \qquad d_+ \le \frac{1}{15}.$$

They contradict respectively the normal cap, phase budget, and strict excess gap. Thus all numerical obligations are explicit: root and split endpoints, four-corner polynomial inequalities, prefix tangent endpoint inequalities, and (99). Induction on these two finite trees proves the outer small-budget exclusion. Together with the large-normal and large-axis arguments, this covers the entire range $u \ge 19/20$ conditionally on the specified finite certificates.

The horizontal cofactor majorants have the form $q_i(z) = K_i[b_i - z/6]_+^7$. For any nonnegative multiplier λ,

$$\sum_i z_i(q_i(z_i) - 1) \le \lambda B_{\mathrm{def}} + \sum_i \max_{z \ge 0}\{z(q_i(z) - 1) - \lambda z\} \quad \text{if } \sum z_i \le B_{\mathrm{def}}.$$

Every maximum occurs in $0 \le z \le 3b_i/4$: the derivative is negative beyond that point, while the second derivative on this interval is $(7K_i/9)(b_i - z/6)^5(2z - 3b_i) \le 0$. A rational tangent therefore bounds each maximum. Convexity in the phase costs and the capacity-filling argument (14) bound all active row counts at once.

For the middle-height certificate there is a small but necessary point concerning an additional cap on each phase-type base. Let subscripts $-, +$ denote rational bounds for a box with $17/18 \le u_- \le u_+ \le 19/20$, $0 \le t_+ \le 1/3$, $E_+ \le 65/81$, and $c_+ > 0$. Its horizontal trace bound is $T_+ = u_-(4 + 6a_+ - (\mathcal{N}_- + V_-)/3) \le 37/9$, where $u_- = (1 + a_+)^{-1}$ and $\mathcal{N}_-, V_- \ge 0$. The phase-independent base is

$$b_\square = \max\Big\{\frac{5777}{1960},\ T_+ - \frac{3}{2}u_- + \frac{10 + E_+}{12} + \frac{u_+ t_+^2}{4} - \frac{25u_-^2 V_-^2}{28c_+}\Big\}.$$

Discarding its nonnegative coupling subtraction gives

$$b_\square \le \frac{37}{9} - \frac{3}{2}\frac{17}{18} + \frac{10 + \frac{65}{81}}{12} + \frac{\frac{19}{20}}{36}.$$

The same bound exceeds $5777/1960$, so it also bounds the maximum. Every phase type satisfies $0 \le \phi \le (23/50)^2 + t_+(23/50)$. Adding $u_+\phi$ to the last bound gives exactly

$$b_\square + u_+\phi \le \frac{12051967}{3037500}.$$

Thus the certificate's replacement of this affine base by its minimum with this number changes no value. Convex capacity filling applies to the original affine dependence; no convexity of a clipped function is required.

The two tail cofactors keep the horizontal trace: with $B_t = T + u(p + 2t^2)$ and $C_\sigma = c + u(t - \sigma)^2$,

$$G_{\sigma\sigma} < \frac{1114}{8} C_\sigma \left(\frac{B_t}{8}\right)^8, \qquad z_\sigma = u[3V + \mathcal{N} + p + 3\sigma t + 4t^2 - 1].$$

Their repair is at most $(z_\sigma)_+(G_{\sigma\sigma}-1)_+$. These follow by deleting the tail coordinate, Schur complementing, and applying AM–GM to the eight-dimensional horizontal block.

One endpoint transfer for all repair regions. The same argument applies throughout $459/500 \le u \le 19/20$. It uses the same-coordinate finite certificate with normal-energy threshold $319/1600 = (11/40)(1-11/40)$ and total scaled normal budget $123/250$. Its soundness proof supplies $|\varepsilon_i| < 11/40$ for every contributing horizontal row, and leaves at most one exception among all eight coordinates. Filling the remaining phase capacities gives $\sum|\varepsilon_i| \le 7/8$ and $A_\varepsilon \le 537/1600$, hence $p \le 4/5$.
Write $E = D_G + J = u(V - a + 2p + 5t^2)$ and introduce a scalar variable $x \in [0, E]$. Hold the actual phases, normal energies, u, V, and all original residual weights fixed. Set

$$c_x = 4u - 3 + \frac{u(2\mathcal{N}+5V)}{3} + \frac{E-x}{3}, \qquad T_x = 6 - 2u - \frac{u(\mathcal{N}+V)}{3} - \frac{2(E-x)}{3}.$$

At $x = D_G$ these equal the actual inverse-axis entry and horizontal trace. The normal budget gives $u(V-a) \le 5(1-u)$, so

$$0 < E \le 5(1-u) + u\left(\frac{8}{5} + \frac{5}{4}\right) = 5 - \frac{43u}{20} \le \frac{30263}{10000} < \frac{1861}{600}.$$

For an original contributing horizontal row, let $z = u\epsilon_i^{\mathrm{rep}} \ge 0$ and retain its signed phase $\chi_i = \varepsilon_i^2 + t\varepsilon_i \le 341/1600$. Its majorant is

$$q_x = \frac{1114u}{8\cdot 7^7} c_x [s_x]_+^7, \qquad s_x = T_x - \frac{3}{2}u + \frac{10+x}{12} + \frac{ut^2}{4} - \frac{25u^2V^2}{28c_x} + u\chi_i - \frac{z}{6}.$$

For completeness, all curvature bounds can be checked with the following fixed rational numbers:

$$c_x \ge c_E \ge \frac{84}{125} + \frac{5uV}{3}, \qquad uV \le \frac{123}{250}, \qquad \frac{uV}{c_x} \le \frac{123}{373},$$
$$\ell = \frac{3}{4} - \frac{25}{84}\left(\frac{123}{373}\right)^2 > 0, \qquad \frac{M}{2} \le 6\cdot\frac{84}{125}\ell^2,$$
$$M = 6 - 2\cdot\frac{459}{500} - \frac{3}{2}\cdot\frac{459}{500} + \frac{10+\frac{1861}{600}}{12} + \frac{\frac{19}{20}}{16} + \frac{19}{20}\cdot\frac{341}{1600}.$$

Indeed $s_x \le M$ and $s_x' = 3/4 - 25u^2V^2/(84c_x^2) \ge \ell$. Where $s_x > 0$, direct differentiation gives

$$q_x'' = \frac{7\cdot 1114u}{8\cdot 7^7} s_x^5\left(6c_x(s_x')^2 - \frac{s_x}{2}\right) \ge 0.$$

The clipped seventh power and its first derivative vanish where $s_x = 0$, so this proves convexity across those points as well. At $x = 0$, since $p = \sum_j(\varepsilon_j^2 + t\varepsilon_j)$, completing the other seven squares gives

$$\chi_i - p = \frac{7t^2}{4} - \sum_{j\ne i}\left(\varepsilon_j + \frac{t}{2}\right)^2 \le \frac{7t^2}{4}.$$

With $ua = 1 - u$ and $E = u(V - a + 2p + 5t^2)$, the definitions give

$$\frac{c_0}{2} + s_0 = \frac{35}{6} - 2u + u(\chi_i - p) - \frac{9ut^2}{4} - \frac{25u^2V^2}{28c_0} - \frac{z}{6} \le \frac{35 - 12u}{6} - \frac{ut^2}{2} \le \frac{35 - 12u}{6}.$$

Weighted AM–GM, including the trivial case $s_0 \le 0$, gives

$$q_0 \le \frac{1114}{4 \cdot 8^8} u \Big(\frac{35 - 12u}{6} \Big)^8 < 1.$$

For the last strict inequality the expression in u decreases on $[459/500, 1]$: its derivative has the sign of $35 - 108u$. Substitution at $459/500$ is a strict rational comparison. Both original tail costs have two majorants. The principal-cofactor majorant is

$$q_x^{(9)} = \frac{557}{512} \Big(\frac{\frac{35}{3} - 3u + \frac{7x}{6} - \frac{z}{3}}{9} \Big)^9.$$

Here $0 \le z \le 4$, by the tail degree and the bounds above. Its base is nonnegative, it is convex, and $q_0^{(9)} \le (557/512)((35/3 - 3(459/500))/9)^9 < 1$. The complementary-block alternative is

$$q_x^{(8)} = \frac{1114}{8} C_x \left(\frac{B_x}{8} \right)^8, \quad C_x = c_x + u(t - \sigma)^2, \quad B_x = T_x + u(p + 2t^2), \quad \sigma = \pm 1.$$

The paired phase budget $4t(1-t) \le 3(\mathcal{N} + V)$ gives $C_E - u \ge 4u - 3 + u(t^2 - 10t)/9 \ge 31u/9 - 3 > 0$, hence $C_x \ge u$. Also $p + 2t^2 \ge 0$ gives $B_0 \ge 6 - 2(19/20) - (123/250)/3 - 2(1861/600)/3 > 0$. Since $C_x' = -1/3$, $B_x' = 2/3$ and

$$B_x \le B_E \le 6 - 2\left(\frac{459}{500} \right) + \frac{19}{20} \left(\frac{4}{5} + \frac{1}{2} \right),$$
$$\left(\frac{2}{3} \right) B_E \le \frac{14}{3} \left(\frac{459}{500} \right) \le \frac{14}{3} C_E,$$

differentiation proves convexity. For a contributing tail its actual nonnegative degree is $d_\sigma = 3V + \mathcal{N} + p + 3\sigma t + 4t^2 - 1 \ge 0$. The identity $C_0/2 + 2B_0 = 35/3 - 3u - ud_\sigma/3$ and weighted AM–GM give $q_0^{(8)} \le (557/512)((35/3 - 3u)/9)^9 < 1$, the bound already proved for $q_0^{(9)}$.

If q is any of these convex majorants with $q(0) \le 1$, then $[q(x) - 1]_+/x$ increases for $x > 0$: the numerator is a nonnegative convex function vanishing at zero, and its secant slopes from zero increase. The same applies to the convex balance function $\mathcal{B}_u(x) = (\sqrt{\max(6, 70 + 7x - 72u)} - \sqrt{6})^2$; it vanishes at zero for $u \ge 8/9$. As a function of $A = 70 + 7x - 72u$, the balance is zero for $A \le 6$; for $A > 6$ its first two derivatives are $1 - \sqrt{6/A}$ and $\sqrt{6}/(2A^{3/2}) \ge 0$. The first derivative is continuous at $A = 6$, proving convexity. Divide (79) by $D_G > 0$, transport every original cost to $E \ge D_G$, and multiply by E. This proves

$$E \le \sum_{i \text{ active}} z_i [q_{i,E} - 1]_+ + \sum_{\sigma = \pm 1} z_\sigma^{\text{act}} \min\{[q_{\sigma,E}^{(9)} - 1]_+, [q_{\sigma,E}^{(8)} - 1]_+\} + \mathcal{B}_u(E),$$

where $z_\sigma^{\mathrm{act}} = u\epsilon_\sigma^{\mathrm{rep}}$. The active horizontal set is selected at the original matrix and is never recomputed along the path. Subsequent phase filling may enlarge these endpoint costs. The tail intervals of the certificate use the full diagonal degree instead of the residual weight. This requires a separate, compatible bound. Let $w_\sigma = u[d_\sigma]_+$, so $z_\sigma^{\mathrm{act}} \le w_\sigma$. If $w_\sigma > 0$, the exact diagonal identity gives

$$w_\sigma = 3(G^{-1})_{\sigma\sigma} + \frac{\operatorname{tr} G^{-1}}{2} - 9u.$$

Deleting that diagonal therefore gives the principal-cofactor majorant $q_x^{(9)}$ with $z = w_\sigma$. Multiplying by the larger nonnegative weight bounds the original cost, and its cost-per-excess ratio transports to E exactly as above. The block alternative is independent of the weight and permits the same enlargement. If $w_\sigma = 0$, the original tail cost is zero. Thus the endpoint repair inequality also holds with both tail terms replaced by $w_\sigma \min\{[q_{\sigma,E}^{(9)}(w_\sigma) - 1]_+, [q_{\sigma,E}^{(8)} - 1]_+\}$. If a box supplies $d_- \le d_\sigma \le d_+$, then

$$u_-[d_-]_+ \le w_\sigma \le u_+[d_+]_+.$$

These are the endpoints used by the tail tangent certificate. A finite repair leaf bounds the appropriate complete endpoint total strictly below E, excluding its box. All accompanying contact tests continue to apply to the actual metric.

The lower heights. The lower-height determinant tests, using Lemma 26, give two complete covers: 146 291 terminals for $419/500 \le u \le 181/200$, and 4 257 for $181/200 \le u \le 459/500$, $\mathcal{N} \ge 31/100$. The lower rational endpoint is valid because $512/729 > (419/500)^2$. The region $\mathcal{N} \le 1/200$ is covered by 15 boxes using the weighted primal-trace perturbation of Section 15. Thus only

$$\frac{181}{200} < u < \frac{459}{500}, \qquad \frac{1}{200} < \mathcal{N} < \frac{31}{100} \tag{100}$$

remains. Every boundary endpoint is already covered (Computational result 16).

17 The final finite cover and the exhaustion argument

The final root, in coordinates $(a, \mathcal{N}, t, p, V)$, is

$$\left[\frac{41}{459}, \frac{53}{447}\right] \times \left[\frac{1}{200}, \frac{106}{149}\right] \times \left[0, \frac{1}{2}\right] \times [-1, 2] \times \left[0, \frac{106}{149}\right].$$

It contains (100): $a < 19/181 < 53/447$, $\mathcal{N}, V \le 6a < 114/181 < 106/149$, and $-\mathcal{N}/2 \le p \le 7\mathcal{N}/5$. A sixth certificate file coordinate is padding; it never restricts or replaces the actual common slack.

Contractions intersect the box with $\mathcal{N} + V \le 6a$, (11), (20), and the signed mass bounds. For example, a phase profile supplying $A_\varepsilon \le S$, $\sum |\varepsilon_i| \le L_\varepsilon$, and $\{\mathcal{N} + A_\varepsilon - (1 - 2t)Z\}/2 \le W$ gives

$$p_+ \leftarrow \min(p_+, S + t_+ L_\varepsilon), \quad V_- \leftarrow \max\Big(V_-, \frac{4t_-(1 - t_-) - \min(\mathcal{N}_+ + p_+, W)}{3}\Big).$$

The full actual $\mathcal{N}_+$ remains in W. The identities $p = A_\varepsilon + tZ$ and $|Z| \le \min(\mathcal{N} + A_\varepsilon, \sqrt{8A_\varepsilon})$ give further necessary intersections. The checker performs four passes and rounds every endpoint outward to the 2^{-28} grid; each pass preserves the feasible set. The unequal-triangle cap is propagated into (12) and the reference determinant before a terminal test is evaluated. Conditional subdivisions in A_ε or Z must cover the entire remaining interval, including endpoints.
The five resumed journals contain 13 985 nodes, 6 992 splits, and 6 993 terminal regions: 61 empty intersections, 3 910 contact or complete conditional-phase exclusions, 34 affine harmonic bounds, 4 triangle–Schur bounds, 362 weighted primal-trace bounds, 1 267 isotropic primal-trace bounds, and 1 355 triangle–axis bounds. These counts sum exactly; the final frontier is empty. Every resumed root equals a pending box of the preceding journal (Computational result 17).
An actual counterexample would have a good centre, survive the height and normal reductions, and belong to this root. Each necessary contraction preserves it. Following the finite tree would therefore reach a terminal box containing it, where one of the proved inequalities contradicts its existence. This proves the exclusion of `0050`, for both first-gluing classes, both coupling signs, and every second gluing.

Proof of Theorem 1. Let $L \supset M \supset N$ be a chain of Definition 2 with $\Delta < 972$ whose rank-9 member is admissibly marked by $P =$ `0050:024468269`. In the notation of Section 2.1 it is an admissible chain for P with $\Delta < 972$; Section 4 writes it in canonical coordinates, with the bounds (10). By Lemma 19 and the bad-centre forest of Section 6 (Computational result 14) its nearest-phase centre is good, and Lemma 20 with Computational result 15 gives (20). The height reduction of Sections 7–16 (Proposition 25, Proposition 28, the covers of Computational result 16) confines it to (100), hence to the final root, and the final cover (Computational result 17) reaches a terminal box containing it where one of the inequalities proved in Sections 10–16 contradicts its existence. Hence no such chain exists, and every chain of Definition 2 admissibly marked by P has $\det L \ge 972$. □

18 What is verified, and where

In the style of the root paper's "What is imported, precisely", we list the finite checks, the records that are retained, and the hypotheses on which Theorem 1 rests. The verification rules are common to the series and are stated in [4, Section 7 and Appendix D.7]; we recall what they require of this case.

Three obligations. A finite computer-assisted argument has three logically separate parts: every actual counterexample must lie in the stated root and satisfy every necessary inequality; a terminal witness must imply the claimed contradiction throughout its closed box; and the complete collection of boxes must cover the root. Sections 4–17 prove the geometric and analytic implications; the certificate archive of the series contains the rational data and the programs that check their coefficient identities, terminal witnesses, and coverage. A count of successful sample points satisfies none of these obligations. The certificate formats are elementary: matrix and polynomial identities, compared exactly coefficient by coefficient (a tensor Bernstein certificate is

expanded back into the power basis); finite vector and cell lists, with an exact bounded LDL^{T} enumeration for every claimed complete short-vector list and a completeness certificate for every cell list; and linear or scalar terminal witnesses, for which the checker reconstructs the necessary inequality system and uses nonnegative rational multipliers (Lemma 10), rational interval arithmetic, polynomial signs, or positive Bernstein coefficients, every square-root and logarithm enclosure carrying its error bound. The rule that discharges a polynomial implication is the following.

Lemma 29 (checking a polynomial implication). *Suppose a domain is given by polynomial inequalities $g_j(x) \geq 0$ for $1 \leq j \leq r$, and equations $h_k(x) = 0$ for $1 \leq k \leq s$. An identity*

$$F = \sum_{\epsilon \in \{0,1\}^r} \sigma_\epsilon \prod_{j=1}^{r} g_j^{\epsilon_j} + \sum_k p_k h_k,$$

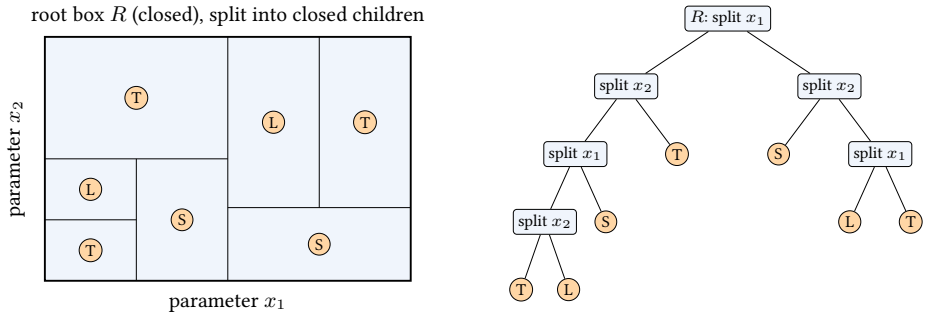
with rational polynomials p_k and each σ_ϵ a sum of positive rational multiples of squares, proves $F \geq 0$ on the domain. Applied to $F - \eta$ with rational $\eta > 0$, it proves the strict margin $F \geq \eta$.

Proof. Each term of the first sum is nonnegative and each term of the second sum vanishes. Exact coefficient comparison checks the identity, including all coefficients that should be zero. □

Covering trees and resumptions. Each root is a product of closed rational intervals. A split stores an axis and an interior rational cut; its two children have union equal to their parent and share the cut boundary. A contraction stores a necessary bound whose intersection contains every counterexample contained in the box. A terminal node stores one of the contradiction witnesses above. A pending node proves nothing. A verifier begins with the root as its pending set and processes each record once; it rejects an unknown address, a repeated node, a wrong parent box, a gap between children, an unsupported contraction, or an invalid terminal witness; at the end the pending set must be empty. For a resumed journal, the next roots must be exactly the preceding pending boxes; for a forest, every substituted child root must equal the parent leaf it replaces, and the substitution graph must be finite and acyclic. These checks prove coverage by induction on the tree. Figure 2 illustrates them. The large centre forest of Section 6 uses checked integer arithmetic with outward intervals on a fixed 2^{-48} grid; each addition, product, quotient and integer square root rounds in the required direction, and overflow rejects the computation; its interval primitives have 12 312 independent rational comparison checks, recorded alongside the complete replay of all terminal inequalities.

The finite checks of this paper. They are exactly Computational results 14–17, together with the printed-form check of Section 3:

– Computational result 14: the cofactor cover of Section 5 (3 978 481 nodes, 1 989 241 leaves, giving (13)), the bad-centre boundary certificates and the 1 249-tree forest with 24 622 334 terminal regions of Section 6, whose terminal inequalities are those of Sections 12–15;



Splits are binary: a box is cut into two closed children that cover it; a node may also carry a *contraction* (a proved shrinking of its box) before it is split. **Leaves** carry exact witnesses, for instance T: a terminal inequality that excludes the whole box ($\Delta \geq 972$, or $d \geq d_{\mathrm{cut}}$) or shows it empty; L: a polynomial certificate (Bernstein coefficients or a Sturm sequence); S: a scalar witness compared in rationals. **The checker** starts with $\{R\}$ as its pending set and processes each record once. It rejects an unknown address, a repeated node, a wrong parent box, a gap between children, an unsupported contraction or an invalid terminal witness; at the end the pending set must be empty. For a resumed journal, the next roots must be exactly the preceding pending boxes; for a forest, every substituted child root must equal the leaf it replaces, and the substitution graph must be finite and acyclic.

Fig. 2. What a covering-tree certificate is (schematic: the tree on the right is the subdivision on the left, read from the root; the boxes and letters are illustrative, not one of the actual trees). The rules are those of this section. The trees of the certificate database of [4] obey the analogous obligations: every cut splits a region into two covering regions, every leaf verifies, and the number of leaves equals the number recorded in the head of the tree.

– Computational result 15: the reference-cell certificate for $\sqrt{2}D_{10}^{+}$ used in Lemma 20;
– Computational result 16: the $u \geq 12/13$ cover and the $323\,437$-leaf strip cover (Section 16), the cover excluding $t \geq 1/3$ and the five-coordinate repair cover (Section 9), the $146\,291$-leaf and $4\,257$-leaf lower covers and the 15-box collar (Section 16); it also includes the four Bernstein certificates of Input 24 and the fourteen radius records of Input 27, the finite inputs of Propositions 25 and 28;
– Computational result 17: the final cover of Section 17, five resumed journals, $13\,985$ nodes, $6\,992$ splits, $6\,993$ terminal regions of the seven stated types, empty frontier.

Where the records are. The entry point of this case in the inventory of the certificate archive is the record `0050_closed_verified.json`, which joins the centre, height, compact-domain, collar and final-cover theorems; only the explicit inequalities and regions certified by each record are used in the assembly. The written argument and verification of this case are: centre obligations, unequal triangles, trace and repair inequalities, and full range assembly; the centre and height replays are archived, and the final upper cover is replayed. The centre-forest record covers the replay of all $24\,622\,334$ leaves. The final checker of this case shares some row-assembly code with its constructor; the derivations of Sections 4–17 are the specification against which that code is reviewed. The reference data (the printed matrix of Appendix A, its con-

tact list, the change of basis of Section 4, and the fixed coordinates and outer ranges bound to the final cover) are checked separately from the covers. The records and their checkers are in the certificate archive `hermite11_local_certificates.zip` of the Zenodo record doi:10.5281/zenodo.23979316. The accompanying `case_map.json` gives the principal record, its size, the dependency paths, checker sources and recorded result; `certificate_manifest.json` gives the file hashes. The archive identifiers and common verification procedure are stated in [3, Section 9].
The centre forest was assembled from 13 union journals and 32 component journals; the archive contains complete replays of all of them, together with the substitution graph (`paper_2026-10-25/audit/provenance_resolution.json`).

The hypotheses. Theorem 1 uses the following inputs.

1. The admissible-chain conditions of Definition 2, including the marking and the exact cutoffs.
2. The statements of Section 2.2, proved in [4, Appendix D].
3. Computational results 14–17 and Inputs 24 and 27, each a finite set of rational data whose checks are specified above.

Lean correspondence. The local analytic statements are `0050`: `Hermite11.case0050_from_finite_input`. Their finite hypotheses, integral-equivalence wrappers and saved reference records are listed in `case_map.json`. Large cover payloads remain external exact checks.

19 Use of computational tools

Refer to Section 10 of [3].

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A The form `0050:024468269`

The data set of [2] records the form by the following integral matrix A_P, in the basis used by the certificates of this paper; the form is $P = (4/s_P)A_P$ with $s_P = \min A_P = 12$, so $P = A_P/3$. This matrix is the record of the data set itself, entry by entry and in the data set's basis: the script `hermite11\_printed\_forms\_check.py` of the Zenodo record [3, Section 9] compares the printed forms of the series with the rows of their keys in the certificate database (which the verification matches to the data set) and reports `PASS` for all eleven, so the chain handed over by [4, Theorem 1], whose rank-9 member is marked by the record, is marked by P.

$$A_P = \begin{pmatrix} 12 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & -17 \\ 3 & 12 & 6 & 6 & 6 & 6 & 6 & 6 & -31 \\ 3 & 6 & 12 & 6 & 6 & 6 & 6 & 6 & -31 \\ 3 & 6 & 6 & 12 & 6 & 6 & 6 & 6 & -31 \\ 3 & 6 & 6 & 6 & 12 & 6 & 6 & 6 & -31 \\ 3 & 6 & 6 & 6 & 6 & 12 & 6 & 6 & -31 \\ 3 & 6 & 6 & 6 & 6 & 6 & 12 & 6 & -31 \\ 3 & 6 & 6 & 6 & 6 & 6 & 6 & 12 & -31 \\ -17 & -31 & -31 & -31 & -31 & -31 & -31 & -31 & 142 \end{pmatrix}.$$

The matrix is symmetric; we recomputed, in exact arithmetic (fraction-free elimination and a bounded LDL^{T} enumeration),

$$\det A_P = 17\,915\,904 = 2^{13} \cdot 3^7, \qquad \det P = \frac{4^9 \det A_P}{12^9} = \frac{8192}{9} = 910.22\ldots,$$
$$\min A_P = 12, \qquad |\operatorname{Min} A_P| = 130,$$

that is, 65 pairs of minimal vectors. The root paper's Table 4 lists the lattice M of rank 10 over this form with $\det M = 3328/3$ and $\min M^* = 32/39$, consistent with $\det N = u \det M = \frac{32}{39} \cdot \frac{3328}{3} = \frac{8192}{9} = \det P$ (the form lies in its own Voronoi domain). In the coordinates of Section 4 the form becomes $\widetilde{P} = C_{\mathrm{can}}^{\mathsf{T}} P C_{\mathrm{can}}$ with $\det C_{\mathrm{can}} = 1/2$ and $\det \widetilde{P} = 2048/9$.