γ_{11}: One Selected Contact, Two Linked Blocks

THE MACADAMIA 0030

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Abstract. This paper is one of five companion papers completing the proof of $\gamma_{11} = 4/\sqrt[11]{972}$ announced in the root paper and assembled in *A Focus on Hermite Constant 11*. The root paper reduces a lattice of rank 11 with minimum 4 and determinant below 972 to a chain of sections of ranks 11, 10 and 9 whose rank-9 member is admissibly marked by one of 4 487 813 perfect forms; ten of them, the macadamias, lie under an actual lattice of rank 10 and need proofs of their own. We settle the first, the form `0030:030301719` of Λ_9 under Λ_{10}, where the bound has little margin. The argument selects one contact of the reference lattice from the actual metric, keeps the off-diagonal column shared by two eight-dimensional blocks of the section metric, applies one determinant–trace theorem to both blocks, and closes with a rational cover of three scalar parameters. The proof is computer-assisted: its finite inputs are three outer-range covers, two middle-range covers with 10 498 and 1 140 leaves, an annulus tree with 3 567 leaves and four smaller trees, every leaf checked in exact rational arithmetic.

Keywords: Hermite constant · Lattice packings · Perfect forms · Laminated lattice Λ_9 · Schur complements · Exact certificates · Branch and bound

1 Introduction

This paper belongs to a series of seven papers, made public concurrently, whose aim is to establish the Hermite constants γ_{10}, γ_{11} and γ_{12}. The root paper [4] proves γ_{10}, reduces γ_{11} to a statement about chains of sections $L \supset M \supset N$ of ranks 11, 10 and 9 lying over a perfect form of rank 9, and derives γ_{12}; the γ_{11} paper [5] proves that statement for every perfect form but ten, the *macadamias*, by exact certificates, and assembles the proof of $\gamma_{11} = 4/\sqrt[11]{972}$ from the root paper, the certificates and five companion papers: the present one and [6,7,8,9].

We settle here the first macadamia of [4, Table 4], the form `0030:030301719`. It is the form of the laminated lattice Λ_9 [2], the matrix P_Λ of [4, Appendix B]: Λ_9 lies under Λ_{10}, a lattice of rank 10 with minimum 4, determinant 768 and dual minimum $2/3$, so no certificate at rank 10 can exclude the form, and the bound 972 must be proved at the level of the rank-11 lattice, where its margin is small. The idea of the proof is the one its title names. A section metric over Λ_9 has two eight-dimensional principal

blocks that share a seven-dimensional block; independent bounds on the two blocks are too weak, but one contact of the reference lattice can be *selected* from the actual metric, the off-diagonal column that the two blocks share is kept in several inequalities at once, the same determinant–trace theorem is applied in dimensions seven and eight, and a rational cover of three scalar parameters finishes the argument.

Theorem 1. *Assume Computational result 17. Let $L \supset M \supset N$ be a chain of lattices as in Definition 2 whose rank-9 member N has a Gram matrix admissibly marked by the form* `0030:030301719`. *Then* $\det L \geq 972$.

Theorem 1 is the statement that [4, Section 6.2] requires of this paper, and that [5] imports for the form `0030:030301719`. Its proof occupies Sections 3 to 11; Section 12 assembles the proof and states exactly which finite checks it rests on. The proof is computer-assisted: it combines analytic inequalities, proved in full, with finite covers of parameter boxes whose leaves are checked in exact rational arithmetic (Computational result 17).

Organisation. Section 2 states the admissible-chain conditions and the common tools used below. Section 3 fixes the reference lattice, the physical coordinates and the first reduction to the range $13/10 < m < 8/5$ of the inverse normal entry m. Section 4 selects a contact and sets up the block coordinates. Sections 5 and 6 prove the common trace profile, the contact inequalities and the directional and short-dual bounds used by every cover. Section 7 proves the bounds common to both signs of the selected inverse entry w; Section 8 treats $w < 0$, Section 9 the low-excess region $T + \tau \leq 1/8$ (where $w > 0$), and Section 10 the branch $w \geq 0$ with $T + \tau \geq 1/8$, which ends with a uniform final lemma. Section 11 says what the finite cover verifies. Section 12 lists what is verified and where, and the hypotheses on which Theorem 1 rests. Appendix A prints the Gram matrix of the form.

2 The local setting and common tools

2.1 Notation and admissible chains

We use the common coordinates of [5, Appendix D]. The determinant of a lattice is the determinant of its Gram matrix, and its minimum is the least nonzero squared length. Its dual is $L^* = \{y \in \operatorname{span} L : y \cdot x \in \mathbb{Z} \text{ for all } x \in L\}$. A sublattice $M \subset L$ is *primitive* if $M = L \cap \operatorname{span} M$; a primitive vector generates a primitive rank-one sublattice. Write $Q[x] = x^{\mathsf{T}} Q x$, with T denoting transpose, and $\operatorname{tr} A$ for the sum of the diagonal entries. For symmetric matrices, $A \succeq 0$ means $A[x] \geq 0$ for every x; $A \succ 0$ means $A[x] > 0$ for $x \neq 0$, and $A \preceq B$ means $B - A \succeq 0$. The nested sections form a flag of ranks 11, 10 and 9; its notation is

$$e = \det N, \qquad d = \det M = \frac{e}{u}, \qquad \Delta = \det L = \frac{e}{uv}, \qquad u = \min M^*, \qquad v = \min L^*. \tag{1}$$

The cutoffs are the positive numbers $d_{\mathrm{cut}}^{11} = 4^{11} 972^9$ and $e_{\mathrm{cut}}^{55} = 2^{246} 3^{197}$.

Definition 2 (admissible residual chain). *A chain $L \supset M \supset N$ is admissible for P if L has rank eleven and minimum at least four, $M = L \cap n_{11}^{\perp}$ and $N = M \cap n_{10}^{\perp}$ for shortest nonzero vectors $n_{11} \in L^{*}$ and $n_{10} \in M^{*}$, its rank-nine Gram matrix is marked by P as defined below, and*

$$\begin{aligned} 768 \le d < d_{\mathrm{cut}} &= 4 \cdot \sqrt[11]{972^{9}}, \\ e < e_{\mathrm{cut}} &= \sqrt[10]{\frac{4096}{3}\, d_{\mathrm{cut}}^{9}} < 1137. \end{aligned} \tag{2}$$

Shortest dual vectors are primitive, so the determinant identities above follow by an adapted basis. Rounding a lift of n_{10} across the last cut gives $v \le u + v/4$, hence $v \le 4u/3$ and $\Delta \ge 3d/(4u)$. The global reduction [4, Proposition 15] supplies such a chain from a hypothetical counterexample.

Definition 3 (marking). *Let Q be the Gram matrix of N in an integral basis, and $X = Q^{-1}$. A perfect form P of minimum four marks Q if $\min Q \ge 4$ and there are weights satisfying*

$$\begin{gathered} X = \sum_{c \in \mathcal{C}(P)} w_c\, cc^{\mathsf{T}}, \qquad w_c \ge 0, \qquad \sum_{c} w_c \le \frac{9}{4}, \qquad w_c \le \frac{1}{4}, \\ \operatorname{tr}(PX) \le 9, \qquad \det P \le e. \end{gathered} \tag{3}$$

Here $\mathcal{C}(P)$ contains one vector from each opposite pair of minimum vectors of P.

Perfect means that the matrices cc^{T} span the symmetric matrices. A marking is transported by an integral change of basis. Its existence for arbitrary Q is proved in [4, Proposition 7]; here the marker is fixed to the printed form.

Gluing coordinates. Vectors $\beta, \alpha \in \mathbb{R}^{9}$ and a scalar $\vartheta \in \mathbb{R}$ give the dual squared lengths at $(z, k, k') \in \mathbb{Z}^{9} \times \mathbb{Z}^{2}$ as

$$X[z] + u(\beta^{\mathsf{T}} z + k)^{2} + v(\alpha^{\mathsf{T}} z + \vartheta k + k')^{2},$$

and the primal squared lengths as

$$Q[x - k\beta - k'(\alpha - \vartheta\beta)] + \frac{(k - \vartheta k')^{2}}{u} + \frac{k'^{2}}{v}.$$

Write $\|t\|_{\mathbb{R}/\mathbb{Z}} = \min_{j \in \mathbb{Z}} |t - j|$. The first shortest-normal condition implies

$$X[z] + u\, \|\beta^{\mathsf{T}} z\|_{\mathbb{R}/\mathbb{Z}}^{2} \ge u \qquad (0 \ne z \in \mathbb{Z}^{9}).$$

The second condition applies to every nonzero vector of M^{*} and every lift to L^{*}.

Physical coordinates. Choose C_P with $P = C_P^{\mathsf{T}} C_P$ and write

$$Q = C_P^{\mathsf{T}} S C_P, \qquad R = S^{-1}. \tag{4}$$

The columns of C_P realise the reference lattice in Euclidean space. Thus S measures squared lengths in these fixed coordinates, with reference metric I, and R is its inverse: $\det S = e/\det P$, $\operatorname{tr} R = \operatorname{tr}(PX) \le 9$; integral vectors c, z become $C_P c, C_P^{-\mathsf{T}} z$. We write $|x| = \|x\|$ for Euclidean length and $\|E\|_F^2 = \operatorname{tr}(E^{\mathsf{T}} E)$ for the squared Frobenius norm, and $\rho_G^2 = \sup_x \min_{a \in \Lambda} G[x-a]$ for the squared covering radius of the reference lattice Λ under discussion, with $x \in \operatorname{span} \Lambda$. Eigenvalues are listed in decreasing order. The notation $I_n, e_i, \mathbf{1}$ denotes the identity, coordinate vectors and all-one vector; Sym_n denotes the symmetric $n \times n$ matrices. We use $(x)_+ = \max(x, 0)$; subscripts $-$ and $+$ on a box coordinate denote its lower and upper endpoints. Schur complements, cofactors and adjugates have the conventions and block identities of [5, Appendix D.2]. Exact LDL^{T} factorisation and bounded short-vector enumeration are described in [5, Section 5.1].

2.2 Statements of [5] used by the argument

The following lemmata are stated and proved in [5, Appendix D]; each statement cites its proof there.

Lemma 4 (one section vector controls both extensions; [5, Lemma 23]). *For every $0 \neq z \in \mathbb{Z}^9$,*

$$\Delta \ge \frac{e}{2X[z]^2}. \tag{5}$$

If three nonzero section covectors sum to zero and all have squared X-length at most x, then $\Delta \ge 16e/(27x^2)$.

The first inequality is the *scalar flag*: applied in physical coordinates to a covector of squared length ℓ, it reads $\Delta \ge e/(2\ell^2)$.

Lemma 5 (inverse-trace eigenvalue bound; [5, Lemma 27]). *Let $B \succ 0$ have size $n \ge 2$, $\operatorname{tr} B^{-1} \le n$, and $\det B < \rho$. Define*

$$F_n(t) = t\Big(\frac{n-1}{n-\frac{1}{t}}\Big)^{n-1} \qquad \Big(t > \frac{1}{n}\Big).$$

Every eigenvalue λ of B satisfies $F_n(\lambda) < \rho$. In particular, if $H > 1$ and $F_n(H) \ge \rho$, then $B \prec HI$.

Lemma 6 (a shared D_n squared-norm bound; [5, Lemma 28]). *Let E be a symmetric $n \times n$ matrix, $n \ge 4$, with $E_{ii} + E_{jj} \ge 2|E_{ij}|$ for $i \neq j$. Then $\|E\|_F^2 \le (n+1)(\operatorname{tr} E)^2/2$.*

Lemma 7 (shared D_n quadratic bounds; [5, Lemma 29]). *Let $3 \le n \le 9$ and let E be symmetric, with $E_{ii} + E_{jj} \ge 2|E_{ij}|$ for $i \neq j$. Then $-(\operatorname{tr} E)I \preceq E \preceq 2(\operatorname{tr} E)I$.*

The proof of Lemma 7 in [5] writes $d_i = E_{ii}$, and, for $x \in \mathbb{R}^n$, sets $L_x = \sum_i |x_i|$, $q_x = \sum_i x_i^2$, $f_i = 2x_i^2 - |x_i|L_x + q_x$; it checks $f_i \geq 0$ and $2f_i \leq \sum_j f_j$, and deduces $\sum_i d_i f_i \geq 0$. This *pair-diagonal dual criterion* ("if $f_i \geq 0$ and $2f_i \leq \sum_j f_j$ then $\sum_i d_i f_i \geq 0$ whenever $d_i + d_j \geq 0$ for $i \neq j$") is used repeatedly in Sections 5, 6 and 7.

Lemma 8 (equal-diagonal determinant comparison; [5, Lemma 31]). *Let B be a symmetric $r \times r$ matrix, $r \geq 1$, with $B_{ii} = d$ and $|B_{ij}| \leq c$ for $i \neq j$, where $c \geq 0$ and $d > (r-1)c$. Then $B \succ 0$ and $\det B \geq (d+c)^{r-1}(d-(r-1)c)$.*

Lemma 9 (a shared D_n determinant bound; [5, Lemma 32]). *Let E be symmetric of size $n \geq 7$, $E_{ii} + E_{jj} \geq 2|E_{ij}|$ for $i \neq j$, and $0 \leq s = \operatorname{tr} E \leq 1$. If $1 + s - (n-1)s^2/4 > 0$, then*

$$I + E \succ 0, \qquad \det(I+E) \geq 1 + s - \frac{n-1}{4}s^2.$$

In particular this applies for $n = 7, 8$ with $s \leq 1$, and for $n = 10$ with $s \leq 3/10$.

The proof of Lemma 9 decomposes the diagonal of E into a convex combination of the *arrow vertices* se_i and the *negative-coordinate vertices* $(s/(n-2))(\mathbf{1} - 2e_i)$, keeping each ratio $2E_{ij}/(E_{ii} + E_{jj})$ fixed; Sections 5 and 9 use this decomposition and its two kinds of vertices.

Lemma 10 (joint determinant and inverse averages; [5, Lemma 33]). *Fix vectors z_j, weights $w_j \geq 0$ and $b \geq 0$, with either $b > 0$ or some $w_j > 0$, $z_j \neq 0$. For $A \succ 0$ let $F(A) = b + \sum_j w_j z_j^{\mathsf{T}} A^{-1} z_j$. If $H = \sum_a \mu_a A_a$, where $A_a \succ 0$, $\mu_a \geq 0$ and $\sum_a \mu_a = 1$, then $\log F(H) \leq \sum_a \mu_a \log F(A_a)$. Consequently, for $q \geq 0$,*

$$\log\det H - q\log F(H) \geq \sum_a \mu_a\big(\log\det A_a - q\log F(A_a)\big).$$

In particular, a common lower bound on $\det A_a / F(A_a)^q$ holds also for $\det H / F(H)^q$.

Lemma 11 (determinant from trace and inverse trace; [5, Lemma 34]). *Let $n \geq 3$, $B \succ 0$, $\operatorname{tr} B = \sigma$, and $\operatorname{tr} B^{-1} = U$. Then $\sigma U \geq n^2$. If equality holds, $B = (\sigma/n)I$. Otherwise define*

$$t = \frac{U\sigma - n(n-2) + \sqrt{(U\sigma - n^2)(U\sigma - (n-2)^2)}}{2U}, \qquad \ell = \frac{\sigma - t}{n-1}. \tag{6}$$

Then $\det B \geq t\ell^{n-1}$. This lower bound increases with σ at fixed U, and decreases with U at fixed σ.

Lemma 12 (a shared trace–inverse logarithm bound; [5, Lemma 38]). *Let $0 < \ell \leq r$, $H > 0$, and let $a_{\log} = 1/(\ell + r)$, $b_{\log} = \ell r/(\ell + r)$. If*

$$q_{\log} \leq \min\left\{\log\ell - a_{\log}\ell + \frac{b_{\log}}{\ell},\ \log H - a_{\log}H + \frac{b_{\log}}{H}\right\},$$

then $\log x \geq a_{\log} x - b_{\log}/x + q_{\log}$ *for* $0 < x \leq H$. *In particular,*

$$x - 1 \leq 3\log x + 2\left(\frac{1}{x} - 1\right) \qquad \left(0 < x \leq \frac{14}{5}\right), \tag{7}$$

$$\log(1+x) \geq \frac{17}{40}x + \frac{23}{40}\frac{x}{1+x} \qquad \left(-1 < x \leq \frac{9}{16}\right). \tag{8}$$

Lemma 13 (rational linear certificates; [5, Lemma 44]). *Suppose a vector* z *satisfies* $Az \leq b$ *and coordinate bounds* $\ell \leq z \leq h$, *all rational. For any rational* $\lambda \geq 0$, *let* $a = A^{\mathsf{T}}\lambda$. *If* $\sum_j \min(a_j l_j, a_j h_j) > \lambda^{\mathsf{T}} b$, *there is no such* z. *More generally, for a linear objective* $q^{\mathsf{T}} z$ *and* $r = q - A^{\mathsf{T}}\lambda$,

$$q^{\mathsf{T}} z \leq \lambda^{\mathsf{T}} b + \sum_j \max(r_j l_j, r_j h_j).$$

For a bilinear term xy with $x \in [\ell_x, h_x]$, $y \in [\ell_y, h_y]$, the four products $(x-\ell_x)(y-\ell_y)$, $(h_x - x)(h_y - y)$, $(x - \ell_x)(h_y - y)$ and $(h_x - x)(y - \ell_y)$ are nonnegative; expanding them gives the four usual linear bounds on a variable representing xy [5, Appendix D]. For $r > 0$ rational, logarithms are enclosed by

$$\log r = 2\sum_{j=0}^{m-1} \frac{z^{2j+1}}{2j+1} + \mathcal{R}_m, \qquad z = \frac{r-1}{r+1}, \qquad |\mathcal{R}_m| \leq \frac{2|z|^{2m+1}}{(2m+1)(1-z^2)}. \tag{9}$$

A polynomial on $[0,1]$ is nonnegative if all its Bernstein coefficients are nonnegative.

Lemma 14 (Lemma 12 of [4]). *Let* Y *be a positive definite* 9×9 *matrix with* $\operatorname{tr}(Y^{-1}) \leq 9$ *and* $(e_i + \sigma e_9)^{\mathsf{T}} Y (e_i + \sigma e_9) \geq 2$ *for* $i = 1, \ldots, 8$ *and* $\sigma = \pm 1$. *Then* $\det Y \geq (Y^{-1})_{99}$.

Lemma 14 is proved in [4, Section 5.5], where it is applied to $Y = \frac{1}{2} C^{-\mathsf{T}} Q C^{-1}$ with $P_\Lambda = 2C^{\mathsf{T}}C$, that is, to the physical section metric S of (4) for the form P_Λ. This is the only place where the argument of this paper meets the rank-10 proof of [4]: the lower endpoint of the first reduction (Section 3) uses exactly this inequality, and no further rank-10 hypothesis.

3 The reference lattice and the first reduction

This case illustrates why a useful reduction must keep the same off-diagonal column in several inequalities. Independent bounds on the two principal blocks are too weak. The proof retains their common column, applies the same determinant–trace theorem in dimensions seven and eight, and finishes by a rational cover of three scalar parameters.

The reference lattice. Let

$$E_8 = \{x \in \mathbb{Z}^8 : \textstyle\sum x_i \in 2\mathbb{Z}\} \;\cup\; \left\{x \in \left(\mathbb{Z} + \tfrac{1}{2}\right)^8 : \textstyle\sum x_i \in 2\mathbb{Z}\right\},$$

$$L_{\mathrm{ref}} = (E_8, 0) + \mathbb{Z}(e_1, 1).$$

Its determinant is one and its minimum is two. Its 272 minimum vectors are the 240 horizontal E_8 roots and the 32 vectors $(\pm e_i, \pm 1)$. The fixed coordinate certificate supplies nine vectors of $\sqrt{2}L_{\mathrm{ref}}$ whose Gram matrix is the printed form (Appendix A). Membership is checked from the displayed parity conditions, and the Gram identity is checked entry by entry. Both Gram determinants are 512, so the generated sublattice has index one; these vectors therefore form a basis. In these physical coordinates

$$e = 512 \det S, \qquad R = S^{-1}, \qquad \operatorname{tr} R \le 9, \qquad S[x] \ge 2 \quad (0 \ne x \in L_{\mathrm{ref}}).$$

Write

$$S = \begin{pmatrix} A & -Az \\ -z^{\mathsf{T}}A & \frac{1}{m} + z^{\mathsf{T}}Az \end{pmatrix}, \qquad m = R_{99}.$$

Then

$$\det S = \frac{\det A}{m}, \qquad \operatorname{tr} A^{-1} + m(1 + |z|^2) \le 9, \qquad \Delta \ge \frac{1024 \det A}{m^3}. \tag{10}$$

The last inequality is the scalar flag (5) applied to the integral vertical covector, whose squared length is $m/2$. Thus

$$\det A < C_A(m) := \min\left(\frac{1137m}{512}, \frac{243m^3}{256}\right).$$

The inverse contact expansion gives $|z|_1 \le 1$ and $|z|_\infty \le 1/(2m)$: if the coefficients of $(e_i, \pm 1)$ are $\kappa_{i,\pm}$, then $m = \sum \kappa_{i,\pm}$, $mz_i = \kappa_{i,+} - \kappa_{i,-}$, and $0 \le \kappa_{i,\pm} \le 1/2$. Combine repeated or opposite contacts before applying this coefficient bound: the combined term still belongs to the same nonnegative rank-one expansion. The cross contacts say $A[\pm e_i - z] \ge 2 - 1/m$.

The horizontal roots give a useful stronger statement. Every horizontal root v satisfies $|v_i v_j| \le v_i^2$. Summing its nonnegative rank-one contributions, for distinct horizontal coordinates i, j, gives

$$|R_{ij}| \le R_{ii} - \kappa_{i,+} - \kappa_{i,-}, \qquad |R_{ij}| + |R_{i9}| \le R_{ii}. \tag{11}$$

Here the first inequality uses the fact that cross contacts contribute only to horizontal diagonal entries; the second uses $|\kappa_{i,+} - \kappa_{i,-}| \le \kappa_{i,+} + \kappa_{i,-}$. This gives a joint bound on the inverse entries.

The lower endpoint. The lower endpoint uses exactly the matrix inequality already proved in Lemma 14. Applied to this S, it gives $\det S \ge R_{99} = m$, or $\det A \ge m^2$. No further rank-ten hypothesis or duplicate computation is needed.

The two analytic ends of the range of m. For $m \le 256/243$, (10) gives $\Delta \ge 1024/m \ge 972$. The upper range needs only the first cap on $\det A$. Positivity and the inverse-trace budget give $m < 9$ and $\operatorname{tr} A^{-1} \le 9 - m$, so inverse AM–GM yields $8^8 \le (\det A)(9-m)^8$. If $m \ge 13/5$, the function $f(m) = m(9-m)^8$ decreases: its derivative is $9(9-m)^7(1-m) < 0$. Consequently

$$8^8 \le (\det A)(9-m)^8 < \frac{1137}{512} m(9-m)^8 \le \frac{1137}{512}\,\frac{13}{5}\left(\frac{32}{5}\right)^8 < 8^8,$$

a contradiction, with the last step a rational comparison. Thus every counterexample has $256/243 < m < 13/5$. The intervening outer ranges are covered by three exact signed-average certificates:

$$\left[\frac{256}{243}, \frac{13}{10}\right], \qquad \left[\frac{8}{5}, \frac{17}{10}\right], \qquad \left[\frac{17}{10}, \frac{13}{5}\right],$$

with respectively 379, 153 and 23 outer nodes (the middle certificate also has 51 inner leaves). The finite trees and scalar witnesses are part of Computational result 17; their analytic implications are as follows.

Signed averages on an outer box. Write $v = |z|^2$, and suppose a box supplies $m \geq m_0 > 0$ and $0 \leq v_0 \leq v \leq v_1$. Then

$$\operatorname{tr} A^{-1} \leq U := 9 - m_0(1 + v_0), \qquad \det A < C := C_A(m_1),$$

where m_1 is the upper endpoint for m. A nonempty box has $U > 0$, since A^{-1} has positive trace. Inverse AM–GM excludes the box if $C(U/8)^8 \leq 1$. Otherwise positive rational spectral bounds $a \leq \lambda \leq b$ are justified by endpoint tests on

$$\det A \geq \lambda\Big(\frac{7}{U - \frac{1}{\lambda}}\Big)^7.$$

Indeed the other seven reciprocal eigenvalues have sum at most $U - 1/\lambda > 0$. After putting $x = U\lambda/8$, the right side, multiplied by $(U/8)^8$, is $7^7x^8/(8x-1)^7$; it decreases for $1/8 < x \leq 1$ and increases for $x \geq 1$. These monotonicities justify the two outward endpoint tests.

Let $\eta = 2 - 1/m_0$ and $W = A[z]$. For $0 \leq \omega \leq 1$ define

$$E_\omega = \sum_i \left(\omega|z_i| - \frac{1}{8}\right)_+.$$

The cross contacts imply the signed-average bound

$$\operatorname{tr} A \geq 8\eta - 8E_\omega(b - a) + 8(2\omega - 1)W. \tag{12}$$

Proof of (12). Set $\ell_i = \omega|z_i|$ and $N = \sum_i (1/8 - \ell_i)_+$. Since $\sum_i l_i \leq 1$, we have $0 \leq E_\omega \leq N$. If $N > 0$, the weights

$$p_i = \frac{1}{8} + \left(\ell_i - \frac{1}{8}\right)_+ - \left(\frac{E_\omega}{N}\right)\left(\frac{1}{8} - \ell_i\right)_+$$

satisfy $p_i \geq \ell_i$, $\sum_i p_i = 1$, and $\sum_i p_i A_{ii} \leq \operatorname{tr} A/8 + E_\omega(b - a)$. If $N = 0$, then $E_\omega = 0$ and $p_i = 1/8$ has the same properties. Average the inequalities $A[\pm e_i - z] \geq \eta$ with the nonnegative weights $(p_i \pm \omega z_i)/2$. Their sum is one, and their average is $\sum_i p_i A_{ii} + (1 - 2\omega)W$. This proves (12). □

Only nine scalar checks are needed to bound E_ω. If k is the number of positive summands in its definition, then

$$E_\omega \leq \min\Big\{\frac{\omega k}{2m_0} - \frac{k}{8},\ \omega\sqrt{kv_1} - \frac{k}{8},\ \omega - \frac{k}{8}\Big\}, \qquad 0 \leq k \leq 8. \tag{13}$$

The three bounds use the coordinate cap, Cauchy–Schwarz, and $|z|_1 \le 1$, respectively. Upper square-root enclosures make all nine checks rational. Also $av_0 \le W \le bv_1$. A stronger version replaces a in (12) by $\max(a, \eta - W)$: the individual contacts give $A_{ii} \ge \eta - W + 2|(Az)_i|$. Consequently $W \ge \eta - b$. The resulting expression is affine on either side of $W = \eta - a$; its minimum on $[\max(av_0, \eta - b), bv_1]$ is checked at the two endpoints and at that one breakpoint when it lies in the interval.

The additional transition trace bound. In the middle outer range the certificate also subdivides an interval $[L_0, L_1]$ containing the actual largest eigenvalue of A. Set $d = \eta - L_1 v_1$, so every $A_{ii} \ge d$, and let $\xi = \operatorname{tr} A - 8d \ge 0$, $a_0 = d - 1$, and $t = \operatorname{tr} A - 8 = 8a_0 + \xi \ge 0$. The root contacts imply

$$\lambda_{\max}(A) \le 1 + \frac{t + \sqrt{t^2 + 7\xi^2}}{2}. \tag{14}$$

Proof of (14). Let $u = \max_i |x_i|$, let v_x be the sum of the other seven absolute coordinates, and set $P = (t + \sqrt{t^2 + 7\xi^2})/2$. The pair-contact inequality gives

$$A[x] \le |x|^2 + a_0(u + v_x)^2 + \xi(u + v_x)u.$$

The quadratic form $P(u^2 + v_x^2/7) - a_0(u + v_x)^2 - \xi(u + v_x)u$ is positive semidefinite: its determinant is zero and its diagonal coefficients are nonnegative, since

$$t^2 + 7\xi^2 - (6a_0 - \xi)^2 = 7(2a_0 + \xi)^2 \ge 0.$$

Finally $v_x^2 \le 7(|x|^2 - u^2)$ and $P \ge 0$, proving (14). This proof permits $d < 1$. □

If $y = L_0 - 1 > 0$, (14) implies $7\xi^2 + 4(8a_0 + \xi)y - 4y^2 \ge 0$. For any rational $r \ge 0$ with $r^2 \le 8y^2 - 56a_0 y$, completing the square therefore gives

$$\operatorname{tr} A \ge 8d + \max\left\{0,\ \frac{2(r - y)}{7}\right\}.$$

If no such enclosure is used, the root bound $\operatorname{tr} A \ge 8$ remains available. A second contact bound is

$$\operatorname{tr} A \ge 8\eta - L_1 \max\{0,\ 8v_1 - 2\sqrt{v_1}\}.$$

Indeed summing the individual contacts gives $\operatorname{tr} A \ge 8\eta - 8W + 2|Az|_1$; $W \le \sqrt{v}\,|Az|_1$ and $W \le L_1 v$ prove the assertion, including $v = 0$, because $(8v - 2\sqrt{v})_+$ is nondecreasing. A lower square-root enclosure preserves this lower trace bound.

The terminal determinant comparison. Every logarithmic witness has the form $\log x \ge q - r/x + sx$, with $r, s \ge 0$, on a positive closed spectral interval. It is enough to test the two endpoints and every interior root of $-sx^2 + x - r$: these are precisely the possible interior minima of $\log x - q + r/x - sx$. There are at most two such roots, and the polynomial is never identically zero. Summing over all eight eigenvalues gives $\log\det A \ge 8q - rU + sT_0$ whenever $\operatorname{tr} A \ge T_0$. For a retained largest eigenvalue in $[L_0, L_1]$, sum instead over the other seven to obtain

$$\log\det A \ge \log L_0 + 7q - r\left(U - \frac{1}{L_1}\right) + s(T_0 - L_1).$$

The strict upper bound $\log\det A < \log C$ contradicts a checked lower bound at least $\log C$. Empty inverse-tail intervals $U - 1/L_1 \le 0$ and impossible trace bounds $T_0 > 8L_1$ are also terminal contradictions. All logarithms and critical-point tests are enclosed outward. Closed box subdivision includes every boundary. Thus only

$$\frac{13}{10} < m < \frac{8}{5} \tag{15}$$

remains.

4 Selecting a contact

Select a horizontal coordinate i maximizing $S_{ii} + 2|S_{i9}|$, and, if necessary, reflect the original normal coordinate so that $S_{i9} \le 0$. This preserves L_{ref}, since $2e_1 \in E_8$. Let $h = S_{99}$, $s = -S_{i9} \ge 0$. Use the ordered basis consisting of the other seven horizontal coordinate vectors, then e_9, then $(e_i + e_9)/\sqrt{2}$. Thus the eighth coordinate of the principal block below is the original normal coordinate. The resulting matrix is

$$\mathcal{B} = \begin{pmatrix} H & Hp \\ p^{\mathsf{T}}H & 1+\tau \end{pmatrix}, \qquad T = \operatorname{tr} H - 8, \quad c = 1 + \tau - p^{\mathsf{T}}Hp, \quad r_p = |p|^2.$$

All retained contacts are actual contacts of the metric. In particular $T, \tau \ge 0$. No off-diagonal entry has been averaged away. Set

$$\begin{gathered} d_{\mathrm{sel}} = R_{ii} = \frac{1}{2c}, \qquad w = R_{i9}, \qquad \kappa_p = 2 - 4cw = 2\sqrt{2}\,p_8 = \max_j 2\sqrt{2}\,|p_j|, \\ \chi = (H^{-1})_{88} = m - 2cw^2, \qquad \rho = \sum_{j \le 8} R_{j9}^2, \qquad \delta = 9 - \operatorname{tr} R. \end{gathered} \tag{16}$$

Block inversion (Schur complements, [10, Section 0.8.5]) proves

$$\begin{aligned} e &= 1024\, c \det H, & \operatorname{tr} H^{-1} &= 9 - \delta - \frac{1 + r_p - \frac{\kappa_p}{2}}{c}, \\ \operatorname{tr} S &= 10 + T + 2\tau - h + 2s, & \operatorname{tr} A &= 10 + T + 2\tau - 2h + 2s, \\ \operatorname{tr} A^{-1} &= 9 - m - \delta - \frac{\rho}{m}. \end{aligned} \tag{17}$$

For an unselected horizontal coordinate j, block inversion gives $p_j = -\sqrt{2}\,cR_{ji}$. Since $R_{ii} = 1/(2c)$, (11) gives $2c|R_{ji}| \le 1 - 2c|w| \le 1 - 2cw$. Together with $p_8 = (1 - 2cw)/\sqrt{2}$, this proves $p_8 = \max_j |p_j| \ge 0$, including the asserted maximum in (16). The sign $w < 0$ is exactly the branch $\kappa_p > 2$. The root bounds are

$$\frac{5}{26} < c < \frac{33}{14}, \qquad |w| \le \frac{1}{2}, \qquad \frac{\kappa_p^2}{8} \le r_p \le \kappa_p^2, \qquad r_p < 3.$$

The bounds on c follow from $5I/13 \prec S \prec 33I/7$; the coefficient bound gives $|w| \le 1/2$, and the maximum-coordinate identity gives $\kappa_p^2/8 \le r_p \le \kappa_p^2$. Here the spectral

bounds follow from Lemma 5 and $\det S < 1137/512$. To justify the radius bound explicitly, let $d = R_{ii} = 1/(2c)$. The contact expansion gives $|w| \le d$, so $0 \le \kappa_p = 2 - 2w/d \le 4$. The inverse spectrum lies in $[7/33, 13/5]$, hence

$$R^2 \preceq \frac{464}{165}R - \frac{91}{165}I.$$

Block inversion, with the same selected coordinate, gives

$$1 + r_p - \frac{\kappa_p}{2} = 2c^2(R^2)_{ii} \le \frac{464c - 182c^2}{165} \le \frac{26912}{15015}.$$

The last inequality is obtained by completing the square at $c = 116/91$. Therefore $r_p \le 41927/15015 < 3$.

Two further necessary bounds. The same matrix identities supply two further necessary bounds. Let $r_{\mathrm{col}} = (R_{j9})_{j\le 8}$ and $b_{\mathrm{col}} = (S_{j9})_{j\le 8}$. The normal column of $SR = I$ gives $Ar_{\mathrm{col}} = -mb_{\mathrm{col}}$ and $b_{\mathrm{col}}^{\mathsf{T}} r_{\mathrm{col}} = 1 - mh$. Hence $r_{\mathrm{col}}^{\mathsf{T}} A r_{\mathrm{col}} = m^2 h - m$. Restricting the quadratic bounds for S to its principal block A yields

$$\frac{5}{13}\rho \le m^2 h - m \le \frac{33}{7}\rho. \tag{18}$$

Also $\chi = (H^{-1})_{88} > 0$. Completing the square gives

$$0 \le \Big(p - \frac{p_8}{\chi}H^{-1}e_8\Big)^{\mathsf{T}} H \Big(p - \frac{p_8}{\chi}H^{-1}e_8\Big) = p^{\mathsf{T}}Hp - \frac{p_8^2}{\chi}.$$

Using $\kappa_p^2 = 8p_8^2$ and $p^{\mathsf{T}}Hp = 1 + \tau - c$ proves

$$\frac{\kappa_p^2}{8\chi} \le 1 + \tau - c. \tag{19}$$

The containing triangle for the selected contact. For $0 < x \le 33/7$,

$$\log x \ge \frac{13}{50}(x-1) + \frac{37}{50}\Big(1 - \frac{1}{x}\Big).$$

Indeed the derivative of the difference is $-13(x-1)(x-37/13)/(50x^2)$, so its minimum is attained at 1 or $33/7$. The value at 1 is zero, and the other value is positive by the exact rational comparison

$$\log\Big(\frac{33}{7}\Big) \ge 2\sum_{j=0}^{5} \frac{\left(\frac{13}{20}\right)^{2j+1}}{2j+1} > \frac{13}{50}\Big(\frac{33}{7} - 1\Big) + \frac{37}{50}\Big(1 - \frac{7}{33}\Big).$$

The first inequality is the positive series $\log((1+t)/(1-t)) = 2\sum_{j\ge 0} t^{2j+1}/(2j+1)$ at $t = 13/20$. For the upper logarithm bound, $(11049/10000)^8 > 1137/512$ and

$$\log\Big(\frac{1137}{512}\Big) < 8\log\Big(\frac{11049}{10000}\Big) \le 4\Big(\frac{11049}{10000} - \frac{10000}{11049}\Big) < \frac{4}{5}.$$

Here $\log x \le (x-1/x)/2$ for $x \ge 1$ follows by differentiating the difference, whose derivative is $(x-1)^2/(2x^2) \ge 0$. Summing the lower logarithm bound over the spectrum therefore gives

$$\operatorname{tr} S + \frac{37}{13}\delta < \frac{157}{13}. \tag{20}$$

The scalar inequality $x+(165/91)/x \le 464/91$ on $[5/13, 33/7]$, applied in the spectral decomposition and evaluated on the normal coordinate, gives $h \le (464-165m)/91$. Writing the normal entry of $SR = I$ and using $|R_{j9}| \le 1/2$ yields

$$mh - 1 = -\sum_{j\le 8} S_{j9}R_{j9} \le \frac{1}{2}\sum_{j\le 8} |S_{j9}|.$$

The choice of i maximizes $S_{ii} + 2|S_{i9}|$ and makes $S_{i9} \le 0$. This choice always exists: maximize over the eight coordinates, then, if necessary, reverse the normal coordinate. Such a sign change is the congruence $S \mapsto DSD$ with $D^2 = I$; it preserves positivity, determinant, both traces, all pair-contact inequalities and all absolute inverse-column bounds. From the exact contact value and principal trace,

$$T + \tau = \operatorname{tr} S + \frac{h}{2} - 9 - \frac{1}{2}\max_{j\le 8}(S_{jj} + 2|S_{j9}|).$$

The maximum is at least the average. Combining the last two displays,

$$T + \tau \le \frac{15}{16}\operatorname{tr} S + \frac{9-4m}{16}h - \frac{35}{4}.$$

On $13/10 \le m \le 8/5$, the coefficient $9-4m$ is positive and $(9-4m)(464-165m)$ decreases: its difference from the value at $13/10$ is

$$\left(m - \frac{13}{10}\right)\left(660\left(m + \frac{13}{10}\right) - 3341\right) \le 0.$$

Using (20) and $\delta \ge 0$ now gives the uniform, strict bound

$$0 \le T + \tau < \frac{46931}{14560} < \frac{13}{4}. \tag{21}$$

This derivation uses the whole matrix and requires no covering data. The sharper bound used in the subsequent cover is

$$\operatorname{tr} S + \frac{14}{5}\delta < U(m) := \min\left(12,\ \frac{1351 + 2000m}{364}\right).$$

Its full derivation is given in the next section. The complete low-excess argument in Section 9 proves $\Delta \ge 972$ whenever $T + \tau \le 1/8$. Thus the remaining covers may retain $T + \tau \ge 1/8$. That argument uses the fixed reference checks and finite covers specified in Computational result 17; its matrix and phase implications are proved there in full.

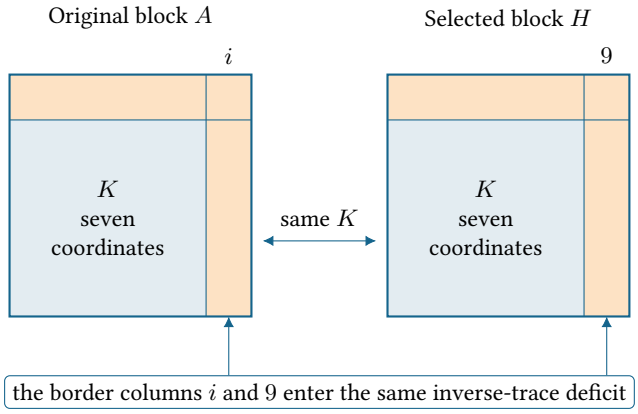

Fig. 1. The two eight-dimensional blocks share the same seven-dimensional block. The proof keeps this relation when bounding both determinants.

5 The common trace profile and contact inequalities

The trace profile. For $n \geq 3$, let $\mathcal{S}_{n,C}(V)$ be the maximum possible trace when $\det B \leq C$ and $\operatorname{tr} B^{-1} \leq V$. By the determinant theorem of Section 2.2 (Lemma 11), the extremum has eigenvalues $t, \ell, \dots, \ell$ with $t\ell^{n-1} = C$ and $1/t + (n-1)/\ell = V$. Let $r_C = C^{1/n}$ and $x = t/\ell \geq 1$. Then

$$V = r_C^{-1}\big((n-1)x^{1/n} + x^{-(n-1)/n}\big), \qquad \mathcal{S}_{n,C}(V) = r_C\big(x^{(n-1)/n} + (n-1)x^{-1/n}\big).$$

Differentiating gives $\mathcal{S}'_{n,C}(V) = r_C^2 x^{(n-2)/n}$, so the profile is convex. Its secant from n/r_C to V_0 has slope

$$k_{n,C,V_0} = \frac{\mathcal{S}_{n,C}(V_0) - nr_C}{V_0 - \frac{n}{r_C}}.$$

For $V \leq V_0$,

$$\operatorname{tr} B + k_{n,C,V_0}(V_0 - \operatorname{tr} B^{-1}) \leq \mathcal{S}_{n,C}(V_0). \tag{22}$$

For a parameter box, apply this inequality directly with its fixed upper determinant cap C and inverse-trace budget V_0. This retains the *same* actual inverse trace in every row. Rounding the slope downward and the trace endpoint upward is safe, since $V_0 - \operatorname{tr} B^{-1} \geq 0$. More generally, a proved row $\operatorname{tr} B + k(V_0 - \operatorname{tr} B^{-1}) \leq S_0$ implies $\operatorname{tr} B + k_1(V_1 - \operatorname{tr} B^{-1}) \leq S_1$ whenever $k_1 \leq k$ and $S_0 + k_1(V_1 - V_0) \leq S_1$. The smaller root ℓ is bracketed by rational signs of $f(x) = x^n - V_0 C x + (n-1)C$; no numerical root is accepted as a proof.

A complete finite criterion for a trace row. Choose positive rationals $r_- \leq r_+$ and $\ell_- \leq \ell_+ < r_-$ satisfying

$$r_-^n \leq C \leq r_+^n, \qquad f(\ell_-) \geq 0 \geq f(\ell_+).$$

The intermediate value theorem gives $r \in [r_-, r_+]$ and $\ell \in [\ell_-, \ell_+]$ with $r^n = C$ and $f(\ell) = 0$. In particular $0 < \ell < r$; setting $t = C/\ell^{n-1}$ gives $t\ell^{n-1} = C$ and $1/t + (n-1)/\ell = V_0$. For a proposed slope $k \geq 0$ and upper bound U_*, it suffices to check the two rational inequalities

$$nr_+ + k\left(V_0 - \frac{n}{r_+}\right) < U_*, \qquad \frac{C}{\ell_-^{n-1}} + (n-1)\ell_+ < U_*.$$

Indeed, $nr + k(V_0 - n/r)$ increases with positive r, so the first inequality bounds the scalar endpoint. Positivity and the bracket for ℓ give $t + (n-1)\ell \leq C/\ell_-^{n-1} + (n-1)\ell_+$, bounding the other endpoint. Subtracting the affine function kV preserves convexity of the trace profile, hence bounds the whole intervening graph by its two endpoint values. Thus every $B \succ 0$ with $\det B < C$ and $\operatorname{tr} B^{-1} \leq V_0$ satisfies

$$\operatorname{tr} B + k(V_0 - \operatorname{tr} B^{-1}) < U_*.$$

Inverse AM–GM puts its inverse trace strictly above n/r, so both endpoints used here bracket the actual inverse trace. This criterion applies in every dimension $n \geq 3$, and in particular to each of the seven-, eight- and nine-dimensional blocks.

The two parts of the sharper trace ceiling $U(m)$. For the constant part, take the rational comparison spectrum

$$r_* = \frac{683}{625}, \qquad \ell_* = \frac{1000}{1099}, \qquad t_* = \frac{r_*^9}{\ell_*^8}, \qquad k = \frac{14}{5}.$$

Direct rational arithmetic gives

$$r_*^9 > \frac{1137}{512}, \qquad \frac{1}{t_*} + \frac{8}{\ell_*} > 9, \qquad \ell_* < r_*,$$
$$9r_* + k\left(9 - \frac{9}{r_*}\right) < 12, \qquad t_* + 8\ell_* + k\left(9 - \frac{1}{t_*} - \frac{8}{\ell_*}\right) < 12.$$

Inverse AM–GM places the actual inverse trace strictly above $9/r_*$. The determinant cap r_*^9 and the convex trace profile therefore bound $\operatorname{tr} S + k(9 - \operatorname{tr} R)$ by the larger of the last two endpoint expressions. This proves $\operatorname{tr} S + (14/5)\delta < 12$. All these comparisons involve only rational numbers; no algebraic endpoint or numerical maximizer is needed. For the affine part, the actual normal flag gives $\Delta \geq 1024 \det S/m^2$. Thus a counterexample satisfies $\det S < (243/256)m^2$. The rational comparisons $(1081/1000)^8 > 11907/6400$ and

$$\log\left(\frac{11907}{6400}\right) < 8\log\left(\frac{1081}{1000}\right) \leq 4\left(\frac{1081}{1000} - \frac{1000}{1081}\right) < \frac{5}{8}$$

prove the required logarithm anchor. Since $\log x \leq x - 1$ for $x > 0$,

$$\log\left(\frac{243}{256}m^2\right) = \log\left(\frac{11907}{6400}\right) + 2\log\left(\frac{5m}{7}\right) < \frac{10}{7}m - \frac{11}{8}.$$

Combining this with the lower logarithm bound used in (20) yields

$$\operatorname{tr} S + \frac{37}{13}\delta < \frac{1351 + 2000m}{364}.$$

Finally $37/13 > 14/5$ and $\delta \geq 0$, so the constant and affine bounds give exactly the asserted $U(m)$.

Elementary contact calculations. We record the elementary contact calculations used to construct the remaining rows. For an n-dimensional D_n block $I + E$, write $T_E = \operatorname{tr} E$. The pair contacts are $E_{ii} + E_{jj} \geq 2|E_{ij}|$. Lemma 9 decomposes its diagonal into trace-one vertices e_i and $(\mathbf{1} - 2e_i)/(n-2)$, while preserving the normalized off-diagonal entries. Convex quadratic maxima therefore reduce to these diagonal vertices and the endpoints of the paired intervals. The same lemma gives, for D_8,

$$\det(I + E) \geq F(T_E) := 1 + T_E - \frac{7}{4}T_E^2 \qquad (0 \leq T_E \leq 1).$$

For D_7 the analogous profile is $1 + x - \frac{3}{2}x^2$.
For the column estimates below, $n \in \{7, 8\}$. If coordinate n of H has diagonal h and inverse diagonal χ, let $H = \left(\begin{smallmatrix} K & g \\ g^{\mathsf{T}} & h \end{smallmatrix}\right)$. Pair contacts and Lemma 7 give

$$4(h-T)\left(h - \frac{1}{\chi}\right) \leq T^2 + (n-2)(h-1)^2, \qquad |g|^2 \leq \frac{T^2 + (n-2)(h-1)^2}{4}. \tag{23}$$

Indeed $K \succeq (1 - \operatorname{tr}(K - I))I = (h - T)I$, while $h - 1/\chi = g^{\mathsf{T}}K^{-1}g$. If $h - T \leq 0$ the first assertion is automatic; otherwise multiply this identity by $h - T$. For the squared-column bound, let $d_i = K_{ii} - 1$ and $a = h - 1$. Then $\sum_i d_i = T - a$. The diagonal-square argument in Lemma 6, applied in dimension $n - 1 \geq 4$, gives $\sum_i d_i^2 \leq (T - a)^2$. The cross contacts therefore imply

$$4|g|^2 \leq \sum_i (d_i + a)^2 \leq (T-a)^2 + 2a(T-a) + (n-1)a^2 = T^2 + (n-2)a^2.$$

Also $2|g| \leq 2\sum_i |g_i| \leq \sum_i (d_i + a) = T + (n-2)a =: E_*$, proving both $E_* \geq 0$ and $|g| \leq E_*/2$. These estimates require no diagonal vertex optimization.
For the projected block $J_K = K - gg^{\mathsf{T}}/h$,

$$\begin{aligned} \det J_K &= \frac{\det H}{h}, \qquad \operatorname{tr} J_K^{-1} = \operatorname{tr} H^{-1} - \chi, \\ \operatorname{tr} J_K &\geq f_n(T,h) := \frac{3n-2}{2} + T - \frac{n+2}{4}h - \frac{T^2 + n - 2}{4h}, \\ \operatorname{tr} J_K &\geq n + T - h - \frac{[T + (n-2)(h-1)]^2}{4h}. \end{aligned} \tag{24}$$

These are just $\operatorname{tr} K - |g|^2/h$ with the two bounds above. The right sides are concave. Triangulating a rectangle in (T, h) therefore supplies affine lower bounds from its corners. If $K \preceq \Lambda I$, a further bound is $\operatorname{tr} J_K \geq n + T - h - \Lambda + \Lambda/(\chi h)$.

The two-column version. Write

$$S = \begin{pmatrix} K & U_2 \\ U_2^{\mathsf{T}} & D \end{pmatrix} > 0, \qquad R = S^{-1} = \begin{pmatrix} R_0 & C \\ C^{\mathsf{T}} & B_2 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} d_{\mathrm{sel}} & w \\ w & m \end{pmatrix}.$$

Here K is obtained by deleting the selected horizontal coordinate and the original normal coordinate, in that order for the complementary two-coordinate block. Block inversion gives

$$B_2^{-1} = D - U_2^{\mathsf{T}} K^{-1} U_2, \qquad C = -K^{-1} U_2 B_2, \qquad R_0 = K^{-1} + C B_2^{-1} C^{\mathsf{T}}. \tag{25}$$

Thus B_2^{-1} is the Schur complement and $D - B_2^{-1} = U_2^{\mathsf{T}} K^{-1} U_2 \succeq 0$. For every real z, evaluation on $(1, z)$ gives

$$S_{ii} + 2zS_{i9} + z^2 h \geq \frac{m - 2wz + d_{\mathrm{sel}} z^2}{d_{\mathrm{sel}} m - w^2} = \frac{2cm - 4cwz + z^2}{\chi}.$$

The denominator is positive because $d_{\mathrm{sel}} m - w^2 = \chi/(2c) > 0$. Substituting $S_{ii} = 2(1+\tau) + 2s - h$ proves the whole family

$$-2\tau + (1 - z^2)h + 2(z-1)s \leq 2 - \frac{2cm - 4cwz + z^2}{\chi}. \tag{26}$$

Thus every rational test parameter used in a finite cover is justified by the same inequality. If $K \preceq \Lambda I$ with $\Lambda > 0$, then

$$C^{\mathsf{T}} C \succeq \frac{B_2 (D - B_2^{-1}) B_2}{\Lambda}, \qquad \operatorname{tr} K^{-1} \leq \operatorname{tr} R - \operatorname{tr} B_2 - \frac{\operatorname{tr}(B_2 D - I_2)}{\Lambda}. \tag{27}$$

Proof of (27). For completeness, apply the upper bound for K to $K^{-1} y$:

$$y^{\mathsf{T}} K^{-1} y = (K^{-1} y)^{\mathsf{T}} K (K^{-1} y) \leq \Lambda |K^{-1} y|^2.$$

Taking $y = U_2 B_2 z$ and using (25) proves the first inequality of (27) for every z. To take the weighted trace, recall that if $M \succeq 0$ and $W \succeq 0$, then $\operatorname{tr}(WM) = \operatorname{tr}(W^{1/2} M W^{1/2}) \geq 0$; the last matrix is positive semidefinite. The square root exists by orthogonal diagonalization. Multiply the first inequality by $B_2^{-1} > 0$ under the trace. Cyclicity and $B_2^{-1} B_2 = I_2$ give

$$\operatorname{tr}(C B_2^{-1} C^{\mathsf{T}}) \geq \frac{\operatorname{tr}(B_2^{-1} B_2 (D - B_2^{-1}) B_2)}{\Lambda} = \frac{\operatorname{tr}(B_2 D - I_2)}{\Lambda}.$$

The identity $\operatorname{tr} R = \operatorname{tr} K^{-1} + \operatorname{tr}(C B_2^{-1} C^{\mathsf{T}}) + \operatorname{tr} B_2$ proves the second inequality. This argument works for arbitrary block dimensions. □

In the present case K has dimension seven and inherits all D_7 pair contacts. Lemma 7 therefore supplies $\operatorname{tr} K - 7 \geq 0$ and the valid choice $\Lambda = 1 + 2(\operatorname{tr} K - 7)$.
For $w \geq 0$ and any $\alpha > 0$, a further estimate retains both column norms:

$$B_2^{-1} \succeq \frac{\operatorname{diag}\bigl(m - w\alpha,\ d_{\mathrm{sel}} - \frac{w}{\alpha}\bigr)}{\det B_2}.$$

Indeed, the difference is $(w/\det B_2)\left(\begin{smallmatrix}\alpha & -1\\ -1 & 1/\alpha\end{smallmatrix}\right)$, whose quadratic value at (x,y) is $w(\alpha x-y)^2/(\alpha\det B_2)\ge 0$. Writing C_1,C_2 for the two columns of C and summing this inequality over its rows yields

$$\operatorname{tr}K^{-1}\le \operatorname{tr}R-\operatorname{tr}B_2-\frac{(m-w\alpha)|C_1|^2+\left(d_{\mathrm{sel}}-\frac{w}{\alpha}\right)|C_2|^2}{\det B_2}.$$

Neither diagonal coefficient is required to be nonnegative. These are the coupled two-column rows used in the certificates; the preceding argument proves their validity.

One trace argument for all coordinate scalings. For any positive matrix M and any $a_{\mathrm{sc}}>0$, let $M_{a_{\mathrm{sc}}}=D_{a_{\mathrm{sc}}}MD_{a_{\mathrm{sc}}}$, where $D_{a_{\mathrm{sc}}}$ is diagonal with the selected entry $\sqrt{a_{\mathrm{sc}}}$ and all other entries one. Direct multiplication gives

$$\det M_{a_{\mathrm{sc}}}=a_{\mathrm{sc}}\det M,\qquad \operatorname{tr}M_{a_{\mathrm{sc}}}=\operatorname{tr}M+(a_{\mathrm{sc}}-1)M_{ii},$$
$$\operatorname{tr}M_{a_{\mathrm{sc}}}^{-1}=\operatorname{tr}M^{-1}+(a_{\mathrm{sc}}^{-1}-1)(M^{-1})_{ii}.$$

Apply this first to H, at its old-normal coordinate, and then to A, at the selected horizontal coordinate. Let $d_{\mathrm{sel}}=1/(2c)$ and $R_p=r_p-\kappa_p^2/8\ge 0$. The exact inverse traces are

$$\operatorname{tr}H_{a_{\mathrm{sc}}}^{-1}=U_H-\delta-\frac{R_p}{c},\qquad U_H=9-d_{\mathrm{sel}}+(a_{\mathrm{sc}}^{-1}-1)m-\frac{2cw^2}{a_{\mathrm{sc}}},$$
$$\operatorname{tr}A_{a_{\mathrm{sc}}}^{-1}=U_A-\delta-\frac{\rho-w^2}{m}-\frac{w^2}{a_{\mathrm{sc}}m},\quad U_A=9-m+(a_{\mathrm{sc}}^{-1}-1)d_{\mathrm{sel}}.$$

Every subtracted term is nonnegative. Their traces are, respectively, $8+T+(a_{\mathrm{sc}}-1)h$ and $8+T+2a_{\mathrm{sc}}\tau-(1+a_{\mathrm{sc}})h+2a_{\mathrm{sc}}s+2a_{\mathrm{sc}}$. Any established cap $c\det H<\mathcal{D}$ gives

$$\det H_{a_{\mathrm{sc}}}<a_{\mathrm{sc}}\frac{\mathcal{D}}{c},\qquad \det A_{a_{\mathrm{sc}}}<2a_{\mathrm{sc}}m\mathcal{D}.$$

Thus a single affine trace profile applies to every positive scaling, including constants and positive multiples of c or $1/m$. More explicitly, if the profile gives $\operatorname{tr}M+k(U_*-\operatorname{tr}M^{-1})<B_*$ with $k\ge 0$, and $\operatorname{tr}M^{-1}\le U-q\le U_*$, then

$$\operatorname{tr}M+k(U_*-U+q)<B_*.$$

This substitution is the complete analytic justification for the scaled trace rows; only their scalar profile endpoints need finite checks.

The same substitution treats the seven-dimensional block K in (25). Here $\operatorname{tr}K=8+T-h$, $\det K=\chi\det H$ and $\operatorname{tr}K^{-1}\le 9-m-d_{\mathrm{sel}}-q_K$. The three proved choices for q_K are

$$q_K=\delta+\frac{m-w\alpha}{c\chi}R_p+\frac{1-\frac{2cw}{\alpha}}{\chi}(\rho-w^2)\qquad (w\ge 0,\ \alpha>0),$$
$$q_K=\delta+\frac{R_p}{c}+\frac{\chi h-1}{\Lambda},$$
$$q_K=\delta+\frac{2d_{\mathrm{sel}}\tau+(m-d_{\mathrm{sel}})h+(2d_{\mathrm{sel}}-2w)s+2d_{\mathrm{sel}}-2}{\Lambda}.$$

For the last two choices require $K \preceq \Lambda I$ and $\Lambda > 0$. The first follows from the diagonal minorant above and the exact norms $|C_1|^2 = R_p/(2c^2)$, $|C_2|^2 = \rho - w^2$. For the second, block inversion in H gives $\operatorname{tr} H^{-1} = \operatorname{tr} K^{-1} + \chi + \chi|K^{-1}g|^2$ and $\chi g^{\mathsf{T}} K^{-1} g = \chi h - 1$; the inequality $g^{\mathsf{T}} K^{-1} g \le \Lambda |K^{-1}g|^2$ proves the claim. The third is exactly (27), since $\operatorname{tr}(B_2 D - I_2) = 2d_{\mathrm{sel}}\tau + (m - d_{\mathrm{sel}})h + (2d_{\mathrm{sel}} - 2w)s + 2d_{\mathrm{sel}} - 2$. The diagonal-minorant correction may have negative terms; the proved upper bound on $\operatorname{tr} K^{-1}$ justifies their sum. For all three choices the actual inverse trace also satisfies $\operatorname{tr} K^{-1} \le 9 - m - d_{\mathrm{sel}}$ directly from (25). Finally $\det K < \chi \mathcal{D}/c$. These are precisely the determinant and inverse-trace inputs needed for the same trace profile.

6 Directional information and the short dual average

The directional bound. For D_n with $n \ge 3$, set $a_p = \max |p_i|$, $b_p = \min |p_i|$ and $S_p = \sum |p_i|$. The following bound retains the direction of p:

$$p^{\mathsf{T}} E p \le T_E S_p \max\Big(a_p,\ \frac{S_p - 2b_p}{n-2}\Big). \tag{28}$$

This is the directional contact inequality of [5, Lemma 30]. For $n = 8$, $a_p = \kappa_p/(2\sqrt{2})$. The directional bound gives $p^{\mathsf{T}} E p \le T J(r_p, \kappa_p)$ with the following profile:

$$J = \begin{cases} 2r_p, & \dfrac{r_p}{\kappa_p^2} \le \dfrac{1}{3}, \\[2mm] \dfrac{r_p + \kappa_p^2}{2}, & \dfrac{1}{3} \le \dfrac{r_p}{\kappa_p^2} \le \dfrac{3}{5}, \\[2mm] \dfrac{4r_p}{3}, & \dfrac{3}{5} \le \dfrac{r_p}{\kappa_p^2} \le \dfrac{7}{8}, \\[2mm] \dfrac{7\kappa_p^2 - 4r_p}{3}, & \dfrac{7}{8} \le \dfrac{r_p}{\kappa_p^2} \le 1. \end{cases}$$

If $\kappa_p = 0$, then $p = 0$ and we set $J = 0$. The proof of the four branches uses three common bounds. Write $F_1 = a_p S_p$ and $F_2 = S_p(S_p - 2b_p)/6$; (28) gives $p^{\mathsf{T}} E p \le T \max(F_1, F_2)$. Summing the eight squares $(|p_j| - a_p)^2$ and applying Cauchy–Schwarz to S_p prove, respectively,

$$F_1 \le \frac{r_p}{2} + 4a_p^2, \qquad F_2 \le \frac{4r_p}{3}.$$

For the second bound, use $S_p^2 \le 8r_p$ and $b_p S_p \ge 0$. Both F_1 and F_2 also satisfy

$$F_1, F_2 \le \frac{56a_p^2 - 4r_p}{3}.$$

Indeed $S_p \le 8a_p$ and $r_p \le 8a_p^2$ prove the assertion for F_1. A coordinate attaining b_p gives $S_p - b_p \le 7a_p$ and $r_p \le 7a_p^2 + b_p^2$. For F_2, use the identity

$$112a_p^2 - 8r_p - S_p(S_p - 2b_p) = 49a_p^2 - (S_p - b_p)^2 + 8(7a_p^2 + b_p^2 - r_p) + 7(a_p^2 - b_p^2) \ge 0.$$

Thus the directional expression is bounded by $\max(r_p/2+4a_p^2, 4r_p/3)$ and by $(56a_p^2-4r_p)/3$. The bound $2r_p$ follows from Lemma 7. Use this first bound when $3r_p \le \kappa_p^2$; in the next two ranges select the larger term of the maximum, whose crossing is $5r_p = 3\kappa_p^2$; use the decreasing final bound in the last range. Substituting $8a_p^2 = \kappa_p^2$ gives exactly the four stated branches, including their common boundary values.
Useful global upper bounds are $2r_p$, $12r_p/13 + 14\kappa_p^2/39$, and $-4r_p/3 + 7\kappa_p^2/3$. For completeness, subtracting the four branch expressions from the middle affine bound gives, in their stated order,

$$\frac{14}{39}(\kappa_p^2 - 3r_p), \qquad \frac{11}{78}(3r_p - \kappa_p^2), \qquad \frac{2}{39}(7\kappa_p^2 - 8r_p), \qquad \frac{11}{39}(8r_p - 7\kappa_p^2).$$

Each is nonnegative on its branch. The other two global bounds were proved above. Consequently $1 + \tau - c \le r_p + TJ$. Lemma 7 also gives the two global bounds $(1-T)r_p \le 1 + \tau - c \le (1+2T)r_p$ directly. A useful refinement applies when $r_p \le 4\kappa_p^2/7$, equivalently $r_p \le (32/7)a_p^2$. Let $A_p = \sum_{j=1}^{7} |p_j|$, so $S_p = a_p + A_p$. Cauchy–Schwarz gives $A_p^2 \le 7(r_p - a_p^2) \le 25a_p^2$; hence $S_p \le 6a_p$. As $b_p \ge 0$, both alternatives in the maximum in (28) are at most a_p. Consequently

$$p^{\mathsf{T}} E p \le T a_p S_p \le T\Big[a_p^2 + \sqrt{7a_p^2(r_p - a_p^2)}\Big].$$

There is a direct proof of the parameter bounds used in the rational cover. For any $z > 0$, summing the seven squares $(z|p_j| - a_p)^2 \ge 0$ gives

$$a_p A_p \le \frac{z}{2}(r_p - a_p^2) + \frac{7}{2z}a_p^2.$$

Adding a_p^2 and using $T \ge 0$ therefore yields

$$p^{\mathsf{T}} E p \le T\Big[\frac{z}{2}r_p + \Big(1 - \frac{z}{2} + \frac{7}{2z}\Big)a_p^2\Big].$$

Every positive rational z gives a valid row. No division by a_p is made: if $a_p = 0$, then $p = 0$ and all these inequalities hold with equality.

The short dual average. Let μ be twice the minimum squared dual length of the rank-nine section, measured in its original integral coordinates. This positive minimum is attained. It is distinct from the rank-ten dual minimum u: every nonzero section covector has a lift whose normal component has squared length at most $u/4$, so shortness in rank ten gives $\mu/2 \ge 3u/4$, or $3u/2 \le \mu$. In the physical coordinates, the corresponding quadratic form is $R/2$. The covector e_9 is integral, so $\mu \le m$. For every horizontal $j \ne i$, both $e_i + e_j$ and $e_i - e_j$ are also integral dual covectors. To verify the inclusion, observe that each row of the displayed E_8 basis has a half-integral first entry and integral remaining entries. Sums and differences of two rows are therefore integral; their pairings with the last basis column $(e_1, 1)$ are integral as well. Choose the better sign for each of the seven pairs. Together with e_9, these eight witnesses give

$$\mu \le \frac{\operatorname{tr} R + 6R_{ii} - 2\sum_{j\ne i,\, j\le 8} |R_{ij}|}{8} = \frac{9}{8} - \frac{\delta}{8} + \frac{3}{8c} - \frac{2\sqrt{2}\sum_{j=1}^{7}|p_j|}{16c}.$$

Here the last sum uses the selected coordinates, whose eighth coordinate is the old normal. The equality follows from $R_{ii} = 1/(2c)$ and $|R_{ij}| = |p_j|/(\sqrt{2}c)$ for the seven unselected horizontal coordinates. A normal reflection leaves these integral inverse entries and R_{99} unchanged.
Dominance of $p_8 = \kappa_p/(2\sqrt{2})$ gives $r_p \le p_8 \sum_{j=1}^{8} |p_j|$. Hence, when $\kappa_p > 0$,

$$\mu \le \min\Big(m,\ \frac{9}{8} + \frac{3 + \frac{\kappa_p}{2}}{8c} - \frac{\delta}{8} - \frac{r_p}{2c\kappa_p}\Big). \tag{29}$$

There is also a bound that includes $\kappa_p = 0$. Let $Z_p = 8r_p - \kappa_p^2 = 8\sum_{j=1}^{7} p_j^2 \ge 0$. Since $\sum_{j=1}^{7} |p_j| \ge \sqrt{\sum_{j=1}^{7} p_j^2}$, the same average gives

$$\mu \le \frac{9}{8} + \frac{3}{8c} - \frac{\delta}{8} - \frac{\sqrt{Z_p}}{16c}.$$

If $\kappa_p = 0$, dominance forces $p = 0$, and this last formula applies directly, with no division by κ_p. For $Z_p \in [a, b]$ with $0 \le a < b$, choose rational $0 \le q_a \le \sqrt{a}$ and $0 \le q_b \le \sqrt{b}$. Then

$$\frac{b - Z_p}{b - a} q_a + \frac{Z_p - a}{b - a} q_b \le \sqrt{Z_p}.$$

Indeed, for $0 \le t \le 1$, squaring the nonnegative expression $(1-t)\sqrt{a} + t\sqrt{b}$ leaves a deficit $t(1-t)(\sqrt{a} - \sqrt{b})^2$ from $(1-t)a + tb$. Lowering the endpoint values preserves the inequality. Their validity is checked by $q_a^2 \le a$, $q_b^2 \le b$; if $a = b$, use the constant q_a instead. Substitution into the preceding upper bound for μ therefore gives every retained affine spread objective. All normal, radius and spread bounds concern the same attained minimum μ. If a linear certificate gives $\mu \le M$, then $M > 0$, so $\mu^2 \le M^2$ and the determinant cap below remains valid with M in place of μ. This justifies repeated tightening of that cap.
Apply the scalar flag to a shortest integral covector, of squared length $\mu/2$. It gives $\Delta \ge e/(2(\mu/2)^2) = 2048c \det H/\mu^2$. Together with $e < 1137$ and $\Delta < 972$, this proves the additional cap

$$c \det H < \min\left(\frac{1137}{1024},\ \frac{243\mu^2}{512}\right).$$

Furthermore $\sqrt{2}\, g^{\mathsf{T}}(p_1, \ldots, p_7) = 2cwh - s$. Together with $|g| \le (T + 6h - 6)/2$ this gives

$$|s - 2cwh|^2 \le (T + 6h - 6)^2 \frac{Z_p}{16}.$$

For a box upper bound $Z_p \le Z_+$, any rational $\kappa \ge \sqrt{Z_+}/4$ yields the necessary linear rows $\pm s + (\mp 2cw - 6\kappa)h - \kappa T \le -6\kappa$.

7 A common bound for both inverse-entry signs

Column bounds. Canonical selection also bounds every primal normal-column entry. For any horizontal j, the cross-contact inequality and the maximizing property give,

respectively,

$$2|S_{j9}| \le S_{jj} + h - 2, \qquad S_{jj} + 2|S_{j9}| \le S_{ii} + 2s.$$

Adding and using $S_{ii} + h - 2s = 2(1+\tau)$ proves $|S_{j9}| \le \tau/2 + s$. The nonnegative contact expansion gives $\sum_j |R_{j9}| \le m$, since each difference of the two signed contact weights has absolute value at most their sum. Remove the selected entry $R_{i9} = w$ and use $S_{i9} = -s$ in $SR = I$. For either sign of w this proves

$$mh - 1 \le (m - |w|)\Big(\frac{\tau}{2} + s\Big) + ws. \tag{30}$$

Alternatively, retaining the selected term and using only $|R_{j9}| \le 1/2$ on the other coordinates gives

$$mh - 1 \le ws + \frac{1}{2}\sum_{j\neq i} |S_{j9}|, \qquad \sum_j |S_{j9}| \ge 2mh - 2 + (1-2w)s.$$

The same averaging of the canonical maximum as above gives $16(T+\tau) \le 15\,\mathrm{tr}\,S + 9h - 144 - 2\sum_j |S_{j9}|$. Substituting this sharper column bound and $\mathrm{tr}\,S = 10 + T + 2\tau - h + 2s$ yields the necessary row

$$-14\tau + (6+4m)h - (28+4w)s + T \le 10. \tag{31}$$

Both column estimates hold for either sign of w.

The piecewise trace floor. Define the piecewise trace floor

$$L_{\mathrm{tr}}(m) = \begin{cases} 4m - 4, & m \le \dfrac{3}{2}, \\ 6 - \dfrac{6}{m}, & m \ge \dfrac{3}{2}. \end{cases}$$

We prove $T \ge L_{\mathrm{tr}}(m) - 4ws$ directly from the same inverse column. For the seven unselected horizontal coordinates let $\eta_h = h - 1$, $d_j = S_{jj} - 1$ and $b_j = |R_{j9}|$. Then

$$d_j + d_k \ge 0 \ (j \neq k), \qquad 2|S_{j9}| \le d_j + \eta_h, \qquad 0 \le b_j \le \frac{1}{2},$$
$$\sum_j b_j \le m - |w|, \qquad \sum_j d_j = T - \eta_h.$$

The normal entry of $SR = I$ first gives

$$mh - 1 - ws \le \sum_j b_j |S_{j9}| \le \frac{T + 6\eta_h}{4}, \qquad T \ge 2 + (4m-6)h - 4ws.$$

For $h \ge 1$ retain the total inverse-column mass. The pair-diagonal dual criterion used in Lemma 7, applied to $f_j = 1/2 - b_j$, gives $\sum_j b_j d_j \le \frac{1}{2}\sum_j d_j$: indeed $f_j \ge 0$ and $2f_j \le 1 \le 7/2 - \sum_j b_j = \sum_j f_j$, since $m \le 5/2$. Consequently

$$mh - 1 - ws \le \frac{\eta_h(m - |w|)}{2} + \frac{T - \eta_h}{4},$$
$$T \ge 4m - 4 + (2m + 2|w| + 1)(h-1) - 4ws.$$

Positivity gives $mh \geq 1$. If $h \leq 1$ and $m \leq 3/2$, the first row gives $T + 4ws \geq 4m - 4$; if $h \leq 1$ and $m \geq 3/2$, use $h \geq 1/m$ in that row to obtain $T + 4ws \geq 6 - 6/m$. If $h \geq 1$, the second row gives $T + 4ws \geq 4m - 4$. For $m \geq 3/2$ this is at least $6 - 6/m$, because their difference is $2(m-1)(2m-3)/m \geq 0$. Thus the floor is proved for either sign of w, throughout $0 < m \leq 5/2$.
On $m \geq 1$ this floor is concave: it is the minimum of the two concave functions $4m - 4$ and $6 - 6/m$. Indeed their difference is $2(m-1)(2m-3)/m$, so their minimum has exactly the displayed branches. For completeness, the minimum of concave functions is concave because each function at a convex combination is at least the same convex combination of their pointwise minimum. Hence, for $1 \leq a < b$ and $a \leq m \leq b$,

$$L_{\mathrm{tr}}(a) + \frac{L_{\mathrm{tr}}(b) - L_{\mathrm{tr}}(a)}{b - a}(m - a) \leq L_{\mathrm{tr}}(m). \tag{32}$$

This proves the affine floor on cells crossing $m = 3/2$ as well as on cells wholly within one branch. There is no division by a cell width when $a = b$: the floor is then evaluated at that endpoint.

The capped-mass rows. A second refinement retains both the cap and the total mass of the seven unselected inverse entries. Let $B_{\mathrm{col}} = \tau/2 + s$ and $a_j = |S_{j9}|$ on these entries. The canonical contact bound gives $0 \leq a_j \leq B_{\mathrm{col}}$, and the cross contacts give $\sum_j a_j \leq (T + 6h - 6)/2$. Choose a nonnegative integer k_{mass} and $0 \leq r_{\mathrm{mass}} \leq 1/2$ such that $m - |w| \leq k_{\mathrm{mass}}/2 + r_{\mathrm{mass}}$. We claim

$$\sum_j (b_j - r_{\mathrm{mass}})_+ \leq k_{\mathrm{mass}} \left(\frac{1}{2} - r_{\mathrm{mass}} \right), \qquad x_+ := \max(x, 0).$$

Let N count the entries with $b_j > r_{\mathrm{mass}}$. If $N \leq k_{\mathrm{mass}}$, each positive summand is at most $1/2 - r_{\mathrm{mass}}$. If $N \geq k_{\mathrm{mass}} + 1$, their sum is at most $k_{\mathrm{mass}}/2 + r_{\mathrm{mass}} - N r_{\mathrm{mass}} \leq k_{\mathrm{mass}}(1/2 - r_{\mathrm{mass}})$. These exhaust the possibilities and include both remainder endpoints. Now $b_j a_j \leq r_{\mathrm{mass}} a_j + (b_j - r_{\mathrm{mass}})_+ B_{\mathrm{col}}$. Summing and using the normal inverse identity proves

$$mh - 1 - ws \leq k_{\mathrm{mass}} \left(\frac{1}{2} - r_{\mathrm{mass}} \right) \left(\frac{\tau}{2} + s \right) + \frac{r_{\mathrm{mass}}}{2}(T + 6h - 6). \tag{33}$$

Thus the capped-mass rows require no ordering or enumeration of these seven entries. On a positive-w cell, the upper endpoint M_{mass} of $m - w$ gives the usual choice $k_{\mathrm{mass}} = \lfloor 2M_{\mathrm{mass}} \rfloor$ and $r_{\mathrm{mass}} = M_{\mathrm{mass}} - k_{\mathrm{mass}}/2$. The inequality itself is valid for either sign of w.

The common caps. Combining this floor and (30) with $\operatorname{tr} S = 10 + T + 2\tau - h + 2s$ and the trace ceiling gives

$$\left(\frac{3}{2} + \frac{|w|}{2m} \right) \tau + \left(1 - 4w + \frac{|w| - w}{m} \right) s + \frac{14}{5}\delta < U(m) - 10 - L_{\mathrm{tr}}(m) + \frac{1}{m}. \tag{34}$$

Division is valid because $m = R_{99} > 0$. For a parameter cell, the same algebra permits any proved lower bound $\ell(m) \leq L_{\mathrm{tr}}(m)$, including (32), and either ceiling 12 or

$(1351 + 2000m)/364$. For the latter, replace the coefficient $37/13$ of δ by $14/5$ using $\delta \geq 0$. On $w \geq 0$ the resulting necessary affine row, before its parameter coefficients are enclosed, is

$$\Big(\frac{3}{2} + \frac{w}{2m}\Big)\tau + (1 - 4w)s + \frac{14}{5}\delta + 10 + \ell(m) - \frac{1}{m} - U(m) \leq 0.$$

It follows from the mass, floor and ceiling rows by multiplying by $m > 0$ and adding; the non-strict form includes cell boundaries. Thus the two sign branches share the same column argument and retain the same inverse-trace deficit. Let $C(m) = U(m) - 10 - L_{\mathrm{tr}}(m) + 1/m$. Its three expressions on the required interval are

$$C(m) = \begin{cases} \dfrac{544m - 833}{364} + \dfrac{1}{m}, & \dfrac{13}{10} \leq m \leq \dfrac{3}{2}, \\ \dfrac{2000m - 4473}{364} + \dfrac{7}{m}, & \dfrac{3}{2} \leq m \leq \dfrac{3017}{2000}, \\ -4 + \dfrac{7}{m}, & \dfrac{3017}{2000} \leq m \leq \dfrac{8}{5}. \end{cases}$$

Their derivatives are $544/364 - 1/m^2 > 0$, $2000/364 - 7/m^2 > 0$, and $-7/m^2 < 0$, respectively. The pieces agree at their endpoints, so $C(m) \leq C(3017/2000) = 1932/3017$. For every $w \leq 1/4$, including all negative w, the coefficients of τ and s in (34) are at least $3/2$ and 0. Since $\tau, s, \delta \geq 0$, this gives the common caps

$$\tau < \frac{1288}{3017}, \qquad \delta < \frac{690}{3017}, \qquad c \leq 1 + \tau < \frac{4305}{3017}.$$

Since $4305/3017 < 10/7$, the entire region $c \geq 10/7$, $w \leq 1/4$ is excluded on this range of m. If $w \leq 0$, the coefficient of s is at least 1, giving also $s < 1932/3017$. These caps on τ and δ apply for $w \leq 1/4$.

A common reduction for the negative sign. If $w \leq 0$, keep the same actual metric and replace w by zero only in the necessary scalar inequalities. The signed column bound (30) gives

$$mh - 1 \leq (m - |w|)\Big(\frac{\tau}{2} + s\Big) + ws \leq m\Big(\frac{\tau}{2} + s\Big).$$

The discarded terms have the required signs because $\tau, s \geq 0$. Likewise, $T \geq L_{\mathrm{tr}}(m) - 4ws$ implies $T \geq L_{\mathrm{tr}}(m)$. The canonical trace and capped-mass rows weaken in the same way; for the latter, choose the mass cap from m, which also bounds $m - |w|$. For the pair tests (26), restrict to $z \geq 0$. The numerator $2cm - 4cwz + z^2$ is at least $2cm + z^2 \geq 0$ and $0 < \chi \leq m$. Thus replacing w by zero decreases the positive quotient and weakens the proof. The phase-inverse row follows similarly from $\kappa_p \geq 2$ and $\chi \leq m$. For the phase-square rows the lower bound becomes $4 \leq 8r_p$; the upper bound $r_p \leq 4$ follows from the already proved $r_p < 3$. The uniform radius bound justifies the last step.

Two families of column Cauchy inequalities. Let J be the seven unselected horizontal coordinates and let

$$B_0 := \frac{\tau}{2} + s, \qquad A_0 := T + 6h - 6, \qquad P_0 := \sum_{j \in J} |S_{j9} R_{j9}|.$$

These quantities are nonnegative. To see the relevant bounds, set $a_j = |S_{j9}|$. The canonical choice and the cross contacts give

$$S_{jj} + 2a_j \le S_{ii} + 2s, \qquad S_{jj} + h \ge 2 + 2a_j, \qquad S_{ii} + h = 2 + 2\tau + 2s.$$

Hence $a_j \le B_0$. Summing the cross contacts over J, and using $\sum_{j \in J} S_{jj} = 8 + T - h$, gives $2 \sum_{j \in J} a_j \le A_0$. Also $\sum_{j \in J} R_{j9}^2 = \rho - w^2$. Cauchy–Schwarz now proves

$$P_0^2 \le (\rho - w^2) \sum_{j \in J} a_j^2 \le \frac{1}{2}(\rho - w^2) B_0 A_0, \qquad mh - 1 - ws \le P_0.$$

The last inequality is the normal entry of $SR = I$. For $w \le 0$ it implies $mh - 1 \le P_0$. For nonnegative P, X, Y with $P^2 \le XY$, every $z > 0$ gives $2P \le zX + Y/z$: square both sides and use $(zX + Y/z)^2 - 4P^2 = (zX - Y/z)^2 + 4(XY - P^2) \ge 0$. Thus any established upper bound $\rho \le R_0$ gives the necessary row

$$2(mh - 1) \le zB_0 + \frac{R_0 A_0}{2z} \qquad (z > 0). \tag{35}$$

For a refined row, choose a_0, b_0 so that either $B_0 \le b_0$ and $a_0 \le A_0$, or $b_0 \le B_0$ and $A_0 \le a_0$. In both cases

$$B_0 A_0 \le L_0 := b_0 A_0 + a_0 B_0 - a_0 b_0,$$

by expanding the product of the two nonnegative differences. Since $L_0 \ge B_0 A_0 \ge 0$, the same square argument gives

$$2(mh - 1) \le z\rho + \frac{L_0}{2z} \qquad (z > 0). \tag{36}$$

Both displayed inequalities are affine in the retained metric coordinates for fixed parameters. Their bounds refer throughout to the same seven columns of the same metric.

Lemma 15 (the negative-sign low-normal strip). *Subject to the three finite polynomial checks stated below, a counterexample with $w \le 0$ must satisfy $m > 133/100$.*

Proof. Suppose $13/10 \le m \le 133/100$. The preceding mass and floor bounds and (17) imply

$$\operatorname{tr} A \ge \sigma_0 := 6 + 4m - \frac{2}{m}, \qquad \operatorname{tr} A^{-1} \le V_0 := 9 - m.$$

Indeed $h \le 1/m + \tau/2 + s$, so $\operatorname{tr} A = 10 + T + 2\tau - 2h + 2s \ge 10 + T + \tau - 2/m \ge \sigma_0$. The normal flag and $\det A = m \det S$ give the strict opposite cap $\det A < (243/256)m^3$. Set

$$\ell_0 = \frac{19}{20} + \frac{13}{200}\left(m - \frac{13}{10}\right), \qquad t_0 = \sigma_0 - 7\ell_0.$$

Here $\sigma_0, \ell_0 > 0$ directly from the interval for m. The required finite checks are positivity, on this one rational interval, of the following polynomials:

$$P_1(m) = 4000000m\{V_0 t_0^2 - (\sigma_0 V_0 - 48)t_0 + \sigma_0\},$$
$$P_2(m) = 250m(8t_0 - \sigma_0),$$
$$P_3(m) = 256000000000000000000000000m\left\{t_0\ell_0^7 - \frac{243}{256}m^3\right\}.$$

The displayed factors clear all denominators. These checks form part of Computational result 17; their degrees and least Bernstein coefficients, after mapping $m = 13/10 + (3/100)x$ with $0 \le x \le 1$, are

	degree	least Bernstein coefficient
P_1	4	511507843/20000
P_2	2	4690
P_3	9	7933619468909373196282552741489 89/10000000000

Only the exact coefficient identities and their positive signs are computational inputs; no metric is sampled. Now $P_2 > 0$ gives $t_0 > \sigma_0/8$, while $\ell_0 > 0$ gives $t_0 < \sigma_0$. Multiplication by $t_0(\sigma_0 - t_0) > 0$ shows that $P_1 > 0$ is exactly $V_0 < 1/t_0 + 7/\ell_0$. Apply Lemma 11 at the comparison trace $t_0 + 7\ell_0 = \sigma_0$ and inverse trace $1/t_0 + 7/\ell_0$. Its monotonicity yields $\det A \ge t_0\ell_0^7$. Finally $P_3 > 0$ contradicts the strict determinant cap above. No separate check of $\sigma_0 V_0 > 64$ is needed: the chosen comparison spectrum already has positive entries and the stated trace and inverse trace. □

The complete negative-sign parameter rectangle. The global spectral floor $S \succeq (5/13)I$ gives $R_{ii} \le 13/5$. Since $R_{ii} = 1/(2c)$, we have $c \ge 5/26$. The nominal phase-inverse inequality and the signed cap on τ give

$$c \le 1 + \tau - \frac{1}{2m} < 1 + \frac{1288}{3017} - \frac{5}{16} < \frac{5}{4}.$$

Together with Lemma 15, this contains every negative-sign counterexample in the closed rectangle

$$(m, c) \in \left[\frac{133}{100}, \frac{8}{5}\right] \times \left[\frac{5}{26}, \frac{5}{4}\right].$$

Using closed endpoints only enlarges the set to be excluded.
The canonical trace estimate also sharpens on a parameter interval $m \in [\ell, u] \subset [13/10, 8/5]$. The two bounds on $\operatorname{tr} S$ give

$$T + \tau \le \min\left\{\frac{5}{2} + \frac{(9 - 4\ell)(464 - 165\ell)}{1456}, \frac{-13991 + 16636u + 2640u^2}{5824}\right\}.$$

Indeed the first expression, viewed as a function of m, decreases on this interval, while the second increases. Their derivatives have numerators $-3341 + 1320m < 0$ and $16636 + 5280m > 0$, respectively. These endpoint bounds, the two Cauchy families, and the scaled trace rows above justify the corresponding refinements on the entire box.

The inverse-diagonal trace floor. The selected horizontal block also gives a lower trace bound depending only on $\chi = (H^{-1})_{88} = m - 2cw^2$. We prove

$$T \geq F(\chi), \qquad F(z) = \begin{cases} 6 - \dfrac{6}{z}, & 0 < z \leq \dfrac{9}{8}, \\ \dfrac{2}{7}\Big(1 + \sqrt{8 - \dfrac{7}{z}}\Big), & \dfrac{9}{8} \leq z \leq 4. \end{cases} \tag{37}$$

Here $0 < \chi \leq m \leq 8/5$, so the stated domain suffices. The proof uses only the pair contacts of H.

Proof of (37). For $0 \leq T \leq 1$ define

$$A_T = 1 - \frac{T}{6}, \qquad B_T = 1 + T - \frac{7T^2}{4}, \qquad q(T) = \min(A_T, B_T),$$
$$D_T = 1 + T - \frac{3T^2}{2}, \qquad E_T = \frac{1 - \frac{2T}{3}}{6}.$$

We first show $H \succeq q(T) e_i e_i^{\mathsf{T}}$ for every coordinate i. We have $q(T) \geq 1/4$, $D_T > 0$, $E_T > 0$; $q(T) = A_T$ for $T \leq 2/3$ and $q(T) = B_T$ for $T \geq 2/3$. Indeed $B_T - 1/4 = (1 - T)(7T/4 + 3/4)$, $D_T - 1/2 = (1 - T)(3T/2 + 1/2)$, and $E_T \geq 1/18$. The two quadratic discriminants needed below are nonnegative:

$$4(1 - q(T))D_T - T^2 = \begin{cases} \dfrac{T(2 - 3T)(1 + T)}{3}, & T \leq \dfrac{2}{3}, \\ 2T B_T(3T - 2), & T \geq \dfrac{2}{3}, \end{cases}$$

$$4\Big(1 + \frac{T}{6} - q(T)\Big)E_T - \frac{T^2}{9} = \begin{cases} \dfrac{T(6 - 7T)}{27}, & T \leq \dfrac{2}{3}, \\ \dfrac{T(-30 + 77T - 42T^2)}{54}, & T \geq \dfrac{2}{3}. \end{cases}$$

In the last case, $-30 + 77T - 42T^2 = 7T - 2 + 42(T - 2/3)(1 - T) \geq 0$. These identities verify the signs throughout $[0, 1]$. A quadratic $ax^2 - bxy + dy^2$ is nonnegative when $d > 0$ and $4ad \geq b^2$: multiply it by $4d$ and write $(2dy - bx)^2 + (4ad - b^2)x^2$. Fix a real test vector x, and let $u_j = |x_j|$, $X = \sum_j u_j^2$, $L = \sum_j u_j$ and $a_j = 2u_j^2 - u_j L$, with all indices from 1 to 8. The following two inequalities hold for every j:

$$X + T a_j \geq q(T) u_i^2, \tag{38}$$
$$6X + T(2X - L^2 - 4u_j^2 + 2u_j L) \geq 6q(T) u_i^2. \tag{39}$$

For (38), if $j = i$, the difference is

$$(B_T - q(T))u_i^2 + \sum_{k \neq i} \left(u_k - \frac{T u_i}{2}\right)^2 \geq 0.$$

If $j \neq i$, the difference is

$$(1-q(T))u_i^2 - Tu_iu_j + D_Tu_j^2 + \sum_{k\neq i,j}\left(u_k - \frac{Tu_j}{2}\right)^2 \geq 0,$$

by the first discriminant above. For (39), divide the difference by 6. If $j = i$, let $V = \sum_{k\neq i} u_k$ and $W = \sum_{k\neq i} u_k^2$; the result is

$$(A_T - q(T))u_i^2 + \left(1+\frac{T}{3}\right)W - \frac{TV^2}{6} \geq (A_T - q(T))u_i^2 + \left(1-\frac{5T}{6}\right)W \geq 0,$$

since $V^2 \leq 7W$. If $j \neq i$, instead sum V, W over $k \neq i, j$. The result is

$$A_Tu_j^2 + \left(1+\frac{T}{6}-q(T)\right)u_i^2 - \left(\frac{T}{3}\right)u_iV + E_TV^2 + \left(1+\frac{T}{3}\right)\left(W - \frac{V^2}{6}\right) \geq 0,$$

using $V^2 \leq 6W$ and the second discriminant.
Write $d_j = H_{jj} - 1$. The pair contacts give $d_j + d_k \geq 0$, $\sum_j d_j = T$, and

$$H[x] \geq X + \sum_j d_ja_j.$$

Indeed, bound each off-diagonal contribution below by $-(d_j + d_k)|x_jx_k|$ and collect the diagonal terms. For $T > 0$, let $f_j = X - q(T)u_i^2 + Ta_j$. The two tests say $f_j \geq 0$ and $2f_j \leq \sum_k f_k$. The pair-diagonal dual criterion therefore gives

$$0 \leq \sum_j d_jf_j = T\Big(X - q(T)u_i^2 + \sum_j d_ja_j\Big).$$

Division by T proves $H[x] \geq q(T)x_i^2$. At $T = 0$, the common pair-contact quadratic floor gives $H \succeq I$, with the same conclusion.
Apply this coordinate bound to $x = H^{-1}e_i$. If $z = (H^{-1})_{ii} > 0$, then $H[x] = z$ and $x_i = z$, hence $q(T) \leq 1/z$. For $z \leq 9/8$ and $T \leq 2/3$, this gives $T \geq 6 - 6/z$; for $T \geq 2/3$ the latter bound is automatic. For $z > 9/8$, the inequality excludes $T < 2/3$ and gives

$$8 - \frac{7}{z} \leq \left(\frac{7T}{2} - 1\right)^2.$$

Since $7T/2 - 1 \geq 0$, taking square roots proves the second branch of (37). Finally $F(z) \leq 1$ for $0 < z \leq 4$: the first branch is at most $2/3$, and in the second $\sqrt{8 - 7/z} \leq 5/2$. Thus $T \geq 1$ also satisfies the asserted floor. □

The floor F also admits a valid affine chord on every positive parameter interval, including intervals crossing $9/8$. Each branch is concave: $-1/z$ is concave on $z > 0$, and the second branch is the composition of the increasing concave square root with $8 - 7/z$ on its nonnegative domain. To check the junction, the line $\ell_F(z) = z - 11/24$ is an upper support at $z = 9/8$, where $F(9/8) = 2/3$. On the lower branch,

$$\ell_F(z) - \left(6 - \frac{6}{z}\right) = \frac{\left(z - \frac{9}{8}\right)\left(z - \frac{16}{3}\right)}{z} \geq 0.$$

On the upper branch, the square $(\sqrt{x} - 4/3)^2 \geq 0$ gives $\sqrt{x} \leq 3x/8 + 2/3$ for $x \geq 0$. Consequently

$$F(z) \leq \frac{4}{3} - \frac{3}{4z}, \qquad \ell_F(z) - \Big(\frac{4}{3} - \frac{3}{4z}\Big) = \frac{\big(z - \frac{9}{8}\big)\big(z - \frac{2}{3}\big)}{z} \geq 0.$$

An outer chord crossing $9/8$ therefore lies below F at the junction. On either side, it lies below the chord joining that junction to the corresponding endpoint, because their difference is affine and nonnegative at both ends. Branch concavity finishes the proof. Thus, for $0 < a < b$ and $a \leq z \leq b$,

$$F(a) + \frac{F(b) - F(a)}{b - a}(z - a) \leq F(z). \tag{40}$$

Either endpoint value may be replaced by any certified rational lower bound, since the interpolation weights are nonnegative. On the square-root branch, a proposed lower root $r \geq 0$ is certified by $r^2 \leq 8 - 7/a$ (or by the corresponding inequality at b). If $a = b$, the constant endpoint bound is used directly.

8 The negative inverse-entry branch

When $w < 0$, let $v_w = -w > 0$ locally; this symbol is distinct from the second dual minimum v. The preceding trace floor gives $T \geq L_{\mathrm{tr}}(m) + 4v_w s$.

The inverse-trace deficit as contact slacks. The inverse-trace deficit has an exact contact interpretation. Write $R = \sum_r \omega_r r r^{\mathsf{T}}$, where $\omega_r \geq 0$, $|r|^2 = 2$ and $S[r] \geq 2$. Taking traces in $SR = I$ and in the expansion of R gives, respectively, $9 = \sum_r \omega_r S[r]$ and $\operatorname{tr} R = 2\sum_r \omega_r$. Therefore

$$\delta = \sum_r \omega_r (S[r] - 2),$$

and every summand is nonnegative. For the selected cross pair, let $\alpha_i = \kappa_{i,+} + \kappa_{i,-} \geq |w|$; its difference of weights is w. Since $S_{ii} + h - 2 = 2\tau + 2s$ and $S_{i9} = -s$, this pair contributes exactly

$$2\alpha_i\tau + 2(\alpha_i - w)s \geq 2|w|\tau + 2(|w| - w)s.$$

Here only $\tau, s \geq 0$ and $\alpha_i \geq |w|$ are used. Discarding the other nonnegative contact slacks and setting $w = -v_w$ proves

$$\delta \geq 2v_w\tau + 4v_w s.$$

Substituting $w = -v_w$ in (34) gives

$$\Big(\frac{3}{2} + \frac{v_w}{2m}\Big)\tau + \Big(1 + \frac{2v_w}{m} + 4v_w\Big)s + \frac{14}{5}\delta < C(m).$$

The common caps proved above therefore apply, with the same actual slack.

Table 1. The complete reading path for `0030`. After the outer ranges, $\frac{13}{10} < m < \frac{8}{5}$. The remaining rows retain all other admissibility conditions.

region	argument	finite input
$m \le \frac{256}{243}$ or $m \ge \frac{13}{5}$	One-vector flag or inverse AM–GM; Section 3	None
$\frac{256}{243} \le m \le \frac{13}{10}$ or $\frac{8}{5} \le m \le \frac{13}{5}$	Signed-average bounds; Section 3	Three outer-range covers
$w < 0$	Negative-entry branch; Section 8	1 140 leaves
$T + \tau \le \frac{1}{8}$	Low-excess argument; Section 9	Reference identities, annulus and intermediate-radius trees
$w \ge 0, T + \tau \ge \frac{1}{8}, r_p \le \frac{1}{100}$	Two-range determinant bound; Section 10	The stated scalar comparisons
$w \ge 0, T + \tau \ge \frac{1}{8}, r_p \ge \frac{1}{100}$	Positive-entry branch; Section 10	10 498 leaves

The square moment of the original block. The remaining sign information comes from the original D_8 block A. Let $T_A = \mathrm{tr}(A - I)$, $D = 1 - h$, and $E_A = T_A + 8h - 8 \geq 0$. Pair contacts imply

$$\mathrm{tr}((A - I)^2) \leq T_A^2 + \frac{7}{2} E_A^2.$$

For a direct proof let $d_i = A_{ii} - 1 = D + \xi_i$. The cross contacts give $\xi_i \geq 0$, $\sum_i \xi_i = E_A$. Squaring the pair-contact bounds yields

$$\mathrm{tr}((A - I)^2) \leq \frac{T_A^2 + 8 \sum_i d_i^2}{2} \leq T_A^2 + \frac{7}{2} E_A^2,$$

where the last step uses $\sum_i \xi_i^2 \leq E_A^2$. Define the nonnegative deficit $\mathcal{E} = T_A^2 + \frac{7}{2} E_A^2 - \mathrm{tr}((A - I)^2)$. The bounds on T_A, E_A and $\mathcal{E}$ used below may come from linear combinations of the current necessary rows. They must apply to these actual quantities. In particular, if $\ell_T \leq T_A \leq u_T$ and $\ell_E \leq E_A \leq u_E$, then

$$\mathcal{E} \leq (\ell_T + u_T) T_A - \ell_T u_T + \frac{7}{2} \big((\ell_E + u_E) E_A - \ell_E u_E \big) - \mathrm{tr}((A - I)^2).$$

This follows twice from $(x - \ell)(u - x) \geq 0$. The right side is affine in the retained metric and moment coordinates, so any nonnegative linear combination certifying its upper bound also bounds the actual deficit. The same procedure certifies bounds on h, T_A, E_A, m.

The signed geometric-mean model. When $D > 0$, let $d_i = A_{ii} - 1 \geq D$ and define the signed geometric-mean model

$$B_{ii} = d_i, \qquad B_{ij} = \sigma_{ij} \sqrt{d_i d_j} \ (i \neq j), \qquad R_0 = A - I - B,$$

where σ_{ij} is the sign of A_{ij}, chosen as $+1$ when $A_{ij} = 0$, and $\sigma_{ii} = 1$. For $i < j$, let

$$a_{ij} = \frac{d_i + d_j}{2}, \qquad g_{ij} = \sqrt{d_i d_j}, \qquad z_{ij} = |A_{ij}|,$$
$$p_{ij} = \max(0, z_{ij} - g_{ij}), \qquad s_{ij} = \max(0, g_{ij} - z_{ij}).$$

The pair contacts give $0 \leq z_{ij} \leq a_{ij}$ and $a_{ij} \geq g_{ij} \geq D$. These imply

$$D s_{ij} \leq a_{ij}^2 - z_{ij}^2, \qquad 8 D p_{ij} \leq (d_i - d_j)^2. \tag{41}$$

For the first inequality, when $z_{ij} \leq g_{ij}$ use $D(g_{ij} - z_{ij}) \leq (g_{ij} + z_{ij})(g_{ij} - z_{ij}) \leq a_{ij}^2 - z_{ij}^2$; otherwise its left side is zero. For the second, when $z_{ij} > g_{ij}$ use

$$8D(z_{ij} - g_{ij}) \leq 8D(a_{ij} - g_{ij}) \leq 4(a_{ij} + g_{ij})(a_{ij} - g_{ij}) = (d_i - d_j)^2;$$

otherwise its left side is zero. Thus no denominator estimate is needed. Summing over unordered pairs and using the diagonal second-moment bound above gives

$$\sum_{i<j} (a_{ij}^2 - z_{ij}^2) = \frac{1}{2} \Big(\frac{T_A^2 + 8 \sum_i d_i^2}{2} - \mathrm{tr}((A - I)^2) \Big) \leq \frac{\mathcal{E}}{2},$$
$$\sum_{i<j} (d_i - d_j)^2 = 8 \sum_i d_i^2 - T_A^2 \leq 7 E_A^2.$$

Consequently

$$\sum_{i<j} s_{ij} \le s_b := \frac{\mathcal{E}}{2D}, \qquad \sum_{i<j} p_{ij} \le p_b := \frac{7E_A^2}{8D}, \qquad \sum_{i<j} |(R_0)_{ij}| \le \varepsilon_R := s_b + p_b.$$

The diagonal of R_0 is zero. For every x,

$$|x^{\mathsf{T}} R_0 x| \le \sum_{i<j} |(R_0)_{ij}|(x_i^2 + x_j^2) \le \varepsilon_R |x|^2.$$

The bad triangle. If all triangle products of signs equal $+1$, fix a vertex k and let $\sigma_i = \sigma_{ik}$. The triangle identity $\sigma_{ij}\sigma_{jk}\sigma_{ki} = 1$ and $\sigma_{ij}^2 = 1$ give $\sigma_{ij} = \sigma_i\sigma_j$, also when indices coincide. Hence $B = q_\sigma q_\sigma^{\mathsf{T}}$, where $(q_\sigma)_i = \sigma_i\sqrt{d_i}$, is rank one. Otherwise choose distinct i, j, k with $\sigma_{ij}\sigma_{jk}\sigma_{ki} = -1$. Let $a = \sigma_{ij}$ and $b = \sigma_{ik}$; then $\sigma_{jk} = -ab$. In the original coordinates of this triangle, set

$$u_\triangle = \big(\sqrt{d_i},\ -a\sqrt{d_j},\ -b\sqrt{d_k}\big)^{\mathsf{T}}.$$

Direct multiplication gives

$$B[\{i,j,k\}] = 2\,\mathrm{diag}(d_i, d_j, d_k) - u_\triangle u_\triangle^{\mathsf{T}}.$$

Here brackets denote a principal submatrix. On the plane $u_\triangle^{\mathsf{T}} x = 0$, its quadratic form is at least $2D|x|^2$. The displayed error bound, applied to the vector extended by zero, therefore gives

$$x^{\mathsf{T}} A[\{i,j,k\}]x \ge (1 + 2D - \varepsilon_R)|x|^2 \qquad (u_\triangle^{\mathsf{T}} x = 0).$$

For completeness, this forces at least two eigenvalues of A to be at least $c = 1+2D-\varepsilon_R$. If at most one were at least c, choose its unit eigenvector w, or any unit eigenvector if there were none. The two homogeneous equations $u_\triangle^{\mathsf{T}} x = 0$ and $w^{\mathsf{T}}\tilde{x} = 0$ in three unknowns have a nonzero solution; $\tilde{x}$ denotes extension by zero to all eight coordinates. Expansion in an orthonormal eigenbasis would then give $\tilde{x}^{\mathsf{T}} A\tilde{x} < c|\tilde{x}|^2$, contradicting the preceding inequality. Thus $\lambda_2(A) \ge 1 + 2D - \varepsilon_R$, with eigenvalues ordered from largest to smallest.

The rank-one branch. In the rank-one branch, splitting positive and negative off-diagonal errors gives, with $s_b = \mathcal{E}/(2D)$ and $p_b = 7E_A^2/(8D)$,

$$\det A \ge \Big(1 - \frac{p_b}{1 - s_b}\Big) \min\Big(1 + T_A,\ e^{-s_b^2/(1-s_b)}(1 + T_A + 2Ds_b)\Big)$$

whenever $s_b < 1$ and $p_b < 1 - s_b$.

Proof of the rank-one bound. After changing coordinate signs, write the error as $Z_+ - Z_-$, with both matrices entrywise nonnegative and zero on the diagonal. Their unordered entry sums are at most p_b and s_b, respectively. Let $s_0 = \sum_{i<j}(Z_-)_{ij} \le s_b$ and $q_i = \sqrt{d_i} \ge \sqrt{D}$. Then

$$\log\det(I - Z_-) \ge -\frac{s_0^2}{1 - s_0}, \qquad q^{\mathsf{T}}(I - Z_-)^{-1} q \ge T_A + 2Ds_0.$$

We give finite proofs of both estimates and of the error factor. For any symmetric zero-diagonal matrix Z, let $s = \frac{1}{2}\sum_{i,j}|Z_{ij}|$. The quadratic estimate above gives $\|Z\| \le s$. Also $|Z_{ij}| \le s$, since the equal terms $|Z_{ij}|$ and $|Z_{ji}|$ both occur in the sum. Consequently $\operatorname{tr} Z^2 = \sum_{i,j}|Z_{ij}|^2 \le 2s^2$. For $-s \le z \le s < 1$ the scalar bound

$$\log(1-z) \ge -z - \frac{z^2}{2(1-s)}$$

follows by differentiating the difference: its derivative is $z(s-z)/((1-z)(1-s))$, nonpositive to the left of zero and nonnegative to the right, and its value at zero is zero. Apply this to the eigenvalues of Z_- and use $\operatorname{tr} Z_- = 0$; this proves the logarithmic determinant bound. For the inverse, ordinary multiplication proves

$$(I - Z_-)^{-1} = I + Z_- + Z_-(I - Z_-)^{-1}Z_-.$$

The last term is positive semidefinite. Since $q_i q_j \ge D$,

$$q^{\mathsf{T}} Z_- q \ge D\sum_{i,j}(Z_-)_{ij} = 2Ds_0.$$

The rank-one determinant formula therefore gives

$$\det K_0 \ge e^{-s_0^2/(1-s_0)}(1 + T_A + 2Ds_0), \qquad K_0 := I - Z_- + qq^{\mathsf{T}} \succeq (1-s_b)I.$$

To bound the effect of adding Z_+, let $p_0 = \sum_{i<j}(Z_+)_{ij} \le p_b$, $\ell_i = \sum_j (Z_+)_{ij}$, and

$$P_+ = \frac{1}{2}(\operatorname{diag}\ell + Z_+), \qquad P_- = \frac{1}{2}(\operatorname{diag}\ell - Z_+).$$

Both matrices are positive semidefinite, because

$$x^{\mathsf{T}}(\operatorname{diag}\ell \pm Z_+)x = \frac{1}{2}\sum_{i,j}(Z_+)_{ij}(x_i \pm x_j)^2 \ge 0.$$

They satisfy $Z_+ = P_+ - P_-$ and $\operatorname{tr} P_- = p_0$. Set $a_0 = 1 - s_b > p_b \ge p_0$. Since every eigenvalue of P_- is at most its trace, $K_0 - P_- \succeq (a_0 - p_0)I \succ 0$. For $W = P_-^{1/2}K_0^{-1}P_-^{1/2} \succeq 0$ we have

$$\operatorname{tr} W = \operatorname{tr}(K_0^{-1}P_-) \le \frac{p_0}{a_0} < 1, \qquad \det(K_0 - P_-) = \det K_0 \det(I - W).$$

The determinant equality follows from $\det(I+UV) = \det(I+VU)$. The trace bound follows by applying $K_0^{-1} \preceq a_0^{-1}I$ to the columns of $P_-^{1/2}$ and summing. If $x_i \ge 0$ are the eigenvalues of W, induction on the number of factors gives $\prod_i(1-x_i) \ge 1 - \sum_i x_i$; every factor is nonnegative because $\sum_i x_i < 1$. Finally, adding $P_+ \succeq 0$ cannot decrease the determinant of the positive matrix $K_0 - P_-$: after congruence by its inverse square root all eigenvalues are at least one. Thus

$$\det(K_0 + Z_+) \ge \left(1 - \frac{p_0}{a_0}\right)\det K_0 \ge \left(1 - \frac{p_b}{a_0}\right)\det K_0.$$

It remains to replace s_0 by its containing interval. If $s_b = 0$ then $s_0 = 0$. Otherwise $t = s_0/s_b \in [0, 1]$ and

$$-\frac{s_0^2}{1-s_0} - t\Big(-\frac{s_b^2}{1-s_b}\Big) = \frac{s_0(s_b - s_0)}{(1-s_b)(1-s_0)} \geq 0.$$

Concavity of $\log$ gives the same chord inequality for $\log(1 + T_A + 2Ds_0)$. Adding the two inequalities bounds their sum below by the smaller endpoint value. Exponentiating gives the stated minimum, including both endpoints. □

The simpler bound

$$\det A \geq (1 - \varepsilon_R)(1 + T_A + \varepsilon_R) \qquad (\varepsilon_R < 1) \tag{42}$$

has the same positive-matrix proof. Split each of Z_-, Z_+ into the two positive semidefinite matrices $(\operatorname{diag} \ell \pm Z)/2$ just used. Collecting signs gives $A = I + qq^{\mathsf{T}} + P - N$, with $P, N \succeq 0$ and $\operatorname{tr} P = \operatorname{tr} N = a_R \leq s_b + p_b = \varepsilon_R$. Since $|q|^2 = T_A$, the nonnegative eigenvalues of $qq^{\mathsf{T}} + P$ give

$$\det(I + qq^{\mathsf{T}} + P) \geq 1 + T_A + a_R.$$

Indeed, expanding the product of one plus these eigenvalues leaves only nonnegative terms beyond one and their sum. The subtraction argument above, now with quadratic floor one, gives $\det A \geq (1 - a_R)(1 + T_A + a_R)$. For $0 \leq a_R \leq \varepsilon_R < 1$ and $T_A \geq 0$, this is at least the right side of (42), since the difference is $(\varepsilon_R - a_R)(T_A + \varepsilon_R + a_R) \geq 0$. Thus either coherent determinant bound exceeding $C_A(m)$ forces an actual bad triangle; the region test follows.

The other branch: a logarithmic minorant. For the other branch, choose $0 < \ell < r < t$ and let

$$c_0 = \frac{1}{2(\ell r + \ell t + rt)}, \qquad a = 2c_0(\ell + r + t), \qquad b = 2c_0 \ell r t.$$

The residual $f(x) = \log x - ax + b/x + c_0 x^2$ satisfies

$$f'(x) = \frac{2c_0(x - \ell)(x - r)(x - t)}{x^2}.$$

Its successive monotonicity intervals are decreasing, increasing, decreasing and increasing, with endpoints ℓ, r, t. Consequently $f(x) \geq \min(f(\ell), f(t))$ for every $x > 0$. A finite check $q_0 \leq f(\ell), f(t)$ therefore proves the global minorant $\log x \geq ax - b/x - c_0 x^2 + q_0$.

For a closed positive interval $I = [L, U]$, a stronger bound $f \geq q_0 + \eta$ on I requires only the checks at L, U and at those of ℓ, t that belong to I. Indeed, partitioning I at ℓ, r, t reduces each piece to its minimum endpoint; the middle critical point r is a maximum. This also covers singleton intervals and critical points at the boundary.

The bad-triangle argument supplies two distinct eigenvalue indices with values at least a threshold B. Choose an index of a largest eigenvalue; at least one of those

two indices remains distinct from it. The largest value is at least $\max(B, \operatorname{tr} A/8)$, the other is at least B, and the contact ceiling bounds both above by $1 + 2T_A$. Let I_1, I_2 contain these respective ranges, and let $\eta_1, \eta_2 \geq 0$ be verified residual improvements on them. Summing the eight scalar inequalities, with an improvement at these two distinct indices, gives

$$\log\det A \geq a \operatorname{tr} A - b \operatorname{tr} A^{-1} - c_0 \operatorname{tr}(A^2) + 8q_0 + \eta_1 + \eta_2.$$

Here $\operatorname{tr}(A^2) = 8 + 2T_A + \operatorname{tr}((A-I)^2)$; substituting the linked trace identities expresses the row in the variables of the cover. The two indices must be distinct; the corresponding eigenvalues may coincide. Every residual check is a finite logarithm comparison with an outward rational bound. The continuous implication uses no approximate eigenvalues.

The sharper two-atom intervals. The sharper two-atom intervals used in the cover follow from the same square moment. Write the eigenvalues of $A - I$ as z_j, with sum T_A and square sum V_A. For any two distinct indices with values $x \geq y$, Cauchy–Schwarz on the other six gives

$$(T_A - x - y)^2 \leq 6(V_A - x^2 - y^2). \tag{43}$$

If $y \leq T_A/8$ there is nothing to prove. Otherwise write $x = y + z$ with $z \geq 0$. Rearranging (43) gives

$$(8y - T_A)^2 + 4(16y - 2T_A)z + 28z^2 \leq 24V_A - 3T_A^2.$$

The last two terms on the left are nonnegative. Hence, if $0 \leq T_A \leq T_+$, $0 \leq E_A \leq E_+$ and a rational $r_+ \geq 0$ satisfies $r_+^2 \geq 21T_+^2 + 84E_+^2$, the smaller of the two eigenvalues of A is at most

$$1 + \frac{T_+ + r_+}{8}.$$

Here we used $V_A \leq T_A^2 + (7/2)E_A^2$ and the nonnegative upper bounds before squaring them.

For the lower threshold, suppose the region gives $0 < D_- \leq D$, $E_A \leq E_+$ and $\mathcal{E} \leq \mathcal{E}_+$. The error bounds already proved imply

$$\lambda_2(A) \geq k_* := 1 + 2D_- - \frac{\mathcal{E}_+}{2D_-} - \frac{7E_+^2}{8D_-}$$

whenever the coherent branch has been excluded. This follows by bounding each error mass separately using $D \geq D_-$; no independent choice of the two eigenvalues is made. Bounds on $\mathcal{E}$ can use secants for T_A^2 and E_A^2: for $a \leq z \leq b$, the difference $(a+b)z - ab - z^2 = (z-a)(b-z)$ is nonnegative. This also covers $a = b$.

To retain the correlation between the two eigenvalues, partition a containing interval for the smaller one into closed cells $[L_j, U_j]$, with $L_j \geq 1$. Let $b_j = L_j - 1$ and $V_+ = T_+^2 + (7/2)E_+^2$. Choose a rational $r_j \geq 0$ with

$$r_j^2 \geq 42V_+ - 6(T_+ - b_j)^2 - 42b_j^2.$$

Then the larger eigenvalue, whenever the smaller belongs to this cell, is at most

$$X_j = 1 + \max\Big(T_+ - b_j,\ \frac{T_+ - b_j + r_j}{7}\Big).$$

To verify this, again write the two eigenvalues as $1 + x \geq 1 + y$. We have $y \geq b_j \geq 0$. If $x \leq T_+ - b_j$ the first term suffices. Otherwise $0 < x + b_j - T_+ \leq x + y - T_A$, so (43) and $V_A \leq V_+$ give

$$(x + b_j - T_+)^2 \leq 6(V_+ - x^2 - b_j^2).$$

Rearranging yields $(7x - (T_+ - b_j))^2 \leq 42V_+ - 6(T_+ - b_j)^2 - 42b_j^2 \leq r_j^2$, which proves the second term. Thus a cell with $X_j < L_j$ contains no actual pair. In every retained cell, the two distinct atoms lie in $[L_j, U_j]$ and $[L_j, X_j]$. The finite residual checks on these two intervals supply a summed improvement. Its minimum over all retained cells is valid globally, because the closed cells cover the entire smaller-atom interval, including every boundary. If all cells are excluded, no actual pair exists. A singleton interval forms one closed cell.
The complete negative-entry cover has 2 279 nodes and 1 140 leaves, with 28 576 exact bounds and no pending region.

9 The complete low-excess region

We retain the selected coordinates of (16) for an actual residual candidate in the central range $13/10 \leq m \leq 8/5$. Write Q_{10} for its first rank-ten Gram matrix; its standing cutoff is $(\det Q_{10})^{11} < 4^{11}972^9$. Throughout this section assume

$$T \geq 0, \qquad \tau \geq 0, \qquad T + \tau \leq \frac{1}{8}, \qquad H = I + E, \qquad c = 1 + \tau - H[p] > 0.$$

Here $H[p] = p^{\mathsf{T}} H p$ and $r_p = |p|^2$. The selected-contact construction gives $\operatorname{tr} H = 8 + T$, $2|H_{jl}| \leq H_{jj} + H_{ll} - 2$ for $j \neq \ell$, and $p_8 \geq 0$. The selected physical metric and reference coordinate map are denoted by

$$\mathcal{B} = \mathcal{B}(H, p, \tau) = \begin{pmatrix} H & Hp \\ p^{\mathsf{T}} H & 1 + \tau \end{pmatrix}, \qquad Q = B^{\mathsf{T}} \mathcal{B} B, \qquad (\det B)^2 = 1024,$$

where Q is the actual rank-nine Gram matrix. Let $\alpha_j = 2\sqrt{2}|p_j|$, so that $\sum_j \alpha_j^2 = 8r_p$ and $\kappa_p = \alpha_8 = \max_j \alpha_j$. The only finite inputs used below are identities and inequalities for fixed reference vectors, rational polynomial certificates, and finite closed-box covering trees.

Two inverse profiles for the same contact cone. Set

$$F(t) = 1 + t - \frac{7}{4}t^2, \qquad \mathcal{I}(t) = 6 + \frac{2 + t}{F(t)},$$
$$\mathcal{D}_-(t) = \Big(1 - \frac{t}{6}\Big)\Big(1 - \frac{5t}{6}\Big)\Big(1 + \frac{t}{3}\Big)^6.$$

For $0 \le t \le 1/8$ all denominators in this paragraph are positive. The decomposition in Lemma 9 has two types of vertices. At an arrow vertex of trace t, write the off-diagonal column as g. Then $|g_j| \le t/2$ and

$$d = 1 + t - |g|^2 \ge F(t) \ge 1.$$

Block inversion gives inverse diagonal $1/d$ at the distinguished coordinate and $1+g_j^2/d$ at another coordinate. Its inverse trace is $7+(1+|g|^2)/d = 6+(2+t)/d$. Consequently the arrow vertices satisfy

$$\det H \ge F(t), \qquad (H^{-1})_{jj} \le 1 + \frac{t^2}{4F(t)}, \qquad \operatorname{tr} H^{-1} \le \mathcal{I}(t). \tag{44}$$

At a negative-coordinate vertex let $v = t/6$. The isolated coordinate is $1 - v$, and the other seven coordinates have diagonal $1 + v$ and all off-diagonal magnitudes at most v. Lemma 8 gives determinant at least $\mathcal{D}_-(t)$. To bound one inverse diagonal in the seven-dimensional block, delete that coordinate. The remaining six-dimensional block is at least $(1 - 4v)I$ and the deleted column has square norm at most $6v^2$. Its Schur complement is therefore at least

$$1 + v - \frac{6v^2}{1 - 4v} = \frac{(1 + 2v)(1 - 5v)}{1 - 4v}.$$

Thus that inverse diagonal is at most $B(v) = (1-4v)/((1+2v)(1-5v))$. The identity

$$(1 + 2v)(1 - 5v) - (1 - 4v)(1 - v) = 2v(1 - 7v) \ge 0$$

shows $B(v) \le 1/(1 - v)$. Moreover

$$\mathcal{I}(6v) - \Big(\frac{1}{1 - v} + 7B(v)\Big) = \frac{v^2(112 - 1204v + 4200v^2 - 3780v^3)}{(1 + 6v - 63v^2)(1 - v)(1 + 2v)(1 - 5v)} \ge 0 : \tag{45}$$

the numerator is nonnegative because $0 \le v \le 1/48$, $112 - 1204v > 0$, and $4200 - 3780v > 0$. These vertices therefore satisfy

$$\det H \ge \mathcal{D}_-(t), \qquad (H^{-1})_{jj} \le \frac{1}{1 - \frac{t}{6}}, \qquad \operatorname{tr} H^{-1} \le \mathcal{I}(t). \tag{46}$$

The arrow diagonal bound is also at most $1/(1 - t/6)$: after multiplication by the positive denominators the required inequality is $t(2 - t - 3t^2)/3 \ge 0$. Lemma 10 transfers the common inverse diagonal and trace bounds to every contact matrix. At $t = 0$ the contact inequalities give $E = 0$ directly. Hence

$$\frac{1}{1 + t} \le (H^{-1})_{jj} \le \frac{1}{1 - \frac{t}{6}}, \qquad \operatorname{tr} H^{-1} \le \mathcal{I}(t). \tag{47}$$

For the lower diagonal bound use $H_{jj} \le 1 + t$ (each contact vertex has this bound), together with $1 \le H_{jj}(H^{-1})_{jj}$. More significantly, for $b \ge 0$, $C > 0$, and $D > 0$, the two *joint* inequalities

$$D \le \frac{F(t)}{\Big[b + C\Big(1 + \frac{t^2}{4F(t)}\Big)\Big]^2}, \qquad D \le \frac{\mathcal{D}_-(t)}{\Big[b + \frac{C}{1 - \frac{t}{6}}\Big]^2} \tag{48}$$

imply $D \le \det H/[b + C(H^{-1})_{jj}]^2$. Indeed each vertex satisfies the appropriate inequality and Lemma 10, with power two, applies to their common convex decomposition.

A global bound on the phase coordinates. Cauchy–Schwarz on the eight positive eigenvalues gives $\operatorname{tr} H^{-1} \ge 64/(8+T)$. Together with (17), this proves

$$1 + r_p - \frac{\kappa_p}{2} \le \beta(T)c, \qquad \beta(T) = \frac{8+9T}{8+T} \le \frac{73}{65}. \tag{49}$$

It also gives the weaker factor $1+T$, since $64/(8+T) \ge 8 - T$. By Lemma 7, $c \le 1 + \tau - (1-T)r_p$. Combining this bound with the weaker version of (49) gives

$$(4 - 2T^2)r_p - 2(T + \tau + T\tau) \le \kappa_p.$$

Since $T^2 \le 1/64$ and $T\tau \le 1/256$,

$$\frac{127}{32} r_p - \frac{33}{128} \le \kappa_p, \qquad \kappa_p^2 \le 8 r_p. \tag{50}$$

These imply $r_p < 2/3$. In fact, if $r_p = 2/3 + x$, $x \ge 0$, then the left side of (50) is positive and

$$\Big(\frac{127}{32} r_p - \frac{33}{128}\Big)^2 - 8 r_p = \frac{16129}{1024} x^2 + \frac{67307}{6144} x + \frac{54457}{147456} > 0,$$

contrary to (50). In particular $\kappa_p < 22/7$. The exact old-normal identity and (47) give

$$m = (H^{-1})_{88} + \frac{(2-\kappa_p)^2}{8c} \le \frac{48}{47} + \frac{(2-\kappa_p)^2}{8c}.$$

If $\kappa_p \ge 6/7$, the preceding upper bound $22/7$ yields $X := (2-\kappa_p)^2 \le 64/49$. The sharper (49) and $\kappa_p^2 \le 8 r_p$ imply $X + 4 \le 8(73/65)c$, so

$$\frac{X}{8c} \le \frac{73}{65} \frac{X}{X+4} \le \frac{73}{65} \frac{16}{65}.$$

But $48/47 + (73/65)(16/65) < 13/10 \le m$, a contradiction. Thus

$$\boxed{0 \le \alpha_j \le \kappa_p < \frac{6}{7} \quad (j = 1, \dots, 8).} \tag{51}$$

In particular $w > 0$ throughout this low-excess region.

The directional flag, including its normalisation. The fixed reference data contain the sixteen integral covectors whose physical images are $q_{j,\varepsilon} = (\varepsilon e_j/\sqrt{2}, 1/2)$, $\varepsilon \in \{-1, 1\}$. Their integrality is a finite vector identity. The squared covolume of the phase-coordinate reference basis is 1024; hence the actual rank-nine determinant is $e = 1024 c \det H$. Block inversion gives

$$q_{j,\varepsilon}^{\mathsf{T}} \mathcal{B}^{-1} q_{j,\varepsilon} = \frac{(H^{-1})_{jj}}{2} + \frac{\left(\frac{1}{2} - \frac{\varepsilon p_j}{\sqrt{2}}\right)^2}{c}.$$

Choose $\varepsilon p_j = |p_j|$ and apply the scalar flag inequality. For every j,

$$\Delta \geq \frac{512 \det H \, c^3}{\left[\frac{c(H^{-1})_{jj}}{2} + \left(\frac{1}{2} - \frac{\alpha_j}{4}\right)^2\right]^2}. \tag{52}$$

For fixed $d \geq 0$, $k > 0$, $b \geq 0$, the function $512dc^3/(kc/2+b)^2$ is nondecreasing for $c > 0$: its logarithmic derivative when $d > 0$ is $(kc/2 + 3b)/(c(kc/2+b)) > 0$; for $d = 0$ the assertion is immediate. Increasing b or k weakens its lower bound.

The whole range $r_p \geq 1/5$. Here let $u_* = 6/5 - \kappa_p/2 > 0$. Equation (49) gives $u_* \leq \beta(T)c$. The identity

$$F(T)(8+T) - \frac{18265}{18688}(8+9T) = \left(\frac{1}{8} - T\right)\left(\frac{7}{4}T^2 + \frac{423}{32}T + \frac{423}{292}\right) \geq 0$$

proves $F(T)/\beta(T) \geq 18265/18688$. Use (52) at $j = 8$, the determinant bound $\det H \geq F(T)$, (47), and the monotonicities just proved. Since $\beta(T) \leq 73/65$ and $(H^{-1})_{88} \leq 48/47$, they give

$$\Delta \geq \frac{512\left(\frac{18265}{18688}\right)u_*^3}{\left[\frac{48}{47}\frac{u_*}{2} + \frac{73}{65}\left(\frac{1}{2} - \frac{\kappa_p}{4}\right)^2\right]^2} \geq 972. \tag{53}$$

The final inequality is a fixed quartic, valid for $0 \leq \kappa_p \leq 6/7$. For an explicit exact check, let

$$P(X) = \frac{48324203937}{485317300} - \frac{203963726349}{1213293250}X + \frac{267617475351}{4853173000}X^2 + \frac{1229253047}{103259000}X^3 - \frac{94531131}{17576000}X^4.$$

Expansion gives

$$P(X) = 512\frac{281}{256}\left(\frac{6}{5} - \frac{X}{2}\right)^3 - 972\frac{73}{65}\Big[\frac{48}{47}\cdot\frac{1}{2}\left(\frac{6}{5} - \frac{X}{2}\right) + \frac{73}{65}\left(\frac{1}{2} - \frac{X}{4}\right)^2\Big]^2.$$

The five degree-four Bernstein coefficients of P on $[0, 6/7]$ are positive. Equivalently, if $P(X) = \sum_{j=0}^{4} p_j X^j$, the five explicit rational numbers

$$b_k = \sum_{j=0}^{k} p_j \left(\frac{6}{7}\right)^j \frac{\binom{k}{j}}{\binom{4}{j}}, \qquad 0 \leq k \leq 4,$$

are positive, a finite rational calculation included in the verified proof. The identity $P((6/7)t) = \sum_{k=0}^{4} b_k \binom{4}{k} t^k (1-t)^{4-k}$ then proves nonnegativity on the entire interval. Multiplying (53) by its positive squared denominator is exactly $P(\kappa_p) \geq 0$, divided by $73/65$.

The intermediate range $1/10 \le r_p \le 1/5$. This part needs a finite two-dimensional cover, but only one general continuous argument. Choose one of the four tuples

$$(a_{\mathrm{d}}, b_{\mathrm{d}}, \lambda_-, \lambda_+) = \left(-\frac{4}{3}, \frac{7}{3}, \frac{7}{8}, 1\right), \quad \left(\frac{4}{3}, 0, \frac{3}{5}, \frac{7}{8}\right), \\ \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{3}{5}\right), \quad \left(2, 0, \frac{1}{8}, \frac{1}{3}\right)$$

for which $\lambda_-\kappa_p^2 \le r_p \le \lambda_+\kappa_p^2$. Such a tuple exists because $\kappa_p^2/8 \le r_p \le \kappa_p^2$. The directional proof in (28) gives $E[p] \le T(a_{\mathrm{d}} r_p + b_{\mathrm{d}}\kappa_p^2)$. Let $a_T = 1 + a_{\mathrm{d}}T > 0$ and $b_T = b_{\mathrm{d}}T$. Then

$$c \ge 1 - a_T r_p - b_T\kappa_p^2, \qquad \beta(T)c \ge 1 + r_p - \frac{\kappa_p}{2}.$$

Eliminating r_p by adding the first inequality to a_T times the second, or using the four available bounds on r_p separately, proves that c is at least each of

$$\frac{1 + a_T - \frac{a_T\kappa_p}{2} - b_T\kappa_p^2}{1 + a_T\beta(T)}, \qquad 1 - (a_T\lambda_+ + b_T)\kappa_p^2, \\ 1 - \frac{a_T}{5} - b_T\kappa_p^2, \qquad \frac{1 + \lambda_-\kappa_p^2 - \frac{\kappa_p}{2}}{\beta(T)}, \qquad \frac{\frac{11}{10} - \frac{\kappa_p}{2}}{\beta(T)}. \tag{54}$$

For the first candidate multiply numerator and denominator by $8 + T$; for the last two multiply both by $8 + T$. Thus every candidate has a specified rational polynomial numerator $N_j(T, \kappa_p)$ and a positive denominator $Q_j(T)$:

j	N_j	Q_j
0	$(1 + a_T - a_T\kappa_p/2 - b_T\kappa_p^2)(8 + T)$	$(8 + T) + a_T(8 + 9T)$
1	$1 - (a_T\lambda_+ + b_T)\kappa_p^2$	1
2	$1 - a_T/5 - b_T\kappa_p^2$	1
3	$(1 + \lambda_-\kappa_p^2 - \kappa_p/2)(8 + T)$	$8 + 9T$
4	$(11/10 - \kappa_p/2)(8 + T)$	$8 + 9T$

Set $q = 1 - T/6$, $L = F(T) + T^2/4$ and $b_0 = (1/2 - \kappa_p/4)^2 > 0$. A leaf chooses an index j_+ for the arrow vertices and, independently, an index j_- for the negative-coordinate vertices. Its two polynomial checks are

$$2048F(T)^3N_{j_+}^3 - 972Q_{j_+}\left(LN_{j_+} + 2F(T)b_0Q_{j_+}\right)^2 \ge 0, \\ 2048\mathcal{D}_-(T)q^2N_{j_-}^3 - 972Q_{j_-}\left(N_{j_-} + 2qb_0Q_{j_-}\right)^2 \ge 0. \tag{55}$$

Each check itself forces its chosen numerator to be positive. A negative numerator makes the positive-coefficient cubic negative; a zero numerator leaves a strictly negative squared term because $b_0, Q_j, F(T), q > 0$. No separate positivity assumption on the five candidates is used. For positive F_0, q, N, Q, L_0, b_0, direct cancellation gives

$$\frac{512F_0\left(\frac{N}{Q}\right)^3}{\left[\frac{NL_0}{2Qq} + b_0\right]^2} = \frac{2048F_0q^2N^3}{Q(L_0N + 2qb_0Q)^2}. \tag{56}$$

Consequently (55) proves the two flag ratios at their respective positive candidates. By (54) and the monotonicity following (52), both remain valid at the *same actual height* c. Apply the joint vertex implication (48), with $C = c/2$, and then (52). This proves $\Delta \geq 972$ on the entire leaf.

The four covering trees have common root $[0, 1/8] \times [0, 6/7]$ in (T, κ_p). If $[\kappa_-, \kappa_+]$ is its interval for κ_p, a leaf can instead be empty: it suffices that $\lambda_+ \kappa_+^2 < 1/10$, or that $\kappa_- \geq 0$ and $1/5 < \lambda_- \kappa_-^2$. Otherwise it supplies the two checks (55). Each polynomial check is an exact rational identity

$$P(T, \kappa_p) = \sum_{j,k} c_{jk} t^j (1-t)^{d-j} s^k (1-s)^{e-k}, \qquad c_{jk} \geq 0,$$

where t, s are the affine coordinates of that closed leaf rectangle and its widths are positive. Binomial factors are absorbed in c_{jk}. Since $0 \leq t, s \leq 1$, this proves $P \geq 0$ everywhere in the rectangle. The reported trees have 188 nodes and 96 leaves; the checker verifies their complete partitions and these identities.

The annulus: shared bounds. It remains to treat $1/100 \leq r_p \leq 1/10$. A box specifies rational intervals

$$T \in [t_0, t_1], \qquad \tau \in [z_0, z_1], \qquad r_p \in [r_0, r_1], \qquad \kappa_p \in [k_0, k_1].$$

Here $0 \leq t_0 \leq t_1 \leq 1/8$, $0 \leq z_0 \leq z_1 \leq 1/8$, $0 \leq k_0 \leq k_1 \leq 6/7$, and $0 < r_0 \leq r_1 \leq 1/10$. Write $\alpha = \sum_j \alpha_j$, where $\alpha_j = 2\sqrt{2}|p_j|$ as above. We first explain every bound used by the box tests.

For nonnegative numbers $a_j \leq k$, the following capped-sum rule will be used repeatedly:

$$\sum_j a_j \leq nk + q, \quad 0 \leq q \leq k \quad \Longrightarrow \quad \sum_j a_j^2 \leq nk^2 + q^2. \tag{57}$$

Here n is any nonnegative integer. To prove it, maximize the sum of squares on the compact set with the prescribed upper bound on the sum. If two coordinates lie strictly between 0 and k, transfer mass from the smaller to the larger until one reaches an endpoint. The sum is unchanged and the sum of squares increases: transferring $d \geq 0$ from a to $b \geq a$ changes it by $2d(b-a) + 2d^2 \geq 0$. Repeat. At most one coordinate remains strictly between 0 and k. Filling the available sum, when there are enough coordinates, gives n full coordinates and one coordinate q; if there are fewer coordinates, the assertion is weaker than their trivial bound. This proves (57), including all zero and equality cases. Consequently, if $k_1 > 0$, $0 \leq q \leq k_1$, and $q^2 \leq 8r_0 - nk_1^2$, then

$$\alpha \geq \max\left\{ \frac{8r_0}{k_1},\ nk_1 + q \right\}. \tag{58}$$

The first term follows from $\alpha_j^2 \leq k_1 \alpha_j$. For the second, a strict smaller sum would give a strict smaller square sum by the same mass-transfer argument, contradicting $\sum \alpha_j^2 = 8r_p \geq 8r_0$. Upper bounds valid without further geometry are $\alpha \leq 8k_1$ and $\alpha \leq 8\sqrt{r_1}$, the latter by Cauchy–Schwarz. A rational upper endpoint for the square root suffices.

There is also a useful parity constraint. Let i be the selected coordinate, so $p_i \geq 0$ and $\alpha_i = \kappa_p$. For every sign vector $\sigma \in \{\pm 1\}^8$ with $\sigma_i = 1$ and $\prod_j \sigma_j = 1$, the finite contact checks assert

$$\big((\sigma - 6e_i) \cdot v_{\mathrm{hor}}\big) v_9 \geq 0 \tag{59}$$

for every contact vector v of the reference form, in the selected physical coordinates. This is a finite list of scalar checks. The contact marking expresses the inverse physical metric as a nonnegative sum of vv^t. Its last column above the diagonal is $-p/c$. Multiplying that column by $\sigma - 6e_i$ in (59) therefore gives $\sigma \cdot p \leq 6p_i$. Choose $\sigma_j p_j = |p_j|$, taking $\sigma_j = 1$ when $p_j = 0$. If this preferred sign vector has even parity, then $\alpha \leq 6\kappa_p \leq 6k_1$ and hence

$$r_p = \frac{1}{8} \sum \alpha_j^2 \leq \frac{1}{8} k_1 \alpha \leq \frac{3}{4} k_1^2.$$

Thus $r_0 > 3k_1^2/4$ forces odd parity. The convention at a zero coordinate causes no exception: the same implication applies to the chosen preferred sign vector.

The fixed odd-spinor data provide an integral lattice vector with physical coordinates $(-\sigma/\sqrt{2}, 1)$ for every odd preferred sign vector with $\sigma_i = 1$. Let $E = H - I$ and

$$C_\sigma = \sum_{j,\ell} E_{j\ell} \sigma_j \sigma_\ell \Big(1 - \frac{\alpha_j + \alpha_\ell}{2}\Big).$$

The value of this vector is exactly $5 + \tau - \alpha/2 + C_\sigma/2$; its minimum condition gives $C_\sigma \geq \alpha - 2 - 2\tau$. Since $\alpha_j \leq k_1 < 1$, all the factors $1 - (\alpha_j + \alpha_\ell)/2$ are nonnegative. Applying the pair-contact inequalities to the off-diagonal terms yields

$$C_\sigma \leq \sum_j E_{jj} \Big(8 - 4\alpha_j - \frac{\alpha}{2}\Big). \tag{60}$$

The diagonal-cone rule behind this and later estimates is

$$\sum_j d_j b_j \leq T \max\Big\{\max_j b_j,\ \frac{\sum_j b_j - 2\min_j b_j}{6}\Big\}, \qquad \sum_j d_j = T, \quad d_j + d_\ell \geq 0. \tag{61}$$

It follows directly from the diagonal decomposition in Lemma 9: its vertices are a single coordinate of mass T, and one coordinate $-T/6$ with the other seven $T/6$. In particular, if every $\alpha_j \geq \ell$, (60)–(61) give

$$C_\sigma \leq T \max\Big\{8 - 4\ell - \frac{\alpha}{2},\ 8 + \frac{4k_1}{3} - \frac{7\alpha}{6}\Big\}. \tag{62}$$

For the trace-only mass bound below, one can also use the first entry with $\ell = 0$ once $\alpha \geq 2k_1$; alternatively substitute $\alpha_j \geq \alpha - 7k_1$ in the first entry of (62). Combining with the lower bound on C_σ shows that α is at most one of the following certified

bounds:

$$\begin{gathered}\max_{t\in\{t_0,t_1\}}\frac{2+2z_1+8t}{1+\frac{t}{2}},\\ \max_{t\in\{t_0,t_1\}}\max\Big\{\frac{2+2z_1+(8-4\ell)t}{1+\frac{t}{2}},\ \frac{2+2z_1+\big(8+\frac{4k_1}{3}\big)t}{1+\frac{7t}{6}}\Big\},\\ \max_{t\in\{t_0,t_1\}}\max\Big\{\frac{2+2z_1+(8+28k_1)t}{1+\frac{9t}{2}},\ \frac{2+2z_1+\big(8+\frac{4k_1}{3}\big)t}{1+\frac{7t}{6}}\Big\}.\end{gathered}\tag{63}$$

These endpoint maxima are sufficient because a ratio of two affine functions with positive denominator is monotone or constant. For the middle bound choose any rational $\ell\ge 0$ with $\ell^2\le\max(0,8r_0-7k_1^2)$: each coordinate then satisfies $\alpha_j\ge\ell$ by subtracting the other seven squares. The first, simpler bound follows from (62) with $\ell=0$, since $8+4k_1/3-7\alpha/6\le 8-\alpha/2$ whenever $\alpha\ge 2k_1$; when this condition fails its proposed upper bound is already greater than α, because that bound is at least 2 and $2k_1<2$. Thus it holds in both cases.

Using the same odd vector gives the following joint Schur bound. For $j\neq\ell$ apply the pair-contact inequalities with the nonnegative weights

$$w_{jl}=\frac{\alpha_j\alpha_\ell}{8}+\frac{\alpha}{32}\Big(1-\frac{\alpha_j+\alpha_\ell}{2}\Big).$$

The row sum, including the diagonal coefficient, is $\alpha/4-\alpha^2/64$, independent of j. Therefore

$$E[p]+\frac{\alpha}{32}C_\sigma\le T\Big(\frac{\alpha}{4}-\frac{\alpha^2}{64}\Big),\qquad E[p]\le\frac{\alpha\big\{2+2\tau+8T-\big(1+\frac{T}{2}\big)\alpha\big\}}{32}.\tag{64}$$

Because $0\le\alpha\le 8k_1<8$, the last expression increases with T and τ. If (58) supplies $\alpha\ge L_*$, let

$$W_*=2+2z_1+8t_1,\qquad V_*=1+\frac{t_1}{2},\qquad a_*=\max\Big\{L_*,\frac{W_*}{2V_*}\Big\}.$$

Then $E[p]\le a_*(W_*-V_*a_*)/32$, by maximizing a concave quadratic on $[L_*,\infty)$. Independently, the four directional bounds in (28) bound $E[p]$ on the box by t_1J, where J is an upper bound for all applicable affine expressions in (r_p,κ_p^2). Such a bound is certified by clipping the rectangle against each of the four linear regime conditions and checking its vertices; all vertices and checks are rational. It may be combined with $E[p]\le 2t_1r_1$ and, when odd parity is forced, with (64). Taking the minimum preserves bounds for the same actual $E[p]$.

Let P_* be any resulting rational upper bound for $E[p]$ and define

$$\begin{gathered}c_0=\max\Big\{1+z_0-r_1-P_*,\ \Big(1+r_0-\frac{k_1}{2}\Big)\frac{8+t_1}{8+9t_1}\Big\},\\ c_1=1+z_1-(1-t_1)r_0,\qquad F_0=F(t_0),\quad F_1=F(t_1),\\ d_1=\frac{1}{1-\frac{t_1}{6}},\qquad K_1=\max\{\mathcal{I}(t_0),\mathcal{I}(t_1)\}.\end{gathered}\tag{65}$$

The Schur identity, the marking bound and the spectral floor imply $c_0 \le c \le c_1$. The preceding common profiles imply $\det H \ge F_0 > 0$, $(H^{-1})_{jj} \le d_1$, and $\operatorname{tr} H^{-1} \le K_1$. For the last assertion no interior maximization is hidden: for a fixed $K_1 \ge 6$, the inequality $\mathcal{I}(T) \le K_1$ is equivalent to

$$(K_1 - 8) + (K_1 - 7)T - \frac{7}{4}(K_1 - 6)T^2 \ge 0.$$

This quadratic is concave and nonnegative at both interval endpoints.

Six annulus tests. A box is rejected immediately if $t_0 + z_0 > 1/8$, if $k_1^2 < r_0$, if $8r_1 < k_0^2$, or if $k_1 = 0$; the latter three contradict $r_p \le \kappa_p^2 \le 8r_p$ and $r_p > 0$. The Schur test is $c_1 < c_0$. For a directional determinant leaf require $c_0 > 0$ and the two checks

$$\begin{aligned} \frac{972}{512c_0^3} &\le \frac{F_0}{\left[\left(\frac{1}{2} - \frac{k_0}{4}\right)^2 + \left(\frac{c_0}{2}\right)\left(1 + \frac{t_1^2}{4F_1}\right)\right]^2}, \\ \frac{972}{512c_0^3} &\le \frac{\left(1 - \frac{t_1}{6}\right)\left(1 - \frac{5t_1}{6}\right)\left(1 + \frac{t_0}{3}\right)^6}{\left[\left(\frac{1}{2} - \frac{k_0}{4}\right)^2 + \frac{\frac{c_0}{2}}{1 - \frac{t_1}{6}}\right]^2}. \end{aligned} \tag{66}$$

All denominators are positive. The arrow diagonal bound increases with T: the derivative of $T^2/F(T)$ has numerator $T(2+T)$. The factors in the negative-vertex bound are bounded separately in the displayed directions. The common joint-ratio result therefore transfers (66) to the actual H; the directional flag and its monotonicity in c give $\Delta \ge 972$.

For an averaged determinant leaf let L_* be certified by (58), require $c_0 > 0$, $K_1 > 0$, $4 + r_1 - L_*/2 \ge 0$, and check

$$972 \le \frac{131072 F_0 c_0^3}{\left[c_0 K_1 + 4 + r_1 - \frac{L_*}{2}\right]^2}. \tag{67}$$

The smaller vector of the pair at coordinate j has doubled squared norm at most $(H^{-1})_{jj} + (1 - \alpha_j + \alpha_j^2/4)/(2c)$. Averaging these eight doubled norms bounds the common shortest-dual quantity μ by $[c \operatorname{tr} H^{-1} + 4 + r_p - \alpha/2]/(8c)$. Thus the same flag used above gives (67), since $c^3/(ac+b)^2$ increases for $c > 0$, $a > 0$, $b \ge 0$. Each averaging vector bounds the same shortest-dual minimum from above.

Finally, an odd-spinor leaf uses $r_0 > 3k_1^2/4$, a bound $\alpha \le A_*$ from (63), and rational q and integer $n \ge 0$ satisfying

$$0 \le q \le k_1, \qquad A_* \le nk_1 + q, \qquad nk_1^2 + q^2 < 8r_0. \tag{68}$$

This contradicts (57). These are all the non-phase annulus tests; each involves only rational endpoint arithmetic after the finite contact identities (59) and the odd-spinor images have been supplied.

One phase localisation argument for three leaf types. Write

$$h_\perp = \frac{1}{u}, \qquad h_* = \frac{\frac{11131}{10}}{1024 F_0 c_0}. \tag{69}$$

The phase rules require $c_0 > 0$. The balanced determinant cutoff gives $\det Q_{10} < 11131/10$: indeed $(11131/10)^{11} > 4^{11}972^9$, an exact integer comparison after clearing 10^{11}. Since $\det Q_{10} = 1024c \det(H) h_\perp$, we have $0 < h_\perp \le h_*$.
For the radical-interval rule, choose rational $L_\phi, J_\phi, a_0, a_1, \ell_-, \ell_+$ with

$$J_\phi > 0, \qquad 0 \le a_0 \le a_1 \le 2, \qquad 0 < \ell_- \le \ell_+ \le 1,$$

and let $b(a) = L_\phi - J_\phi a^2$. Require

$$\begin{aligned} b(a_1) - 4J_\phi a_1 > 0, \qquad & b(a_1) + 4J_\phi a_1 \le \frac{1}{4}, \\ 2b(a_1) \le \ell_-^2, \qquad & (\ell_-^2 - 2b(a_1))^2 \le 4\{b(a_1)^2 - 16J_\phi^2 a_1^2\}, \\ 2b(a_0) \le \ell_+^2, \qquad & 4\{b(a_0)^2 - 16J_\phi^2 a_0^2\} \le (\ell_+^2 - 2b(a_0))^2. \end{aligned} \tag{70}$$

For every $a \in [a_0, a_1]$, the radicals

$$P(a) = \sqrt{b(a) + 4J_\phi a}, \qquad Q(a) = \sqrt{b(a) - 4J_\phi a}$$

then satisfy $0 < Q(a) \le P(a) \le 1/2$ and $\ell_- \le P(a) + Q(a) \le \ell_+$. Indeed, the plus radicand increases and the minus radicand decreases on $[0, 2]$. Also $b(a)$ decreases, and

$$\begin{aligned} &\{b(a)^2 - 16J_\phi^2 a^2\} - \{b(b')^2 - 16J_\phi^2 (b')^2\} \\ &\quad = ((b')^2 - a^2)\{2L_\phi J_\phi + 16J_\phi^2 - J_\phi^2(a^2 + (b')^2)\} \ge 0 \end{aligned}$$

for $a \le b' \le a_1$; the bracket is nonnegative because $L_\phi - J_\phi (b')^2 > 0$. Thus $(P+Q)^2 = 2b(a) + 2\sqrt{b(a)^2 - 16J_\phi^2 a^2}$ decreases. The last four checks in (70), with their stated nonnegative sides, are precisely its lower and upper endpoint bounds after squaring.
For a phase leaf use

$$\begin{aligned} a_1 = k_1, \qquad a_0^2 \le \max(0, 8r_0 - 7k_1^2), \qquad \ell_- > \frac{1}{2}, \\ J_\phi = \frac{h_*}{16 c_0}, \qquad L_\phi + h_* \frac{D_*}{2} = 1 - \frac{h_*}{4c_0} \end{aligned} \tag{71}$$

where $D_* \ge d_1$ is rational. The inverse diagonal $d_j = (H^{-1})_{jj}$ lies between $d_* = 1/(1+t_1)$ and D_*. For the preferred signs $\sigma_j p_j = |p_j|$, the two integral covectors $(\pm\sigma_j e_j/\sqrt{2}, 1/2)$ have squared inverse norms $d_j/2 + (1/2 \mp \alpha_j/4)^2/c$. The shortest-dual condition in the first extension implies

$$\operatorname{dist}(\theta_j^{\pm}, \mathbb{Z})^2 \ge 1 - h_\perp \{ \frac{d_j}{2} + \frac{1}{c}\left(\frac{1}{2} \mp \frac{\alpha_j}{4}\right)^2 \}. \tag{72}$$

Here $\theta_j^{\pm}$ is the phase of the actual gluing on the corresponding integral row. This is simply the inverse Schur formula applied to its closest integral lift, whose value is at least the shortest value u. Using (69)–(71), the right side of (72) is at least

$$L_\phi + \frac{h_*}{2}(D_* - d_j) - J_\phi \alpha_j^2 \pm 4J_\phi \alpha_j.$$

Consequently the errors from the middle of the containing unit interval, $\varepsilon(\theta) = \theta - \lfloor \theta \rfloor - 1/2$, satisfy

$$\begin{aligned} |\varepsilon(\theta_j^+)| &\le \frac{1}{2} - \sqrt{L_\phi + \frac{h_*}{2}(D_* - d_j) - J_\phi \alpha_j^2 + 4J_\phi \alpha_j}, \\ |\varepsilon(\theta_j^-)| &\le \frac{1}{2} - \sqrt{L_\phi + \frac{h_*}{2}(D_* - d_j) - J_\phi \alpha_j^2 - 4J_\phi \alpha_j}. \end{aligned} \tag{73}$$

For zero sensitivity it suffices to drop the nonnegative diagonal gain. For positive sensitivity the additional interval checks below ensure that all the displayed square roots are at most $1/2$.

The sixteen errors come from one displacement, as follows. Every opposite pair of physical covectors has sum $(0, 1)$, so all $\theta_j^+ + \theta_j^-$ are equal. The sum of the two absolute errors is at most $1 - P(\alpha_j) - Q(\alpha_j) \le 1 - \ell_- < 1/2$. The difference between the error sums of two pairs is an integer of absolute value less than 1, hence zero. Write their common sum as z and let $x_j = \sigma_j\{\varepsilon(\theta_j^+) - \varepsilon(\theta_j^-)\}/\sqrt{2}$. Then the displacement $\xi = (x, z)$ in physical coordinates has exactly these sixteen phases. Subtracting $B^{-1}\xi$ from the original gluing constructs a centre β_* at which all sixteen phases are half-integers. This proves compatibility. Moreover $z^2 \le (1 - \ell_-)^2$.

For one coordinate let $a = \alpha_j$, $X = \sigma_j x_j/\sqrt{2} + z/2$, and $Y = z - X$. If $|X| \le A$, $|Y| \le B'$ and $0 \le a \le 5/4$, then

$$\begin{aligned} 8x_j^2 + 16p_j x_j z + z^2 &= (5 + 4a)X^2 + (5 - 4a)Y^2 - 6XY \\ &\le \Phi_a(A, B') := (5 + 4a)A^2 + (5 - 4a)(B')^2 + 6AB'. \end{aligned} \tag{74}$$

Both square coefficients are nonnegative, which proves the inequality directly. Averaging over eight coordinates bounds the complete model energy $\mathcal{E}_0 = \mathcal{B}(I, p, 0)[\xi]$. There is no estimate of mixed terms by unrelated coordinate errors.

To bound the base envelope by $e_0 + e_1 a^2$, set $x = P(a) + Q(a)$. The identities

$$(P - Q)x = 8J_\phi a, \qquad a^2 = \frac{x^2(4L_\phi - x^2)}{4J_\phi(x^2 + 16J_\phi)}$$

give, by direct expansion,

$$\begin{aligned} &\left(e_0 + e_1 a^2 - \Phi_a\left(\frac{1}{2} - P, \frac{1}{2} - Q\right)\right)(x^2 + 16J_\phi) \\ &\quad = \left(4 - \frac{e_1}{4J_\phi}\right)x^4 + \left(e_0 - 4 - 32L_\phi - 48J_\phi + \frac{e_1 L_\phi}{J_\phi}\right)x^2 \\ &\qquad + (128J_\phi + 32L_\phi)x + 16J_\phi(e_0 - 4 - 4L_\phi). \end{aligned} \tag{75}$$

Five nonnegative Bernstein coefficients on $[\ell_-, \ell_+]$ prove this quartic nonnegative. Explicitly, for a polynomial $\sum_{j=0}^{4} q_j x^j$, expand $\sum q_j(\ell_- + (\ell_+ - \ell_-)y)^j = \sum_{j=0}^{4} v_j y^j$; its Bernstein coefficients are $\sum_{j=0}^{k} v_j \binom{k}{j} / \binom{4}{j}$ for $0 \le k \le 4$. The identity with $\sum \binom{4}{k} y^k (1-y)^{4-k}$ proves the claim, including endpoints. When the interval has zero width, evaluation at that endpoint is enough.

Keeping the inverse-trace gain. The preceding envelope may be sharpened by a rational $\eta \ge 0$. For $a \in [a_0, a_1]$, let P_0, Q_0 be the radicals at the largest level $L_0 = L_\phi + h_*(D_* - d_*)/2$. When a positive gain is used, impose the same finite rule (70) at level L_0, with the same J_ϕ, a_0, a_1 and possibly different sum endpoints. In particular, $0 < Q_0 \le P_0 \le 1/2$. Define

$$\begin{aligned}\mathcal{S}(a) = {} & \frac{h_*}{4P_0}\left\{2(5+4a)\left(\frac{1}{2} - P_0\right) + 6\left(\frac{1}{2} - Q_0\right)\right\} \\ & + \frac{h_*}{4Q_0}\left\{2(5-4a)\left(\frac{1}{2} - Q_0\right) + 6\left(\frac{1}{2} - P_0\right)\right\}.\end{aligned} \tag{76}$$

Suppose the finite checks below establish $\eta \le \mathcal{S}(a)$ throughout the coordinate interval. Increasing either squared radical by $h_*(D_* - d_j)/2$ increases that radical by at least $h_*(D_* - d_j)/(4P_0)$ or $h_*(D_* - d_j)/(4Q_0)$, respectively: use $p - q = (p^2 - q^2)/(p+q)$ and $p + q \le 2P_0$ or $2Q_0$. All coefficients of Φ in its nonnegative arguments are nonnegative. Expanding its difference between the base arguments and the smaller actual arguments, and retaining the derivatives at the smallest arguments $(1/2 - P_0, 1/2 - Q_0)$, now gives

$$\Phi_{\alpha_j}(A_j, B_j) \le e_0 + e_1 \alpha_j^2 - \eta(D_* - d_j).$$

Averaging, using $e_1 \ge 0$ and $\sum d_j \le K_1$, proves

$$\mathcal{E}_0 \le e_0 + e_1 r_p - \eta\left(D_* - \frac{K_1}{8}\right) \le E_* := \max\left\{0,\ e_0 + e_1 r_1 - \eta\left(D_* - \frac{K_1}{8}\right)\right\}. \tag{77}$$

A rational upper bound for E_* can be used in every subsequent check. If $\eta = 0$, no second radical interval or sensitivity check is needed.

For completeness, the positive sensitivity checks have three forms; all are elementary scalar certificates. Let $x = P_0 + Q_0$ and $y = P_0 Q_0$. Expansion of (76) gives

$$\mathcal{S}(a) = \frac{h_*}{2}\left\{\frac{4x - 3x^2 - \frac{16 J_\phi a^2}{x}}{y} - 4\right\}. \tag{78}$$

First, eliminate a^2 as in (75). The denominator $D(x) = x^4 + 32 J_\phi x^2 - 64 J_\phi L_0 = 4y(x^2 + 16 J_\phi)$ is positive, and

$$\mathcal{S}(a) = \frac{-8h_* x^4 + 16 h_* x^3 - 160 h_* J_\phi x^2 + (128 h_* J_\phi - 32 h_* L_0)x + 128 h_* J_\phi L_0}{D(x)}.$$

Nonnegative Bernstein coefficients for numerator minus $\eta D(x)$ therefore suffice. Second, on a certified sum interval $x \in [b_0, b_1]$, choose $N_*, Y_* \geq 0$ with

$$N_* \leq 4b_0 - 3b_0^2, \qquad N_* \leq 4b_1 - 3b_1^2,$$
$$(L_0 - J_\phi a_0^2)^2 - 16J_\phi^2 a_0^2 \leq Y_*^2, \qquad N_* - \frac{16J_\phi a_1^2}{b_0} \geq 0.$$

Concavity of $4x - 3x^2$, the radical-product monotonicity proved above, and $a \leq a_1$, $x \geq b_0$ give

$$\mathcal{S}(a) \geq \frac{h_*}{2}\left\{\frac{N_* - \frac{16J_\phi a_1^2}{b_0}}{Y_*} - 4\right\}.$$

Here $Y_* > 0$ follows from $P_0 Q_0 > 0$; the nonnegative-numerator check makes the denominator comparison valid. Third, if $L_0/4 \leq J_\phi$, then

$$\mathcal{S}(a) \geq \frac{4h_*}{\sqrt{L_0}} - 8h_*. \tag{79}$$

A rational positive upper bound for $\sqrt{L_0}$ gives a rational check. Here is an algebraic proof of (79), so no derivative estimate is implicit. Write $t = \sqrt{L_0}$, $0 < t \leq 1/2$, and $0 < x \leq 2t$. With

$$Q_* = J_\phi(-48t^2 - 24tx + 32t + 32x) - 4t^2 x - 2tx^2 + x^3$$

one checks

$$(x^2 + 16J_\phi)\big[t\Big\{4x - 3x^2 - \frac{16J_\phi a^2}{x} + 12y\Big\} - 8y\big] = 2(2t - x)Q_*.$$

The coefficient of J_ϕ is nonnegative. At $J_\phi = t^2/4$, write $v = x/(2t) \in [0, 1]$; then $Q_* = t^3\{2 + 2v(2v-1)^2 + 6(1-2t)(1+v)\} \geq 0$. The right side is therefore nonnegative, and (78) gives (79) after division by positive factors. Each of the three certificates proves the same uniform sensitivity statement used in (77).

From model energy to actual energy. Set $\mathcal{E} = \mathcal{B}(H, p, \tau)[\xi]$. Either of the following finite bounds may be chosen:

$$\mathcal{E} \leq (1 + 2t_1)E_* + z_1(1 - \ell_-)^2, \tag{80}$$

or, when $L_\phi/4 \leq J_\phi$,

$$\mathcal{E} \leq E_* + \frac{49t_1}{12}g_*^2 + (z_1 + t_1 r_1)(1 - \ell_-)^2, \qquad g_* = 1 - \frac{\ell_-(16J_\phi + 4L_\phi)}{\ell_-^2 + 16J_\phi}. \tag{81}$$

For (80), the spectral inequalities $-TI \preceq E \preceq 2TI$ imply $\mathcal{B}(H, p, \tau)[\xi] \leq (1+2T)\mathcal{E}_0 + \tau z^2$ when $r_p \leq 2/3$. To see the point directly, subtract the left side from the right side. The difference without τz^2 is $(2TI - E)[x + pz] + (2T + E[p] - 2Tr_p)z^2$, nonnegative because $E[p] \geq -Tr_p$ and $r_p \leq 2/3$.

For (81), the coordinate phase bounds give

$$|x_j + p_j z| \le \frac{\left(1+\frac{\alpha_j}{2}\right)\left(\frac{1}{2}-P\right) + \left(1-\frac{\alpha_j}{2}\right)\left(\frac{1}{2}-Q\right)}{\sqrt{2}} = \frac{1-\frac{x(16J_\phi+4L_\phi)}{x^2+16J_\phi}}{\sqrt{2}} \le \frac{g_*}{\sqrt{2}}.$$

The last comparison follows from $\ell_- \le x$, $x^2 \le 4L_\phi \le 16J_\phi$ and the exact difference

$$f(\ell_-) - f(x) = \frac{(16J_\phi + 4L_\phi)(x-\ell_-)(16J_\phi - \ell_- x)}{(\ell_-^2 + 16J_\phi)(x^2+16J_\phi)} \ge 0,$$
$$f(x) = 1 - \frac{x(16J_\phi + 4L_\phi)}{x^2 + 16J_\phi}.$$

The diagonal-cone rule (61) gives, for $|v_j| \le b$,

$$E[v] \le \frac{49}{6} T b^2. \tag{82}$$

Indeed, the pair inequalities bound it by $\sum_j E_{jj}|v_j| \sum_\ell |v_\ell|$. Each coefficient is at most $8b^2 \le 49b^2/6$; its negative-vertex value is at most $49b^2/6$, because $(\sum_\ell |v_\ell|)^2 - 2|v_j| \sum_\ell |v_\ell| = (\sum_{\ell \ne j} |v_\ell|)^2 - |v_j|^2 \le 49b^2$. Finally

$$\mathcal{E} = \mathcal{E}_0 + E[x + pz] + (\tau - E[p])z^2$$

and $E[p] \ge -T r_p$ prove (81). Choose a rational E_1 at least one of the right sides in (80)–(81), and impose the final localisation check

$$1 + z_1 + h_* + E_1 < 4. \tag{83}$$

This completes the common phase argument. Every inequality about the actual displacement has now been derived from finite endpoint checks.

The common reference centre and the three phase terminals. We spell out the fixed geometry required here. Let A_{ph} be the integral matrix of nine independent selected phase rows. Its index-two certificate consists of integral matrices $\mathsf{N}_{\mathrm{ph}}, \mathsf{M}_{\mathrm{ph}}$ and integral vectors $\mathbf{b}_{\mathrm{ph}}, \mathbf{c}_{\mathrm{ph}}$ satisfying

$$\mathsf{N}_{\mathrm{ph}}\mathsf{A}_{\mathrm{ph}} = 2I, \qquad \mathsf{N}_{\mathrm{ph}} = \mathbf{b}_{\mathrm{ph}}(\mathbf{c}_{\mathrm{ph}})^t + 2\mathsf{M}_{\mathrm{ph}}, \qquad 2\gamma = \mathbf{b}_{\mathrm{ph}}. \tag{84}$$

The fixed base centre β_0 has half-integral selected phases and physical image $(0,1)$; $2\beta_0$ is integral. If a centre β_* has half-integral phases, then $\mathsf{A}_{\mathrm{ph}}(\beta_* - \beta_0) = k$ is integral and

$$\beta_* - \beta_0 = \gamma((\mathbf{c}_{\mathrm{ph}})^t k) + \mathsf{M}_{\mathrm{ph}} k.$$

Splitting the integer $(\mathbf{c}_{\mathrm{ph}})^t k$ into its even part and its residue modulo two proves that β_* is congruent to either β_0 or $\beta_0 + \gamma$ modulo the original lattice coordinates. The

integral part is absorbed by a unimodular change of the gluing basis, preserving minima and determinants.

The base coset is impossible under (83). In fact the two integral first-layer vectors corresponding to 0 and $2\beta_0$ have average squared norm $1+\tau+\mathcal{E}+h_\perp<4$; at least one violates the minimum. For the other coset, the fixed reference gluing of $\beta_0+\gamma$ with perpendicular height 1 is the stated D_{10}^+ reference lattice of determinant 1024. In its physical coordinates the trace of the actual rank-ten metric is exactly

$$9+T+\tau+h_\perp+\mathcal{E}=10+T+\tau+\zeta, \qquad \zeta:=h_\perp+\mathcal{E}-1. \tag{85}$$

This identity follows by inverting the reference block matrix, or by writing the changed gluing as its reference value plus $B^{-1}\xi$; the resulting trace adds precisely its squared displacement energy. Thus a phase leaf satisfying

$$t_1+z_1+h_*-1+E_1\le\frac{3}{10} \tag{86}$$

has rank-ten trace at most $103/10$. The D_{10}^+ trace criterion, proved as [5, Lemma 43] (for a minimum-four metric G on $\sqrt{2}D_{10}^+$ with $\det G<557/512$, either of $\operatorname{tr}G\le 103/10$ and $\operatorname{tr}G^{-1}\le 151/15$ implies that every minimum-four primitive extension has determinant at least 1216), with the same determinant cutoff, gives $\Delta\ge 1216$.

The remaining two phase terminals use the same fixed list of 32 integral first-layer points, with physical differences from $\beta_0+\gamma$ equal to $(\sigma\sqrt{2}e_j,\epsilon)$ for $j=1,\dots,8$ and $\sigma,\epsilon\in\{\pm1\}$. Their actual minimum inequalities are

$$4\le\mathcal{B}(H,p,\tau)[(\sigma\sqrt{2}e_j,\epsilon)-\xi]+h_\perp. \tag{87}$$

No inequality in (87) is part of the finite certificate: it follows from the integral identities and the lattice minimum. If $T+\tau+\zeta\le 3/10$, the trace criterion has already finished the proof. Otherwise

$$\zeta_0:=\max\left(0,\frac{3}{10}-t_1-z_1\right)\le\zeta\le\zeta_1:=h_*-1+E_1. \tag{88}$$

Positivity of ζ follows from $T+\tau\le 1/8$. Writing $v=Hp$ and $w=-\mathcal{B}(H,p,\tau)\xi$, the four sign choices in (87) are equivalent to the two inequalities

$$|2\sqrt{2}w_j\pm 2w_9|\le\zeta+\tau+2E_{jj}\pm 2\sqrt{2}v_j. \tag{89}$$

Moreover $\mathcal{E}=\mathcal{B}(H,p,\tau)^{-1}[w]$ exactly.

Uniform contact slacks. Let A_* be any mass upper bound established above and let

$$d_*'=z_1+t_1\left\{2+\frac{A_*+k_1}{2}\right\}.$$

The root-cone row bound $|(Ea)_j|\le T(\sum_\ell|a_\ell|+\max_\ell|a_\ell|)/2$ follows by evaluating the sixteen diagonal vertices and their allowed off-diagonal entries. Also $E_{jj}\le T$. Apply this to $a=2\sqrt{2}p$; then (89) gives

$$|2\sqrt{2}w_j\pm 2w_9|\le\zeta+d_*'\pm 2\sqrt{2}p_j. \tag{90}$$

We prove the scalar estimate used here. Suppose $|x_j + Z| \le s + a_j$, $|x_j - Z| \le s - a_j$, $s \le 5/4$, and let $r = \mathrm{mean}(a_j^2) < 1$, $q = \mathrm{mean}(x_j^2)$, $b = \mathrm{mean}(a_j x_j)$. Then

$$q + \frac{\left(\frac{Z}{2} - b\right)^2}{1 - r} \le \frac{s^2 - rs + \frac{r}{4}}{1 - r}. \tag{91}$$

Proof of (91). Changing the signs of a_j, x_j together permits $a_j \ge 0$, and changing all x_j, Z together permits $Z \ge 0$. The constraints imply $0 \le a_j \le s$ and $0 \le Z \le s$. For one coordinate the residual

$$s^2 - x^2 + Z a_j x - \frac{Z^2}{4} - s a_j^2 + \frac{a_j^2}{4}$$

is concave in x, whose allowed interval is $[-s + a_j + Z,\ s - |a_j - Z|]$. At the lower endpoint it is a concave quadratic in $Z \in [0, s]$. At the upper endpoint it is a concave quadratic on each of $[0, a_j]$ and $[a_j, s]$. The required endpoint values are $a_j\{2s - a_j(s + 3/4)\}$, 0, and $3(s^2 - a_j^2)/4$; they are all nonnegative because $a_j \le s \le 5/4$. Thus the residual is nonnegative throughout. Average it to get $q - Zb + Z^2/4 \le s^2 - rs + r/4$, and use $b^2 \le rq$ to obtain (91). □

Apply (91) to (90), with $a_j = 2\sqrt{2}p_j$, $x_j = 2\sqrt{2}w_j$, $Z = 2w_9$, and $s = \zeta + d'_*$. The means are exactly r_p, $|w_{\mathrm{hor}}|^2$ and $p \cdot w_{\mathrm{hor}}$, so the left side is $\mathcal{B}(I, p, 0)^{-1}[w]$. Furthermore

$$\mathcal{B}(H, p, \tau) \succeq (1 - T)\mathcal{B}(I, p, 0) \quad \text{if } r_p \le \frac{1}{3}. \tag{92}$$

Indeed $H \succeq (1-T)I$ and $c \ge (1-T)(1-r_p)$, the latter because $c - (1-T)(1-r_p) = \tau + T - Tr_p - E[p] \ge \tau + T(1 - 3r_p) \ge 0$. Use the Schur energy expression in the vector $x + pz$ to prove (92). Inversion reverses this comparison. For a direct proof, if $a > 0$, $B \succeq aA$, and A, B are positive definite, let $y = B^{-1}w$; Cauchy–Schwarz for A gives $(w^t B^{-1} w)^2 \le A[y]A^{-1}[w] \le a^{-1}B[y]A^{-1}[w]$, and cancel the positive energy; the zero vector is immediate.
Consequently, provided $\zeta_1 + d'_* \le 5/4$,

$$\mathcal{E} \le \Theta_{\mathrm{hom}}(\zeta) := \frac{(\zeta + d'_*)^2 - r_1(\zeta + d'_*) + \frac{r_1}{4}}{(1 - t_1)(1 - r_1)}. \tag{93}$$

The radius replacement is valid because

$$\frac{s^2 - Rs + \frac{R}{4}}{1 - R} - \frac{s^2 - rs + \frac{r}{4}}{1 - r} = \frac{(R - r)\left(s - \frac{1}{2}\right)^2}{(1 - R)(1 - r)} \ge 0.$$

The replacement of T by t_1 then uses $\mathcal{E} \ge 0$.
Separate contact slacks. A sharper alternative retains every E_{jj} in (89). For a free rational $U \ge 1/2$, define, when $0 \le r < \min(1, U^2)$,

$$\Lambda_U(r) = \frac{4U^2 - r}{4(1 - r)(U^2 - r)}, \qquad \omega_U(r) = \frac{2U^2 + 3U - 2r}{4U^2 - r}. \tag{94}$$

Here $\Lambda_U(r) \ge 1$ and $\omega_U(r) > 0$; the numerator of $\Lambda_U(r) - 1$ is $r\{4(U^2 - r) + 3\}$. For general separate slacks S_j, assume $|x_j \pm Z| \le S_j \pm a_j$ and $\omega_U(r)|a_j| \le 1$, where again $r = \mathrm{mean}(a_j^2)$. Squaring the two inequalities with weights $(1 - \omega_U(r)a_j)/2$ and $(1 + \omega_U(r)a_j)/2$ gives

$$x_j^2 + Z^2 - 2\omega_U(r)a_j x_j Z \le S_j^2 + a_j^2 - 2\omega_U(r)S_j a_j^2.$$

With q, b as above, a completion of squares gives

$$\begin{aligned} &\Lambda_U(r)\{q + Z^2 - 2\omega_U(r)bZ\} - q - \frac{\left(\frac{Z}{2} - b\right)^2}{1 - r} \\ &\quad = (\Lambda_U(r) - 1)\left(q - \frac{b^2}{r}\right) + \frac{3(b - UZ)^2}{4(1 - r)(U^2 - r)} \ge 0. \end{aligned} \tag{95}$$

For $r = 0$, Cauchy–Schwarz gives $b = 0$ and the same inequality follows directly, without division by r. Averaging the weighted squares proves the corresponding bound with right side $\Lambda_U(r)\,\mathrm{mean}(S_j^2 + a_j^2 - 2\omega_U(r)S_j a_j^2)$.

In (89) use $a_j = 2\sqrt{2}v_j$, $x_j = 2\sqrt{2}w_j$, $Z = 2w_9$, and $S_j = \zeta + \tau + 2E_{jj}$. The diagonal cone gives $E_{jj} \ge -T/6$ and $\sum_j E_{jj}^2 \le T^2$: both follow from its sixteen vertices, using convexity for the square sum. Hence, with $S = \zeta + \tau$ and $r = |v|^2$,

$$\mathrm{mean}(S_j^2) \le S^2 + \frac{ST}{2} + \frac{T^2}{2}, \qquad \mathrm{mean}(S_j a_j^2) \ge \left(S - \frac{T}{3}\right)r. \tag{96}$$

The identity shear metric $\left(\begin{smallmatrix} I & v \\ v^t & 1 \end{smallmatrix}\right)$ has inverse energy $|w_{\mathrm{hor}}|^2 + (w_9 - v \cdot w_{\mathrm{hor}})^2/(1 - r)$. It compares with the actual metric by the factor $1 - T/(1 - r)$. Indeed, putting $a = T/(1 - r)$, the difference of the quadratic forms of $\mathcal{B}(H, p, \tau)$ and $(1 - a)\left(\begin{smallmatrix} I & v \\ v^t & 1 \end{smallmatrix}\right)$ at (x, z) is

$$(E[x] + T|x|^2) + a\{r|x|^2 - (v \cdot x)^2\} + a(v \cdot x + z)^2 + \tau z^2 \ge 0. \tag{97}$$

This proves the inverse comparison whenever $T/(1 - r) < 1$. Together (95)–(97) give

$$\mathcal{E} \le \frac{\Lambda_U(r)}{1 - \frac{T}{1 - r}}\left\{S^2 + \frac{ST}{2} + \frac{T^2}{2} + r - 2\omega_U(r)\left(S - \frac{T}{3}\right)r\right\}. \tag{98}$$

Here are the exact box checks for this alternative. Let $v_0 = (1 - t_1)^2 r_0$ and $v_1 = (1 + 2t_1)^2 r_1$. The spectral interval for H gives $v_0 \le |Hp|^2 \le v_1$. The root-cone row estimate used above gives $|2\sqrt{2}(Hp)_j| \le k_1 + t_1(A_* + k_1)/2$. Require

$$\begin{aligned} &U \ge \frac{1}{2}, \qquad v_1 < 1, \qquad v_1 < U^2, \qquad \frac{t_1}{1 - v_1} < 1, \\ &\zeta_0 + z_0 - \frac{t_1}{3} \ge 0, \qquad \omega_U(v_0)\left\{k_1 + \frac{t_1(A_* + k_1)}{2}\right\} \le 1. \end{aligned} \tag{99}$$

For $r \le R$, direct subtraction gives

$$\omega_U(r) - \omega_U(R) = \frac{3U(2U - 1)(R - r)}{(4U^2 - r)(4U^2 - R)} \ge 0,$$

$$\Lambda_U(R) - \Lambda_U(r) = \frac{(R - r)\{4(U^2 - R)(U^2 - r) + 3(U^2 - rR)\}}{4(1 - R)(U^2 - R)(1 - r)(U^2 - r)} \ge 0.$$

Thus the weight condition in (98) holds, and its right side is at most the explicit quadratic

$$\begin{aligned}\Theta_{\mathrm{sep}}(\zeta) = \frac{\Lambda_U(v_1)}{1-\frac{t_1}{1-v_1}}\Big\{&(\zeta+z_1)^2 + (\zeta+z_1)\frac{t_1}{2} + \frac{t_1^2}{2} + v_1 \\ &- 2\omega_U(v_1)\Big(\zeta + z_0 - \frac{t_1}{3}\Big)v_0\Big\}.\end{aligned} \tag{100}$$

In passing to this bound, keep the negative term as a single product: all three lower factors are nonnegative by (99). The actual bracket is nonnegative, because it bounds a nonnegative energy with a positive multiplier. This justifies increasing that multiplier and decreasing the positive denominator. No monotonicity of the whole bracket in r is assumed.

The same two endpoint checks finish both contact rules. For $\Theta = \Theta_{\mathrm{hom}}$ or Θ_{sep}, require

$$\frac{11131}{10} < 1024F_0c_0(1+\zeta_j-\Theta(\zeta_j)), \qquad j=0,1. \tag{101}$$

The function $1+\zeta-\Theta(\zeta)$ is concave, since the leading coefficient of Θ is nonnegative. Its minimum on $[\zeta_0,\zeta_1]$ occurs at an endpoint (also when the interval is a single point). Therefore (101) holds at the actual ζ. But $h_\perp = 1+\zeta-\mathcal{E} \ge 1+\zeta-\Theta(\zeta)$, so

$$\det Q_{10} = 1024c\det H\, h_\perp \ge 1024F_0c_0h_\perp > \frac{11131}{10},$$

contradicting (69). This proves both remaining phase terminals.

The remaining small-radius range. For $0 \le r_p \le 1/100$ the phase argument has one uniform choice of constants, and no covering certificate is needed. We first retain the determinant compensation in the two inverse terms. For $0 \le T \le 1/4$ and $0 \le r \le 1/10$, let $c' = 1-r-2Tr$. Expansion gives

$$\begin{aligned}F(T)c'\Big(1-\frac{T}{6}\Big) - (1-r) &= T\Big\{\frac{5-17r}{6} + T\Big(-\frac{23}{12}+\frac{r}{4}\Big) \\ &\qquad + T^2\Big(\frac{7+85r}{24} - \frac{7rT}{12}\Big)\Big\}, \\ F(T)(c')^2 - (1-r)^2 &= T\Big\{(1-r)(1-5r) + T\Big(\frac{25}{4}\Big(r-\frac{1}{25}\Big)^2 - \frac{44}{25}\Big) \\ &\qquad + T^2r(7-3r-7rT)\Big\}.\end{aligned} \tag{102}$$

The first brace is at least $11/20 - 23/48 = 17/240$; the last term is nonnegative since $Tr \le 1/40$. The second brace is at least $9/20 - 11/25 = 1/100$, with its last term again nonnegative. Since $c \ge 1-r_p-2Tr_p > 0$, (102) proves

$$F(T)c\Big(1-\frac{T}{6}\Big) \ge 1-r_p, \qquad F(T)c^2 \ge (1-r_p)^2.$$

Together with $(1 - T/6)(H^{-1})_{jj} \le 1$ and with the cutoff $1024\det(H)\,c\,h_\perp < 11131/10$, these yield, for every real b,

$$\begin{aligned} h_\perp\left\{\frac{(H^{-1})_{jj}}{2} + \frac{b^2}{c}\right\} &\le \frac{\frac{11131}{10}}{1024(1-r_p)}\left\{\frac{1}{2} + \frac{b^2}{1-r_p}\right\} \\ &\le \frac{11}{10}\left\{\frac{1}{2} + \frac{b^2}{\frac{99}{100}}\right\}. \end{aligned} \tag{103}$$

For example the first compensation inequality bounds $h_\perp(H^{-1})_{jj}$ by $\frac{11131/10}{1024(1-r_p)}$; the second bounds $h_\perp/c$ by $\frac{11131/10}{1024(1-r_p)^2}$. Equation (103) follows term by term.
Every $\alpha_j^2 \le 8r_p \le 2/25 < 4/49$, so $\alpha_j \le 2/7$. Taking $b = 1/2 \mp \alpha_j/4$ in (103) and applying (72) gives phase radicands

$$L_\phi - J_\phi a^2 \pm 4J_\phi a, \qquad L_\phi = \frac{31}{180}, \quad J_\phi = \frac{5}{72}, \quad 0 \le a \le \frac{2}{7}. \tag{104}$$

The radical-interval checks (70) hold with $a_0 = 0$, $a_1 = 2/7$, $\ell_- = 79/100$, $\ell_+ = 83/100$. For orientation, the endpoint radicands at $2/7$ are $241/980$ and $769/8820$, both positive and less than $1/4$; the remaining four inequalities are direct rational comparisons. In (75) choose $e_0 = 3/25$ and $e_1 = 1$. The quartic becomes

$$q(x) = \frac{2}{5}x^4 - \frac{461}{45}x^2 + \frac{72}{5}x - \frac{2056}{405}.$$

Its full positive Bernstein expansion, with $y = 25(x - 79/100) \in [0,1]$, is

$$\begin{aligned} 20250000000\,q(x) = {} & 1249411561(1-y)^4 + 4(1047446197)y(1-y)^3 \\ & + 6(798249169)y^2(1-y)^2 + 4(502230013)y^3(1-y) \\ & + 159819001y^4. \end{aligned} \tag{105}$$

The paired-centre construction and (74)–(75) therefore give

$$\mathcal{E}_0 \le \frac{3}{25} + r_p \le \frac{13}{100}, \qquad z^2 \le \frac{441}{10000}. \tag{106}$$

This proof also covers $r_p = 0$.
It remains to bound the actual trace in (85). Since $c \ge 1 + \tau - (1+2T)/100$ and $\det H \ge F(T)$, (80) and (106) show that $T + \tau + h_\perp - 1 + \mathcal{E}$ is strictly below

$$T + \tau - 1 + \frac{\frac{11131}{10}}{1024F(T)\left(1 + \tau - \frac{1+2T}{100}\right)} + (1+2T)\frac{13}{100} + \tau\frac{441}{10000}. \tag{107}$$

Here is one rational argument for the whole triangle $T, \tau \ge 0$, $T + \tau \le 1/8$. Let $w_0 = 1 - 8T - 8\tau$, $w_1 = 8T$, $w_2 = 8\tau$ and

$$d_0 = \frac{99}{100}, \qquad d_1' = \frac{22199}{20480}, \qquad d_2 = \frac{223}{200},$$
$$D = w_0 d_0 + w_1 d_1' + w_2 d_2.$$

The exact difference

$$F(T)\left(1+\tau-\frac{1+2T}{100}\right)-D$$
$$=T\left(\frac{1}{8}-T\right)\left\{\frac{701}{400}-\frac{7}{200}\left(T+\frac{1}{8}\right)\right\}+\tau(F(T)-1)\geq 0$$

is nonnegative. For any positive d_j and nonnegative weights summing to one,

$$\Big(\sum_j w_j d_j\Big)\Big(\sum_j \frac{w_j}{d_j}\Big)=1+\sum_{j<k}\frac{w_j w_k (d_j-d_k)^2}{d_j d_k}\geq 1.$$

Hence the reciprocal in (107) is at most $w_0/d_0+w_1/d_1'+w_2/d_2$. The resulting affine upper bound for (107) is exactly

$$w_0\frac{288911}{1267200}+w_1\frac{515617}{1775920}+w_2\frac{8399461}{35680000}\leq\frac{2909}{10000}<\frac{291}{1000}<\frac{3}{10}. \tag{108}$$

The three displayed vertex comparisons are strict rational arithmetic. The base coset is consequently impossible as before, and the other coset satisfies the D_{10}^{+} trace criterion. This proves $\Delta\geq 1216$ throughout the small-radius low-excess range.

Coverage and the precise finite inputs. The annulus is covered in the four coordinates $(T,\tau,\sqrt{r_p},\kappa_p)$ by a binary closed-box tree with root

$$\left[0,\frac{1}{8}\right]\times\left[0,\frac{1}{8}\right]\times\left[\frac{1}{10},\frac{8}{25}\right]\times\left[0,\frac{6}{7}\right].$$

Each internal node splits one rational interval into two closed intervals meeting at the split point, leaving the other coordinates unchanged. Induction on the tree proves that its leaves cover the entire root, including all boundaries. At a leaf with radius interval $[a,b]$, the arithmetic rules use $r_0=\max(1/100,a^2)$ and $r_1=\min(1/10,b^2)$. Nonnegative radius endpoints justify this squaring. The recorded tree has 7 133 nodes and the following 3 567 leaves:

proved leaf rule	number
joint directional determinant, (66)	1 755
averaged dual determinant, (67)	425
odd-spinor mass contradiction, (68)	683
Schur interval contradiction	285
empty radius/maximum shape	103
separate contact slacks, (99)–(101)	294
uniform contact slacks, (93), (101)	15
localized trace, (86)	3
outside the trace triangle	4

What is required is the actual tree, the rational parameters at each leaf, and the listed rational comparisons and polynomial identities. For the intermediate-radius forest the analogous input is its four root trees and the two polynomial certificates at each retained leaf, as specified above. No certificate is asked to assert an inequality for an arbitrary real matrix or an arbitrary gluing: all such implications have been proved in this section.

The fixed reference data required for the phase and spinor arguments are also explicit finite data: the sixteen integral phase covectors and their physical images; nine independent selected rows; the index-two identities (84), the base centre and its integral double; the rank-ten reference Gram identity and the fixed D_{10}^{+} root and cell data used in the trace criterion; the 32 integral images in (87); the complete contact catalogue and the even-parity signs (59); and the 64 odd-spinor integral images. The determinant of the rank-nine reference Gram matrix is 1024. Every reference-coordinate image or sign is an exact finite algebraic check. These phase and reference-cell identities are checked for each of the eight possible selected horizontal indices and for both orientations of the normal coordinate. Consequently the index and orientation chosen from an actual metric always have the required fixed geometry. Completeness of a catalogue is verified by the finite minimum-vector enumeration bounds specified for contact catalogues; the analytic contact-marking theorem then supplies its nonnegative weights. These reference checks and the two box forests are the computational dependencies of the low-excess argument.

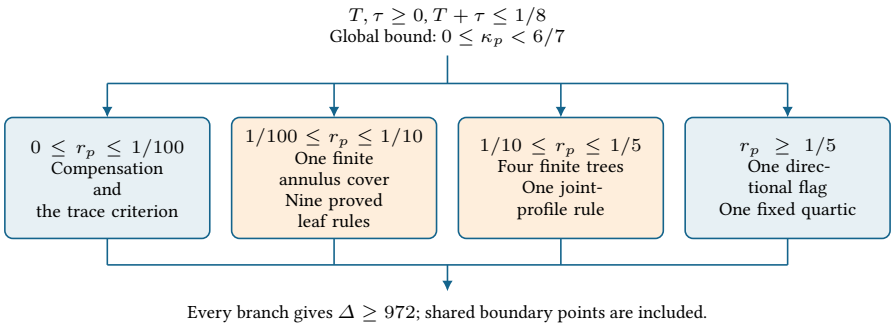

Fig. 2. The low-excess reduction. Orange boxes require finite covering data; all leaf implications are proved in the text. The other two ranges require no covering subdivision.

We can now conclude the whole reduction. An actual central-range candidate with $T+\tau \leq 1/8$ has $0 \leq \kappa_p < 6/7$ by the global coordinate argument above. If $r_p \leq 1/100$, (102)–(108) give the trace terminal. If $1/100 \leq r_p \leq 1/10$, a leaf of the annulus tree applies, and its proved rule gives $\Delta \geq 972$ (or 1216). If $1/10 \leq r_p \leq 1/5$, one of the four directional regimes and its intermediate-radius tree applies. If $r_p \geq 1/5$, the large-radius quartic argument applies. All inequalities at the three dividing radii are included on both adjacent closed ranges, so no boundary case is omitted. Thus, conditional only

on the specified finite checks, every such low-excess candidate satisfies $\Delta \geq 972$. This justifies the restriction $T + \tau \geq 1/8$ in the remaining `0030` cover.

10 The positive inverse-entry branch and a uniform final lemma

For $w \geq 0$, the selected-row trace floor and (34) give

$$T \geq L_{\mathrm{tr}}(m) - 4ws, \qquad \Big(\frac{3}{2} + \frac{w}{2m}\Big)\tau + (1 - 4w)s + \frac{14}{5}\delta < C(m).$$

The common caps apply when $w \leq 1/4$. For $w > 0$, write $w = \omega_+ - \omega_-$ for the two nonnegative weights of the selected signed contacts. The selected contact r has $S[r] = 2(1 + \tau)$, and $\omega_+ \geq w$. For any term $\omega r r^{\mathsf{T}}$ in the nonnegative expansion of R, evaluation on Sr gives $S[r] = (Sr)^{\mathsf{T}} R(Sr) \geq \omega S[r]^2$; division by $S[r] > 0$ proves $\omega S[r] \leq 1$. Thus $2w(1 + \tau) \leq 1$, or $\tau \leq 1/(2w) - 1$. A normal reflection interchanges the two weights, so the argument applies to either orientation. Similarly the selected contribution to the inverse-trace deficit gives $2w\tau \leq \delta$: the selected pair contributes at least $2|w|\tau + 4\max(0, -ws)$, and every discarded contact contribution is nonnegative. Apply (22) to A, H, their selected-coordinate rescalings, and the seven-dimensional blocks. For example, rescaling the last coordinate of H by squared factor $a_* > 0$ multiplies its determinant by a_* and changes the traces by $(a_* - 1)h$ and $(a_*^{-1} - 1)\chi$. Taking $a_* = qc$ leaves a constant determinant cap $q \min(1137/1024, 243\mu^2/512)$. The same operation on A, with $a_* = q/m$, has cap $q \min(1137/512, 243m^2/256)$. Every trace still contains the original ρ, δ, r_p.

The domain $r_p \leq 1/100$. The domain $r_p \leq 1/100$ has a uniform rational proof. Write

$$G(c) = \frac{131072c^3}{(9c + 3)^2}, \qquad b_0 = \frac{217}{250}, \qquad U(c) = 9 - \frac{b_0}{c}.$$

The ambient spectral ceiling gives $H[p] \leq (33/7)r_p$. Since $c + H[p] = 1 + \tau$ and $\tau \geq 0$, it follows that $c \geq 667/700$. Also $p_i^2 \leq r_p$, and

$$\left(r_p + \frac{33}{250}\right)^2 - 2r_p \geq \frac{1}{15625} > 0.$$

Thus $\sqrt{2}p_i \leq r_p + 33/250$, even if $p_i < 0$, and the exact inverse-trace identity gives $\operatorname{tr} H^{-1} \leq U(c)$. The short-dual average, with its nonnegative correction discarded, and the one-vector flag give

$$\Delta \geq G(c) \det H. \tag{109}$$

For completeness, the average bounds the squared dual length ν by $2\nu \leq (9c+3)/(8c)$; since $\det Q = 1024c \det H$, substituting this into $\Delta \geq \det Q/(2\nu^2)$ gives (109). The contact ceiling and $r_p \leq 1/100$ give the common height bound

$$c \geq \frac{99}{100} + \tau - \frac{T}{50}, \qquad T + \tau \geq \frac{1}{8}. \tag{110}$$

First suppose $0 \le T \le 7/16$, so $\det H \ge F(T)$. Both $G(c)$ and $G(c)/c = 131072[c/(9c+3)]^2$ increase for $c > 0$. For $0 \le T \le 1/8$, let $a_0 = 79/80$. Equation (110) implies $c \ge 223/200 - 51T/50 \ge a_0$; concavity gives $F(T) \ge 1 + 25T/32$. Since $G(a_0) \ge 890$,

$$G(c)\det H \ge \frac{890}{a_0}\Big(\frac{223}{200} - \frac{51T}{50}\Big)\Big(1 + \frac{25T}{32}\Big).$$

The right side is a concave quadratic. Its endpoint values are $79388/79$ and $125045/128$, both greater than 972. The two remaining intervals use separate lower bounds on c and $F(T)$; all entries of the following table are exact rational bounds:

T	c at least	$G(c)$ at least	$F(T)$ at least	$G(c)F(T)$ at least
$[1/8, 3/8]$	$393/400$	886	$281/256$	$124483/128$
$[3/8, 7/16]$	$157/160$	884	$1129/1024$	$249509/256$

The height bounds follow from (110) and $\tau \ge 0$, the G bounds by substitution and monotonicity, and the F bounds by checking its two endpoints. Both final entries exceed 972.

For $T \ge 7/16$, the changing inverse-trace budget can be handled by scaling. For $c \ge a \ge b_0 > 0$, $U(c) \ge U(a) > 0$, and

$$\frac{G(c)}{U(c)^8} = 131072\Big(\frac{c}{9c+3}\Big)^2 c\Big(\frac{c}{9c-b_0}\Big)^8$$

is increasing. The first factor increases, and the derivative of the last factor is $9c^8(c - b_0)/(9c - b_0)^9 \ge 0$. Let $q = U(c)/U(a) \ge 1$ and $\widehat{H} = qH$. Then $\operatorname{tr}\widehat{H} \ge \operatorname{tr} H$ and $\operatorname{tr}\widehat{H}^{-1} \le U(a)$. Whenever $0 < \ell \le t$ satisfy

$$t + 7\ell \le \operatorname{tr} H, \qquad U(a) \le \frac{1}{t} + \frac{7}{\ell},$$

Lemma 11 and its monotonicities give $\det\widehat{H} \ge t\ell^7$. Since $G(c)/q^8 \ge G(a)$, this proves $G(c)\det H \ge G(a)t\ell^7$. Only two fixed choices are needed:

range	a	t	ℓ	$t + 7\ell$
$c \ge 39/40$	$39/40$	2	$103/112$	$135/16$
$c < 39/40$	$667/700$	$12/5$	$127/140$	$35/4$

The first trace condition is $T \ge 7/16$. In the second range, (110) forces $T > 3/4$, proving its trace condition. Direct rational comparisons give, for the respective rows,

$$U\Big(\frac{39}{40}\Big) = \frac{7907}{975} < \frac{1671}{206},$$
$$U\Big(\frac{667}{700}\Big) = \frac{26977}{3335} < \frac{12395}{1524},$$

where the right sides are $1/t + 7/\ell$, and

$$G\left(\frac{39}{40}\right) 2\left(\frac{103}{112}\right)^7 > 972, \qquad G\left(\frac{667}{700}\right) \frac{12}{5}\left(\frac{127}{140}\right)^7 > 972.$$

These displayed rational inequalities complete the small-phase argument; no numerical minimization, logarithm comparison or covering certificate is needed for this region. The last boxes admit the following single lemma; it is preferable to 122 separate numerical arguments.

Lemma 16 (a whole small-column rectangle). *In the region $r_p \geq 1/100$ and $T + \tau \geq 1/8$, suppose $0 < c_- \leq c \leq c_+$ and $0 \leq w_- \leq w \leq w_+$. Set*

$$\kappa_{p,-} = \max(0, 2 - 4c_+ w_+), \qquad \kappa_{p,+} = \max(0, 2 - 4c_- w_-),$$
$$r_- = \max\left(\frac{1}{100}, \frac{\kappa_{p,-}^2}{8}\right), \qquad r_+ = \min(3, \kappa_{p,+}^2),$$
$$t_0 = \max\Big(0, \min\Big(\frac{1}{8}, \frac{\frac{9}{8} - r_+ - c_+}{1 + 2r_+}\Big)\Big), \qquad a_* = \frac{7}{16},$$
$$b_* = 1 + r_- - \frac{\kappa_{p,+}}{2}, \qquad V_* = 9 - \frac{b_*}{c_+}.$$

Let $\sigma = \max(8 + a_*, 64/V_*)$, and choose a rational upper bound s_* on

$$\sqrt{(\sigma V_* - 64)(\sigma V_* - 36)}.$$

Let $z_* = (\sigma V_* - 48 + s_*)/(2V_*)$ and $G_* = 131072 c_-^3/(9c_- + 3)^2$. If $b_*, V_* > 0$, $\sigma/8 \leq z_* < \sigma$, and

$$G_* \min(F(t_0), F(a_*)) \geq 972, \qquad G_* z_* \left[\frac{\sigma - z_*}{7}\right]^7 \geq 972,$$

the entire rectangle is excluded, for every m in (15).

Proof. The relation $\kappa_p = 2 - 4cw \geq 0$ and the rectangle bounds imply $\kappa_{p,-} \leq \kappa_p \leq \kappa_{p,+}$. The dominant-coordinate bounds $\kappa_p^2/8 \leq r_p \leq \kappa_p^2$, the standing bound $r_p \leq 3$ and the present lower bound $r_p \geq 1/100$ imply $r_- \leq r_p \leq r_+$. Using $T \geq 0$, the contact inequality and trace condition give

$$1 + \tau - c \leq r_p(1 + 2T) \leq r_+(1 + 2T), \qquad \frac{9}{8} - r_+ - c_+ \leq (1 + 2r_+)T.$$

Thus $T \geq t_0$. Also $1 + r_p - \kappa_p/2 \geq b_* > 0$. Since $0 < c \leq c_+$, the inverse-trace identity gives $\operatorname{tr} H^{-1} \leq 9 - b_*/c_+ = V_*$.
Monotonicity gives $G(c) \geq G_*$. For $t_0 \leq T \leq a_*$, concavity gives

$$F(T) \geq \min(F(t_0), F(a_*)).$$

The first finite comparison and (109) prove the claim. For $T \geq a_*$, Cauchy–Schwarz gives $\operatorname{tr} H \geq 64/\operatorname{tr} H^{-1} \geq 64/V_*$, hence $\operatorname{tr} H \geq \sigma$. The choice of s_* implies the polynomial inequality

$$V_* z_*^2 - (\sigma V_* - 48) z_* + \sigma \geq 0,$$

because four times V_* times its left side equals $s_*^2 - (\sigma V_* - 64)(\sigma V_* - 36)$. Let $\ell_* = (\sigma - z_*)/7$. The assumed range of z_* gives $0 < \ell_* \leq z_*$; multiplying by $z_*(\sigma - z_*) > 0$ shows that the polynomial inequality is exactly $V_* \leq 1/z_* + 7/\ell_*$. Lemma 11, with its monotonicities, therefore gives $\det H \geq z_* \ell_*^7$. The second finite comparison and (109) again give $\Delta \geq 972$. □

All 122 last boxes lie in

$$c \in \left[\frac{367377}{372736}, \frac{184575}{186368}\right], \qquad w \in \left[\frac{243}{512}, \frac{977}{2048}\right].$$

For this one rectangle the two certified bounds exceed 972.0233 and 1033.2195. The decimal numbers are downward summaries of strict rational comparisons.

11 What the finite cover verifies

The remaining parameters are (m, c, w). Every parameter box uses one common vector of nonnegative metric variables,

$$y = (\tau, h, s, T, \rho, \delta, V_A, r_p), \qquad V_A = \operatorname{tr}((A - I)^2).$$

Thus V_A is the square moment of the original horizontal block A. Rows that do not involve this moment have coefficient zero in that slot; rows involving it use the same actual matrix as every other row. The identities $T_A = 2\tau - 2h + 2s + T + 2$ and $E_A = 2\tau + 6h + 2s + T - 6$ specify its linked trace parameters. The necessary rows are the identities and inequalities above, affine envelopes of their quadratic terms, and the trace profiles with rational root bounds. Parameter-dependent coefficients are enclosed by a first-order expansion with an interval bound on each derivative. Explicitly, let x and a centre x_0 lie in a parameter box with $|x_i - x_{0,i}| \leq b_i^{\mathrm{box}}$. If f is differentiable there and $|\partial_i f(z) - a_i| \leq R_i$ throughout the box, then

$$\left| f(x) - f(x_0) - \sum_i a_i (x_i - x_{0,i}) \right| \leq \sum_i R_i |x_i - x_{0,i}| \leq \sum_i R_i b_i^{\mathrm{box}}. \tag{111}$$

To prove this, restrict f to $x_0 + t(x - x_0)$, subtract $t \sum_i a_i (x_i - x_{0,i})$, and apply the one-variable mean-value bound on $0 \leq t \leq 1$. The derivative is $\sum_i (\partial_i f - a_i)(x_i - x_{0,i})$, whose absolute value has the displayed bound. The resulting upper and lower affine functions bound f on the whole box, including its boundary. For rational coefficient expressions, all bounds required here reduce to finite scalar interval checks. Addition adds interval endpoints; multiplication takes the smallest and largest of the four endpoint products. Indeed, a bilinear function on a rectangle is a convex combination of its corner values; a zero-width interval reduces this to the one-variable statement. On

an interval $0 < \ell \le g \le u$, reciprocation gives $1/u \le 1/g \le 1/\ell$. A strictly negative denominator is handled by $1/g = -1/(-g)$. The derivative expressions are obtained exactly from

$$\partial_i(fg) = (\partial_i f)g + f\partial_i g, \qquad \partial_i\left(\frac{1}{g}\right) = -\frac{\partial_i g}{g^2}.$$

Induction over the expression proves both its enclosure and each derivative enclosure. The positive denominator checks also prove differentiability on the whole parameter box.

To see explicitly how these coefficient bounds give necessary rows, write a constraint as

$$F(x,y) = f_0(x) + \sum_{j=1}^{q} f_j(x)y_j \le 0, \qquad y_j \ge 0.$$

If $f_j^- \le f_j \le f_j^+$ are the affine enclosures just proved, then

$$f_0^-(x) + \sum_j f_j^-(x)y_j \le F(x,y) \le f_0^+(x) + \sum_j f_j^+(x)y_j.$$

The two inequalities follow by multiplying each coefficient bound by $y_j \ge 0$ and summing. Retaining the products $x_i y_j$ makes both bounding expressions linear; their four product inequalities are those of Section 2.2. The lower expression gives a necessary row, and the upper expression justifies objective bounds used in contractions. Alternatively, keep each $f_j(x_0)y_j$ and apply (111) to $F(\,\cdot\,, y)$ itself. Its derivative is $\partial_i f_0 + \sum_j (\partial_i f_j)y_j$; the same four-corner rule bounds it over the current metric intervals. This gives both affine bounds without retaining products, even for signed metric variables. In either construction every actual metric satisfies the resulting linear rows. The domain conditions of each chosen row require verification too. Write these conditions as conjunctions or alternatives of inequalities between rational expressions in (x, y); their constant parts concern only the chosen row data. The same interval rules prove each inequality on the entire current box. To prove $f \le g$ it suffices to enclose $f - g$ by an interval whose upper endpoint is nonpositive; for $f < g$ the endpoint must be strictly negative. Minimum operations are enclosed by taking the minima of the respective endpoints. Thus checking a row at its centre is never sufficient: both its domain and its affine coefficient enclosure must hold throughout the box. When contractions change a determinant or spectral bound, their proved bounds enter the same domain checks before the row is appended. Logarithm rows use Lemma 12 with rational choices of ℓ, r, H and a rational lower bound for the two endpoint residuals.

The complete positive-entry tree has 20 995 nodes, 10 498 leaves, and 500 225 checked bounds. Its leaf types are 6 170 quarter-range, 1 080 sign-sensitive, 1 374 projected-block, 1 555 parameter-product, one positive-face, 196 global small-shear, and 122 final-lemma leaves. Every split and contraction is checked, and the terminal frontier is empty. Together with the negative-entry and outer-range covers, this proves the exclusion of `0030`.

12 What is verified, and where

The finite checks. The following statement collects every finite input of the proof; the trees, leaves and scalar checks it names are those described in Sections 3–11.

Computational result 17 (the complete projected-contact cover for `0030`). The printed marker is the record `0030:030301719` of the data set entry by entry (the script `hermite11\_printed\_forms\_check.py` of the Zenodo record [4, Section 9] checks the eleven printed forms of the series against the certificate database) and is integrally equivalent to $\sqrt{2}L_{\mathrm{ref}}$, the reference model of Section 3 (Appendix A: the matrix P_Λ of [4, Appendix B], the first macadamia of [5, Table 3]). The three outer-range signed-average covers and both middle-range covers are exhaustive on their stated roots. Every terminal witness contradicts the contact, inverse-trace or determinant inequalities derived in Sections 3–11, using the positive auxiliary references and rational scalar bounds specified there. The three polynomial positivity checks P_1, P_2, P_3 of Lemma 15 and the Bernstein check of the quartic of (53) hold. The middle covers have $10\,498$ positive-entry leaves and $1\,140$ negative-entry leaves. The low-excess inputs are the fixed contact, phase and reference-cell identities listed in Section 9, its $3\,567$-leaf annulus tree, and its four intermediate-radius trees with 96 leaves in total. Their scalar checks are exactly those proved in Section 9. The sign boundary, parameter boundaries, and each conditional subdivision are included; no symmetry of the actual metric is assumed. The exact tests are the exact arithmetic, enumeration and covering rules stated below.

In detail, the finite inputs of the proof, in the order in which the argument uses them, are:

- a the fixed coordinate certificate of Section 3: nine vectors of $\sqrt{2}L_{\mathrm{ref}}$ whose Gram matrix is the printed form, checked by membership and entry by entry (Appendix A);
- b the three outer-range signed-average covers of Section 3, on $[256/243, 13/10]$, $[8/5, 17/10]$ and $[17/10, 13/5]$, with 379, 153 and 23 outer nodes (and 51 inner leaves for the middle one), whose leaves are the signed-average, transition-trace and terminal determinant tests of that section;
- c the three polynomial positivity checks P_1, P_2, P_3 of Lemma 15, by Bernstein coefficients on $[13/10, 133/100]$;
- d the negative-entry cover of Section 8: $2\,279$ nodes, $1\,140$ leaves, $28\,576$ exact bounds, no pending region, on the rectangle $(m, c) \in [133/100, 8/5] \times [5/26, 5/4]$;
- e the low-excess inputs of Section 9: the fixed contact, phase and reference-cell identities (the sixteen integral phase covectors, nine selected rows, the index-two identities (84), the base centre, the D_{10}^{+} reference data, the 32 integral first-layer images, the contact catalogue with the even-parity signs (59), the 64 odd-spinor images), checked for all eight selected indices and both orientations; the Bernstein check of the quartic P of (53); the four intermediate-radius trees with 188 nodes and 96 leaves and their two polynomial certificates per leaf (55); and the annulus tree with $7\,133$ nodes and $3\,567$ leaves, with the leaf rules of the table in that section;

`f` the positive-entry tree of Sections 10–11: 20 995 nodes, 10 498 leaves, 500 225 checked bounds, with the leaf types listed in Section 11, its 122 final-lemma leaves closed by Lemma 16.

The exact arithmetic, enumeration and covering rules under which these are checked are those stated in [5, Appendix D.7] (the rational linear certificates of Lemma 44 there, which is Lemma 13 here, the three obligations of a finite cover, and the rules for products, square roots, logarithms and Bernstein coefficients), with the exact minimum-vector enumeration of [5, Section 5.1]; they are the rules of the verifier of [5, Section 7]: matrix and polynomial identities compared coefficient by coefficient, finite vector and cell lists checked by exhaustive bounded enumeration with an exact LDL^{T} decomposition, and linear or scalar terminal witnesses reconstructed by the checker with Lemma 13, rational interval arithmetic, polynomial signs (Sturm's theorem [1]) or positive Bernstein coefficients; every square-root and logarithm enclosure carries its error bound.

Where the records are. The final record of this case is the file

`PLambda_0030_closed_verified.json`

It joins both signs of the selected inverse entry and the outer ranges, that is, the finite checks `a`–`f` above. The written argument of this case consists of the selected-coordinate reduction, the joint inverse profiles, the complete low-excess proof and all outer and main-cover implications; the fixed reference identities and the rational covers are its finite dependencies. The record and its checkers are in the certificate archive `hermite11_local_certificates.zip` of the Zenodo record doi:10.5281/zenodo.23979316. Its `case_map.json` identifies each principal record, its size, the underlying checker sources and recorded result; `certificate_manifest.json` binds the stored files and compressed logical streams. The common archive identifiers and verification procedure are given in [4, Section 9].

The three small certificates used for Lemma 15 are assembled in `Hermite11.case0030VerifiedLowStrip`, from the saved root, mean and determinant Bernstein certificates in `Hermite11/Certificates/Case0030Scalars.lean`. The quartic of (53) is also proved directly in Lean by `Hermite11.d8_large_radius_quartic`; the resulting ratio bound is `Hermite11.d8_large_radius_ratio`, both in `Hermite11/D8LargeRadiusScalar.lean`. The Lean development also contains these analytic proofs.

The proof of Theorem 1. Proof. Let $L \supset M \supset N$ be a chain as in Definition 2 whose rank-9 member is admissibly marked by the form `0030:030301719`, and suppose $\Delta = \det L < 972$. In the notation of Section 2.1 this is an admissible residual chain with marker $P = P_\Lambda$ and $\Delta < 972$; by (2), $768 \le d < d_{\mathrm{cut}}$ and $e < 1137$. Transport the marking to the basis of Appendix A and pass to the physical coordinates of Section 3, in which S is the section metric, $R = S^{-1}$, $\operatorname{tr} R \le 9$, $S[x] \ge 2$ on L_{ref}, and $m = R_{99}$. Section 3 gives $\Delta \ge 972$ for $m \le 256/243$ (by (10)) and a contradiction for $m \ge 13/5$ (by inverse AM–GM), and the three outer-range covers of Computational result 17 exclude $[256/243, 13/10]$ and $[8/5, 13/5]$; so $13/10 < m < 8/5$ by (15). Section 4 selects a contact and defines c, w, κ_p, r_p, T, τ; Sections 5–7 prove the trace profile, the contact, directional and short-dual inequalities, and the caps common to both signs of w.

If $w < 0$, Lemma 15 gives $m > 133/100$, the rectangle of Section 7 contains (m, c), and the negative-entry cover of Computational result 17 (Section 8) excludes it. If $T + \tau \le 1/8$, then $w > 0$ by (51) and Section 9 gives $\Delta \ge 972$ in each of the four ranges of r_p, using the annulus and intermediate-radius trees of Computational result 17. If $w \ge 0$ and $T+\tau \ge 1/8$, the domain $r_p \le 1/100$ is excluded by the rational argument of Section 10, and the positive-entry tree of Computational result 17 (Sections 10–11), whose 122 final-lemma leaves are closed by Lemma 16, excludes $r_p \ge 1/100$. The stated regions cover every possibility, including their endpoints. Hence every admissible chain has $\Delta \ge 972$. □

The hypotheses. Theorem 1 uses the following inputs.

1. The admissible-chain conditions of Definition 2, including the marking and the exact cutoffs.
2. Lemma 14 ([4, Lemma 12]), for the lower endpoint $\det A \ge m^2$ of Section 3, and the lemmata of Section 2.2, proved in [5].
3. The D_{10}^{+} trace criterion used by the three phase terminals of Section 9, [5, Lemma 43], whose reference-cell data for $\sqrt{2}D_{10}^{+}$ are among the fixed data of `e`.
4. Computational result 17: the finite checks `a`–`f` listed above, re-verified from the archived record by the exact rules recalled above.

Lean correspondence. The local analytic statements are `0030`: `Hermite11.case0030_from_finite_input`. Their finite hypotheses, integral-equivalence wrappers and saved reference records are listed in `case_map.json`. Large cover payloads remain external exact checks.

13 Use of computational tools

Refer to Section 10 of [4].

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A The form `0030:030301719`

The form is the record at position 30 301 719 of the file `perfect_0030.nc` of the data set of [3], scaled to minimum 4 [4, Section 6.1]. Two bases of it are used in this series.

The basis of the certificates. In the basis used by the certificates of Computational result 17, the integral Gram matrix A_P of the record, with $P = (4/s_P)A_P$, $s_P = 4$, $\det P = 512$ and 136 contact pairs, is

$$A_P = \begin{pmatrix} 4 & 2 & 2 & 0 & 2 & 2 & 2 & -6 & -2 \\ 2 & 4 & 2 & 2 & 0 & 0 & 0 & -4 & 0 \\ 2 & 2 & 4 & 2 & 0 & 0 & 0 & -4 & 0 \\ 0 & 2 & 2 & 4 & -2 & -2 & -2 & 0 & 2 \\ 2 & 0 & 0 & -2 & 4 & 2 & 2 & -4 & -2 \\ 2 & 0 & 0 & -2 & 2 & 4 & 2 & -4 & -2 \\ 2 & 0 & 0 & -2 & 2 & 2 & 4 & -4 & -2 \\ -6 & -4 & -4 & 0 & -4 & -4 & -4 & 12 & 3 \\ -2 & 0 & 0 & 2 & -2 & -2 & -2 & 3 & 4 \end{pmatrix}.$$

We checked, in exact integer arithmetic, that this matrix is symmetric and positive definite (its leading principal minors are $4, 12, 32, 64, 128, 256, 512, 256, 512$), that $\det A_P = 512$, and, by enumeration, that $\min A_P = 4$ with exactly 272 minimal vectors, that is, 136 opposite pairs; these are the invariants of Λ_9. The first reduction of Section 3 exhibits nine vectors of $\sqrt{2}L_{\mathrm{ref}}$ with this Gram matrix (the fixed coordinate certificate).

The basis of the root paper. In [4, Appendix B] the same form is the matrix $P_\Lambda = 2C^{\mathsf{T}}C$, where the columns of C are the simple roots $b_1, \dots, b_8$ of E_8, with a ninth coordinate 0 appended, and $b_9 = e_1 + e_9$:

$$P_\Lambda = \begin{pmatrix} 2G_{E_8} & f_1 + 2f_2 - 2f_3 \\ (f_1 + 2f_2 - 2f_3)^{\mathsf{T}} & 4 \end{pmatrix},$$

that is,

$$P_\Lambda = \begin{pmatrix} 4 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 4 & 0 & -2 & 0 & 0 & 0 & 0 & 2 \\ -2 & 0 & 4 & -2 & 0 & 0 & 0 & 0 & -2 \\ 0 & -2 & -2 & 4 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 4 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 4 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 4 & 0 \\ 1 & 2 & -2 & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix},$$

with $b_1 = \frac{1}{2}(1,-1,-1,-1,-1,-1,-1,1)$, $b_2 = f_1 + f_2$, $b_k = f_{k-1} - f_{k-2}$ ($3 \leq k \leq 8$), and G_{E_8} the Gram matrix of $b_1, \ldots, b_8$. Here $C\mathbb{Z}^9 = L_{\mathrm{ref}}$ is the reference lattice of Section 3, so that P_Λ is the Gram matrix of a basis of $\sqrt{2}L_{\mathrm{ref}}$ and the physical section metric S of Section 3 is the matrix $Y = \frac{1}{2}C^{-\mathsf{T}}QC^{-1}$ of [4, Section 5.5]. We checked that $\det P_\Lambda = 512$ and that $\min P_\Lambda = 4$ with 272 minimal vectors. An integral change of basis U with $U^{\mathsf{T}} P_\Lambda U = A_P$ is part of the fixed coordinate certificate of Computational result 17; one such matrix, with $\det U = 1$, is

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & -3 & 1 \\ -1 & 1 & 1 & 1 & 0 & 0 & -1 & -3 & 1 \\ -1 & 0 & 1 & 1 & 0 & 0 & -1 & -2 & 1 \\ -1 & 0 & 1 & 1 & 0 & -1 & -1 & -1 & 1 \\ -1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 & -1 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and we checked $U^{\mathsf{T}} P_\Lambda U = A_P$ in exact integer arithmetic (any other such U differs from this one by an automorphism of P_Λ).