γ_{11}: Reduction by Sections

THE MACADAMIAS 0019 AND 0017

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Abstract. The root paper of this series reduces the value $\gamma_{11} = 4/\sqrt[11]{972}$ of the Hermite constant in dimension 11 to a statement about chains $L \supset M \supset N$ of lattices of ranks 11, 10 and 9 whose rank-9 member lies over one of 4 487 813 perfect forms of rank 9; the γ_{11} paper settles all of them but ten, the *macadamias*. This paper settles two of the ten: `0019:030884313` (the form P_S of the root paper, an extension of E_8 by a simplex vector) and `0017:001746552` (the form P_K, KAPPA9.2). We prove that every admissible chain whose rank-9 member is admissibly marked by the form has $\det L \geq 972$. The method is a reduction by sections: for `0019` the nine simplex contacts turn the problem into one about an eight-dimensional covariance matrix, and a tree of 17 347 nodes with 7 743 scalar determinant witnesses covers the remaining box; for `0017` a two-vector flag bound and the decomposition of the section metric along a short-dual plane and its rank-seven kernel reduce the problem to five parameters, covered by a tree of 3 010 595 nodes. The proof is computer-assisted: every finite check is stated as a Computational result, and the verification records in the certificate archive are named.

Keywords: Hermite constant · Lattice packings · Perfect forms · Sections of lattices · Exact certificates · Branch and bound · Verified computation

1 Introduction

This paper belongs to a series of seven papers, made public concurrently, whose aim is to establish the Hermite constants γ_{10}, γ_{11} and γ_{12}: the root paper [4] proves γ_{10}, reduces γ_{11} to a statement about chains $L \supset M \supset N$ of lattices of ranks 11, 10 and 9 whose rank-9 member lies over one of 4 487 813 perfect forms of rank 9, and derives γ_{12} from γ_{11}; the γ_{11} paper [5] settles every one of these forms but ten, the *macadamias*, by exact certificates. Nine macadamias are settled in five companion papers, grouped by method; `0052` is settled in [5]. The present paper treats two cases by *reduction by sections*:

- `0019:030884313`, the form P_S of [4], an extension of E_8 by a simplex vector;
- `0017:001746552`, the form P_K of [4], the form KAPPA9.2 of [12].

These are the macadamias numbered 2 and 3 in [4, Table 4]: the lattice of rank 10 over P_S has 294 minimal vectors, determinant $25600/27$ and dual minimum $3/5$, the one over P_K has 270 minimal vectors, determinant 1024 and dual minimum $3/4$; both determinants are below $d_{\mathrm{cut}} = 4 \cdot 972^{9/11}$, so that no certificate at rank 10 can exist for these forms and the bound 972 must be proved at the level of the rank-11 lattice.

Theorem 1. *Assume Computational results 11 and 13, and the reference data of Appendix A. Let P be one of the forms* `0019:030884313` *and* `0017:001746552`*, scaled to minimum* 4*, and let $L \supset M \supset N$ be a chain with the properties listed in Definition 2 below whose rank-9 member N has a Gram matrix admissibly marked by P. Then $\Delta = \det L \geq 972$.*

The idea is the same for both forms, and it gives the paper its name. The quantity to bound, Δ, depends on the rank-9 lattice N and on the way M and L are glued over it; the two cuts of the chain are orthogonal to shortest dual vectors, and the gluing parameters enter Δ only through a few *sections* of the dual lattices. For `0019` the nine simplex contacts of P_S provide a coordinate system in which the problem becomes one about an eight-dimensional metric A and a probability vector p (Section 3): the one-vector flag bound $\Delta \geq e/(2X[z]^2)$ applied to the covector $z = (1, \dots, 1, -5)$ gives $\Delta \geq \frac{1024}{9} \det K/m^2$ in these coordinates, two outer ranges of the parameter m are excluded by AM–GM, and a tree of 17 347 nodes with 7 743 scalar determinant witnesses covers the remaining box. For `0017` a two-vector flag bound (Section 4) applied to the plane spanned by the three shortest dual vectors of P_K gives $\Delta \geq 1024\, g^2/k$, where g is the determinant of the section metric and k that of its rank-seven kernel block; the problem is reduced to five parameters $(k, T, \theta, \lambda, \Lambda)$ and a tree of 3 010 595 nodes covers their closed root (Section 5). Figure 1 shows the two cases side by side.

Organisation. Section 2 gives the local setting and common tools: the chain, its constants, the marking, and the statements of the common toolbox of the series, whose proofs are given in [5]. Sections 3–5 are the two cases and the flag bound between them: each case is a reduction of the chain to a few real parameters, a set of necessary inequalities on them, and a finite covering tree whose terminal boxes are excluded by exact scalar witnesses; the finite checks are stated as Computational results. Section 6 says what is verified by computation, where the records are, and on which hypotheses Theorem 1 rests. Appendix A prints the two Gram matrices in the bases used by the certificates and relates them to the root paper's P_S and P_K.

2 The local setting and common tools

2.1 Notation and admissible chains

We use the common coordinates of [5, Appendix D]. The determinant of a lattice is the determinant of its Gram matrix, and its minimum is the least nonzero squared length. Its dual is $L^* = \{y \in \operatorname{span} L : y \cdot x \in \mathbb{Z} \text{ for all } x \in L\}$. A sublattice $M \subset L$ is *primitive* if $M = L \cap \operatorname{span} M$; a primitive vector generates a primitive rank-one sublattice. Write $Q[x] = x^{\mathsf{T}} Q x$, with T denoting transpose, and $\operatorname{tr} A$ for the sum of the

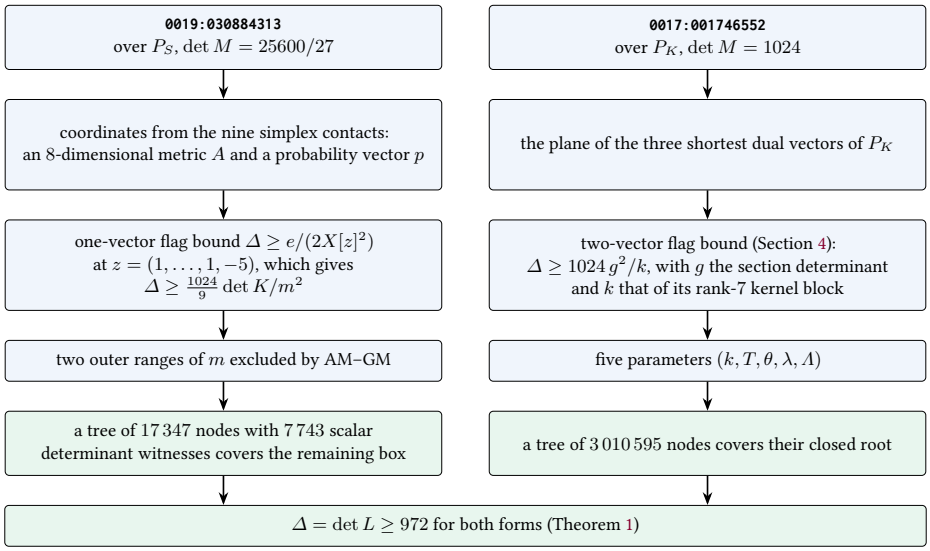

Fig. 1. The two cases of this paper. In both, the gluing parameters enter Δ only through a few sections of the dual lattices; a flag bound reduces the chain to a few real parameters, and a finite covering tree with exact scalar witnesses excludes what remains.

diagonal entries. For symmetric matrices, $A \succeq 0$ means $A[x] \geq 0$ for every x; $A \succ 0$ means $A[x] > 0$ for $x \neq 0$, and $A \preceq B$ means $B - A \succeq 0$. The nested sections form a flag of ranks 11, 10 and 9; its notation is

$$\begin{gathered} e = \det N, \qquad d = \det M = \frac{e}{u}, \qquad \Delta = \det L = \frac{e}{uv}, \\ u = \min M^*, \qquad v = \min L^*. \end{gathered} \tag{1}$$

The cutoffs are the positive numbers $d_{\mathrm{cut}}^{11} = 4^{11}972^9$ and $e_{\mathrm{cut}}^{55} = 2^{246}3^{197}$.

Definition 2 (admissible residual chain). *A chain $L \supset M \supset N$ is admissible for P if L has rank eleven and minimum at least four, $M = L \cap n_{11}^{\perp}$ and $N = M \cap n_{10}^{\perp}$ for shortest nonzero vectors $n_{11} \in L^*$ and $n_{10} \in M^*$, its rank-nine Gram matrix is marked by P as defined below, and*

$$\begin{gathered} 768 \leq d < d_{\mathrm{cut}} = 4\,972^{9/11}, \\ e < e_{\mathrm{cut}} = \Big(\frac{4096}{3}\Big)^{1/10} d_{\mathrm{cut}}^{9/10} < 1137. \end{gathered} \tag{2}$$

Shortest dual vectors are primitive, so the determinant identities above follow by an adapted basis. Rounding a lift of n_{10} across the last cut gives $v \leq u + v/4$, hence $v \leq 4u/3$ and $\Delta \geq 3d/(4u)$. Likewise, if $t_* = \min N^*$, rounding across the first cut gives $u \leq 4t_*/3$. The global reduction [4, Proposition 15] supplies such a chain from a hypothetical counterexample.

Definition 3 (marking). *Let Q be the Gram matrix of N in an integral basis, and $X = Q^{-1}$. A perfect form P of minimum four marks Q (and Q, or N, is then said to be admissibly marked by P) if* $\min Q \geq 4$ *and there are weights satisfying*

$$X = \sum_{c \in \mathcal{C}(P)} w_c \, cc^{\mathsf{T}}, \qquad w_c \geq 0, \qquad \sum_c w_c \leq \frac{9}{4}, \qquad w_c \leq \frac{1}{4}, \\ \operatorname{tr}(PX) \leq 9, \qquad \det P \leq e. \tag{3}$$

Here $\mathcal{C}(P)$ contains one vector from each opposite pair of minimum vectors of P.

Perfect means that the matrices cc^{T} span the symmetric matrices. A marking is transported by an integral change of basis. Its existence for arbitrary Q is proved in [4, Proposition 7]; here the marker is fixed to the printed form.

Gluing coordinates. Vectors $\beta, \alpha \in \mathbb{R}^9$ and a scalar $\vartheta \in \mathbb{R}$ give the dual squared lengths at $(z, k, k') \in \mathbb{Z}^9 \times \mathbb{Z}^2$ as

$$X[z] + u(\beta^{\mathsf{T}} z + k)^2 + v(\alpha^{\mathsf{T}} z + \vartheta k + k')^2, \tag{4}$$

and the primal squared lengths as

$$Q[x - k\beta - k'(\alpha - \vartheta\beta)] + \frac{(k - \vartheta k')^2}{u} + \frac{k'^2}{v}. \tag{5}$$

Write $\|t\|_{\mathbb{R}/\mathbb{Z}} = \min_{j \in \mathbb{Z}} |t - j|$. The first shortest-normal condition implies

$$X[z] + u \, \|\beta^{\mathsf{T}} z\|^2_{\mathbb{R}/\mathbb{Z}} \geq u \qquad (0 \neq z \in \mathbb{Z}^9). \tag{6}$$

The second condition applies to every nonzero vector of M^* and every lift to L^*.

Physical coordinates. Choose C_P with $P = C_P^{\mathsf{T}} C_P$ and write

$$Q = C_P^{\mathsf{T}} S C_P, \qquad R = S^{-1}. \tag{7}$$

The columns of C_P realise the reference lattice in Euclidean space. Thus S measures squared lengths in these fixed coordinates, with reference metric I, and R is its inverse: $\det S = e/\det P$, $\operatorname{tr} R = \operatorname{tr}(PX) \leq 9$; integral vectors c, z become $C_P c, C_P^{-\mathsf{T}} z$. We write $|x| = \|x\|$ for Euclidean length and $\|E\|_F^2 = \operatorname{tr}(E^{\mathsf{T}} E)$ for the squared Frobenius norm. Eigenvalues are listed in decreasing order. The notation $I_n, e_i, \mathbf{1}$ denotes the identity, coordinate vectors and all-one vector; Sym_n denotes the symmetric $n \times n$ matrices. We use $(x)_+ = \max(x, 0)$; subscripts $-$ and $+$ on a box coordinate denote its lower and upper endpoints. Schur complements, cofactors and adjugates have the conventions and block identities of [5, Appendix D.2]. Exact LDL^{T} factorisation and bounded short-vector enumeration are described in [5, Section 5.1].

2.2 Statements of the common toolbox used here

The two cases use the following statements of the common toolbox of the series, the determinant and flag inequalities. The statements below cite their proofs in the common toolbox [5, Appendix D].

Lemma 4 (Circle triangle; [5, Lemma 22]). *If $t_0 + t_1 + t_2 = 0$ in $\mathbb{R}/\mathbb{Z}$, then $\sum_i \|t_i\|_{\mathbb{R}/\mathbb{Z}} \le 1$. Consequently, if $y_0 + y_1 + y_2 = 0$ are nonzero vectors of M^* of squared lengths b_i, then the squared length v of the last normal satisfies*

$$\sum_{i=0}^{2} \sqrt{\max\left\{0, 1 - \frac{b_i}{v}\right\}} \le 1. \tag{8}$$

In particular $v \le (9/8)\max_i b_i$.

Lemma 5 (One section vector controls both extensions; [5, Lemma 23]). *For every $0 \ne z \in \mathbb{Z}^9$,*

$$\Delta \ge \frac{e}{2X[z]^2}. \tag{9}$$

If three nonzero section covectors sum to zero and all have squared X-length at most x, then

$$\Delta \ge \frac{16e}{27x^2}. \tag{10}$$

Lemma 6 (One quadratic interpolation rule; [5, Lemma 26]). *Let f be three times continuously differentiable on an interval, and let $a \ne b$ belong to it. The quadratic q determined by $q(a) = f(a)$, $q'(a) = f'(a)$, and $q(b) = f(b)$ satisfies*

$$f(x) - q(x) = \frac{f'''(\xi)}{6}(x-a)^2(x-b) \tag{11}$$

for every x in the same interval with $x \ne a, b$, and some ξ between the smallest and largest of a, b, x. Thus $q \le f$ on $x \le b$ if $f''' \le 0$ throughout the interval, and on $x \ge b$ if $f''' \ge 0$ throughout it. Its quadratic coefficient has the sign of f'' whenever that sign is strict throughout the interval.

Lemma 7 (Inverse-trace eigenvalue bound; [5, Lemma 27]). *Let $B \succ 0$ have size $n \ge 2$, $\operatorname{tr} B^{-1} \le n$, and $\det B < \rho$. Define*

$$F_n(t) = t\Big(\frac{n-1}{n-\frac{1}{t}}\Big)^{n-1} \qquad \Big(t > \frac{1}{n}\Big).$$

Every eigenvalue λ of B satisfies $F_n(\lambda) < \rho$. In particular, if $H > 1$ and $F_n(H) \ge \rho$, then $B \prec HI$.

Lemma 8 (Determinant from trace and inverse trace; [5, Lemma 34]). *Let $n \ge 3$, $B \succ 0$, $\operatorname{tr} B = \sigma$, and $\operatorname{tr} B^{-1} = U$. Then $\sigma U \ge n^2$. If equality holds, $B = (\sigma/n)I$. Otherwise define*

$$t = \frac{U\sigma - n(n-2) + \sqrt{(U\sigma - n^2)(U\sigma - (n-2)^2)}}{2U}, \qquad \ell = \frac{\sigma - t}{n-1}. \tag{12}$$

Then $\det B \geq t\ell^{n-1}$. *This lower bound increases with* σ *at fixed* U, *and decreases with* U *at fixed* σ.

Lemma 9 (A shared trace–inverse logarithm bound; [5, Lemma 38]). *Let* $0 < \ell \leq r$, $H > 0$, *and let* $a_{\log} = 1/(\ell + r)$, $b_{\log} = \ell r/(\ell + r)$. *If*

$$q_{\log} \leq \min\left\{\log \ell - a_{\log}\ell + \frac{b_{\log}}{\ell},\ \log H - a_{\log}H + \frac{b_{\log}}{H}\right\},$$

then $\log x \geq a_{\log}x - b_{\log}/x + q_{\log}$ *for* $0 < x \leq H$. *In particular,*

$$x - 1 \leq 3\log x + 2\left(\frac{1}{x} - 1\right) \qquad \left(0 < x \leq \frac{14}{5}\right), \tag{13}$$

$$\log(1 + x) \geq \frac{17}{40}x + \frac{23}{40}\frac{x}{1 + x} \qquad \left(-1 < x \leq \frac{9}{16}\right). \tag{14}$$

Lemma 10 (Concentrating bounded mass; [5, Lemma 40]). *Let* $0 \leq x_i \leq r$, $\sum_i x_i \leq m$, *and* $r > 0$. *If* f *is convex and nondecreasing on* $[0, r]$, *with* $f(0) = 0$, *then*

$$\sum_i f(x_i) \leq jf(r) + f(m - jr), \qquad j = \lfloor \frac{m}{r} \rfloor.$$

In particular, define $\mathcal{C}_r(m) = jr^2 + (m - jr)^2$; *then* $\sum_i x_i^2 \leq \mathcal{C}_r(m)$. *For zero capacity use* $\mathcal{C}_0(m) = 0$.

The logarithm enclosure. For $r > 0$ rational, logarithms are enclosed, as in [5, Appendix D.7], using

$$\log r = 2\sum_{j=0}^{m-1} \frac{z^{2j+1}}{2j + 1} + \mathcal{R}_m, \qquad z = \frac{r - 1}{r + 1}, \qquad |\mathcal{R}_m| \leq \frac{2|z|^{2m+1}}{(2m + 1)(1 - z^2)}. \tag{15}$$

This follows by integrating the geometric series for $2/(1 - z^2)$. Square-root enclosures are verified by squaring rational endpoints. A polynomial on $[0, 1]$ is nonnegative if all its Bernstein coefficients are nonnegative, because the basis functions $\binom{n}{k}x^k(1 - x)^{n-k}$ are nonnegative. Interval subdivision applies the same fact on arbitrary rational intervals. These facts, and the rules for finite covers recalled in Section 6, are what the exact replays of Sections 3 and 5 use.

3 Case `0019`: eight-dimensional covariance

This section proves Theorem 1 for `0019`. The form is $P = \frac{2}{3}A_P$ with A_P the matrix of Appendix A (the basis in which the certificates are written); it is equivalent to P_S (Appendix A). Throughout, Q is a Gram matrix of N in this basis, $X = Q^{-1}$, and the letters are those of Section 2.1.

Computational result 11 (The simplex contact data and complete scalar cover). The printed form has the nine simplex contact pairs and the remaining 120 pairs (240 minimum vectors) described below. On the whole root (18), a finite tree of 17 347 nodes covers every point satisfying the necessary probability and matrix inequalities. Its 931 empty leaves have contradictions to those inequalities; its other 7743 leaves have the scalar witnesses (20) (or the aggregate variant below) and a resulting bound $\log\det A \geq \log K_*$. The scalar inequality is checked over its complete two-variable domain, by the endpoint and critical-point procedure below. Every split, contraction and terminal witness is checked exactly.

The simplex contacts and the covariance coordinates. Let $z = (1, \dots, 1, -5)^{\mathsf{T}} \in \mathbb{Z}^9$, with eight ones. The nine simplex contacts form an integral matrix B satisfying

$$B^{\mathsf{T}} P B = 2(I + \mathbf{1}\mathbf{1}^{\mathsf{T}}), \qquad B^{\mathsf{T}} z = \mathbf{1}, \qquad (\det B)^2 = 9.$$

Explicitly, the first eight columns of B are $e_1, \dots, e_8$, and its last column is $(2 \cdot \mathbf{1}_8, 3)^{\mathsf{T}}$. These identities can therefore be checked directly from the printed Gram matrix (Appendix A, where $\det B = 3$ is checked as well). Let

$$K = \frac{1}{2} B^{\mathsf{T}} Q B, \qquad V = K^{-1}, \qquad w = V\mathbf{1}, \qquad m = \mathbf{1}^{\mathsf{T}} V \mathbf{1}.$$

Then

$$e = \frac{512}{9} \det K, \qquad X[z] = \frac{m}{2}, \qquad \Delta \geq \frac{1024}{9} \frac{\det K}{m^2}, \qquad \det K < \frac{10233}{512}. \tag{16}$$

The last two statements follow from Lemma 5 and $e < 1137$. All nine contacts give $K_{ii} \geq 2$. The remaining 240 minimum vectors lie in the hyperplane perpendicular to $\mathbf{1}$ in contact coordinates, so the inverse-cone expansion gives

$$V = \operatorname{diag}(w) + V_0, \qquad w \geq 0, \qquad V_0 \succeq 0, \qquad V_0\mathbf{1} = 0, \qquad \operatorname{tr} V + m \leq 9.$$

The last inequality is exactly $\operatorname{tr}(PX) \leq 9$.
Set $p = w/m$, order its nine entries decreasingly, and write $p_{\max} = p_1$, $P_* = p_2$, $v_p = \sum_i p_i^2$. Choose any orthonormal frame H for $\mathbf{1}^{\perp}$, and define

$$A = H^{\mathsf{T}} K H, \qquad C = H^{\mathsf{T}} (\operatorname{diag} p - p p^{\mathsf{T}}) H.$$

Here C is the restriction to $\mathbf{1}^{\perp}$ of the covariance of the random coordinate vector e_i chosen with probability p_i. Block inversion in $[H, \mathbf{1}/3]$ gives

$$\begin{aligned} \det K &= \frac{9 \det A}{m}, \qquad A^{-1} = mC + H^{\mathsf{T}} V_0 H, \\ \operatorname{tr} A^{-1} &\leq 9 - m(1 + v_p), \qquad K_{ii} = \frac{1}{m} + [H^{\mathsf{T}}(e_i - p)]^{\mathsf{T}} A\, [H^{\mathsf{T}}(e_i - p)]. \end{aligned} \tag{17}$$

Multiplying the last identity by p_i and summing gives $\operatorname{tr}(AC) \geq 2 - 1/m$. Since $V \succeq w_i e_i e_i^{\mathsf{T}}$, $w_i K_{ii} \leq 1$, hence $0 \leq p_i \leq 1/(2m)$. Also $\operatorname{tr} C = 1 - v_p$.

The two outer ranges of m. AM–GM applied to $(I + \mathbf{1}\mathbf{1}^{\mathsf{T}})V$ gives $\det K \geq 10$. Thus $m \leq 27/25$ yields $\Delta \geq 6400000/6561 > 972$. At the other end, $v_p \geq 1/9$ gives $\operatorname{tr} A^{-1} \leq 9 - 10m/9$. If $m \geq 9/4$, the eight eigenvalues of $(16/13)A^{-1}$, together with the ninth number $4m/9$, are positive and have sum at most

$$\frac{16}{13}\Big(9 - \frac{10m}{9}\Big) + \frac{4m}{9} = \frac{144 - 12m}{13} \leq 9.$$

AM–GM therefore gives

$$\det A \geq \frac{4m}{9}\Big(\frac{16}{13}\Big)^8 > \frac{1137m}{512},$$

contradicting $e < 1137$ and (17). Every remaining counterexample therefore belongs to

$$(m, p_{\max}, P_*, v_p) \in \left[\frac{27}{25}, \frac{9}{4}\right] \times \left[\frac{1}{9}, \frac{25}{54}\right] \times \left[0, \frac{25}{54}\right] \times \left[\frac{1}{9}, \frac{25}{54}\right], \tag{18}$$

and satisfies

$$\begin{gathered} A^{-1} \succeq mC, \qquad \operatorname{tr} A^{-1} \leq U := 9 - m(1 + v_p), \qquad \operatorname{tr}(AC) \geq h := 2 - \frac{1}{m}, \\ \det A < K_* := \min\left(\frac{1137m}{512}, \frac{243m^3}{256}\right). \end{gathered} \tag{19}$$

The probability information. No individual probability is enumerated. Cauchy–Schwarz, the upper cap on the remaining entries, and direct multiplication give

$$\begin{gathered} P_* \leq p_{\max}, \qquad 1 - p_{\max} \leq 8P_*, \qquad 2mp_{\max} \leq 1, \\ p_{\max}^2 + P_*^2 + \frac{(1 - p_{\max} - P_*)^2}{7} \leq v_p \leq p_{\max}^2 + P_*(1 - p_{\max}), \\ \operatorname{tr} C^2 = v_p + v_p^2 - 2\sum_i p_i^3, \qquad \sum_i p_i^3 \geq v_p^2. \end{gathered}$$

Lemma 10 gives $v_p \leq \mathcal{C}_{p_{\max}}(1)$ and $v_p \leq p_{\max}^2 + \mathcal{C}_{P_*}(1 - p_{\max})$.
The extrema used on probability boxes also have finite exact descriptions. For $f(s,t) = s^2 + t^2 + (1 - s - t)^2/7$ on a rectangle $[a,b] \times [c,d]$, each vertical edge $s = s_0$ has its minimum at $t = \min(d, \max(c, (1 - s_0)/8))$; exchange the coordinates for a horizontal edge. Include $(1/9, 1/9)$ if it belongs to the rectangle. These four edge tests and the possible centre suffice: a minimum exists by compactness; on the boundary, completing the square gives the stated clipped point; in the interior, the equations $16s + 2t - 2 = 16t + 2s - 2 = 0$ force $s = t = 1/9$. The corner tests are already included in the edge minima.
For the retained-first-probability upper bound, write $M = p_{\max}$ and let $C > 0$ bound the other eight probabilities. On the closed cell $1 - (k+1)C \leq M \leq 1 - kC$, Lemma 10 gives

$$v_p \leq M^2 + kC^2 + (1 - M - kC)^2.$$

This is a convex quadratic in M. Its maximum on the intersection with a box interval $[a,b]$ occurs at an endpoint. Consequently only a, b and the breakpoints $1 - kC$ that lie in

$[a,b]$ need to be tested. Neighbouring pieces agree at $1-kC$, with value $(1-kC)^2+kC^2$, so equality at a cell boundary causes no gap. Every feasible residual mass belongs to a cell with $0 \le k \le 8$. If $C=0$, all eight residual probabilities vanish, and $v_p = M^2 \le b^2$. A retained prefix with residual sum $q>0$ and square sum s gives cube lower bounds s^2/q and $(3qs-q^3)/2$. The first is Cauchy–Schwarz; the second follows by expanding $q^3-3qs+2\sum p_i^3$ as six times a sum of nonnegative triple products. If $q=0$, all residual entries vanish. These bounds give rational upper bounds for $\operatorname{tr} C^2$ on every box.

The covariance spectrum. The rank-one formula for $\operatorname{diag} p - pp^{\mathsf T}$ gives $\lambda_1(C) \le p_{\max}(1-p_{\max}+P_*)$: for $P_* < x < p_{\max}$, its secular equation is $p_{\max}/(p_{\max}-x) = \sum_{i>1} p_i/(x-p_i)$, whose right side is at most $(1-p_{\max})/(x-P_*)$. For sharper bounds retain upper endpoints $\bar p_{\max}$, $\bar P$ and set $q = 1-\bar p_{\max}-\bar P$, $r = v_{p,+} - p_{\max,-}^2 - P_{*,-}^2$. The seven residual probabilities have sum at least q and square sum at most r. Each residual probability is at most

$$\min\Big\{\bar P,\ \sqrt{r},\ \frac{q+\sqrt{6(7r-q^2)}}{7}\Big\},$$

using the last entry only when $r < q^2$; the condition $7r < q^2$ is already impossible. The last bound follows by applying Cauchy–Schwarz to the other six entries. Choose a rational common ceiling a at least this minimum. Replacing the square roots by rational upper enclosures gives a valid choice. Equivalently, the last bound is certified directly by $7a \ge q$ and $7a^2-2qa+q^2-6r \ge 0$. For $0 \le b < a < x$, apply Lemma 6 to $f_x(y) = y^2/(x-y)$, tangent at b and interpolating at a. Since $f_x'''(y) = 6x^2/(x-y)^4 > 0$, the quadratic lies above f_x for $0 \le y \le a$. Explicitly,

$$\frac{y^2}{x-y} \le \frac{x^2y^2 - b(bx+2xa-ba)y + xb^2a}{(x-b)^2(x-a)}.$$

Continuity includes $b=a$. Summing gives the residual resolvent bound $T(x) = Ar+Bq+7E$, where A, B, E are the three quadratic coefficients. They satisfy $A>0$, $B \le 0$, $E \ge 0$, so substituting the containing moments is safe. The simple bound $r/(x-a)$ is also valid. The Schur complement in $xI - \operatorname{diag} p + pp^{\mathsf T}$ now proves

$$\lambda_1(C) \le x \quad \text{if} \quad \frac{\bar p_{\max}^2}{\bar p_{\max}-x} \ge 1 + \frac{\bar P^2}{x-\bar P} + T(x), \quad \max(\bar P, a) < x < \bar p_{\max} \le 2x;$$

$$\lambda_2(C) \le x \quad \text{if} \quad \frac{\bar p_{\max}^2}{\bar p_{\max}-x} + \frac{\bar P^2}{\bar P - x} \ge 1 + T(x), \quad a < x < \bar P,\ \ \bar p_{\max} \le 2x.$$

For the second assertion, eliminate the last seven coordinates; the remaining two-dimensional Schur complement has nonpositive determinant and hence at most one negative eigenvalue. If the actual $P_* \le x$, the assertion follows already from $C \preceq \operatorname{diag} p$. These tests are exact rational inequalities; bisection only proposes a value of x.

If the resulting bounds are $\lambda_1(C) \le u_1$, $\lambda_2(C) \le u_2 \le u_1$, then in *any* orthonormal basis its diagonal entries c_i satisfy

$$\sum_i (c_i - u_2)_+ \le u_1 - u_2.$$

Indeed each diagonal entry is a convex combination of eigenvalues, with squared orthogonal entries as weights; apply convexity of the positive-part function and sum.

The scalar determinant witness. Let x_i be the eigenvalues of A and c_i the diagonal entries of C in the same basis. Then $0 \le c_i \le u_1$, $mx_ic_i \le 1$, $\sum c_i = 1 - v_p$, $\sum c_i^2 \le F$, $\sum x_i c_i \ge h$, and $\sum x_i^{-1} \le U$, where F is the proved covariance-square bound. A leaf supplies rational coefficients $\alpha, \lambda, \mu, \nu, \rho, \sigma$, with $\lambda, \mu, \rho, \sigma \ge 0$, for the scalar inequality

$$\log x \ge \alpha - \frac{\lambda}{x} + \mu xy + \nu y - \rho y^2 - \sigma(y - u_2)_+. \tag{20}$$

It must hold on the whole containing interval for x and $0 \le y \le \min(u_1, 1/(m_- x))$. Summation gives

$$\log\det A \ge 8\alpha - \lambda U + \mu h + \nu(1 - v_p) - \rho F - \sigma(u_1 - u_2).$$

The sign of ν determines which trace endpoint is used. A lower bound at least $\log K_*$ contradicts (19). Witnesses without the last correction use the special case $\sigma = 0$ or use weaker versions of the moment bounds. An aggregate witness follows instead from $\operatorname{tr} A \ge 18 - 9/m - (9v_p - 1)x_+$ and the scalar inequality $\log x \ge \alpha - \lambda/x + \mu x$.
Every x_i is bounded by AM–GM on the other seven inverse eigenvalues: $z((U - z)/7)^7 K_* > 1$ for $z = 1/x_i$; rational endpoint comparisons give a positive enclosing interval $[L_x, U_x]$.
Here are all scalar checks needed for (20). Write $m_0 = m_- > 0$. Split at $y = u_2$: the two branches have

$$(a, b, \alpha_0, \nu_0) = (0, u_2, \alpha, \nu) \quad \text{or} \quad (u_2, u_1, \alpha + \sigma u_2, \nu - \sigma).$$

On either branch the right side is $f(x, y) = \alpha_0 - \lambda/x + \mu xy + \nu_0 y - \rho y^2$, with $a \le y \le \min(b, 1/(m_0 x))$. Set $X_a = U_x$ when $a = 0$, and $X_a = \min(U_x, 1/(m_0 a))$ otherwise. Feasibility implies $x \le X_a$. For fixed x, the quadratic in y attains its maximum at an endpoint, or at $y = (\mu x + \nu_0)/(2\rho)$ when $\rho > 0$ and this point is feasible. Indeed, for $\rho > 0$ complete the square; for $\rho = 0$ the function is affine. Subtracting these candidate values from $\log x$ gives the following Laurent coefficients:

candidate	d_{-2}	d_{-1}	d_0	d_1	d_2
$y = t$	0	λ	$-\alpha_0 - \nu_0 t + \rho t^2$	$-\mu t$	0
$y = \frac{1}{m_0 x}$	$\frac{\rho}{m_0^2}$	$\lambda - \frac{\nu_0}{m_0}$	$-\alpha_0 - \frac{\mu}{m_0}$	0	0
$y = \frac{\mu x + \nu_0}{2\rho}$	0	λ	$-\alpha_0 - \frac{\nu_0^2}{4\rho}$	$-\frac{\mu\nu_0}{2\rho}$	$-\frac{\mu^2}{4\rho}$

The first row with $t = a$ is checked on $[L_x, X_a]$. If $b > 0$, its instance $t = b$ is checked on $[L_x, \min(X_a, 1/(m_0 b))]$, and the reciprocal row is checked on $[\max(L_x, 1/(m_0 b)), X_a]$. If $b = 0$, only $y = a = b = 0$ is feasible. An interval whose left endpoint exceeds its right endpoint is empty and needs no scalar test.
For the stationary row assume $\rho > 0$. If $\mu > 0$, choose a rational $s \ge 0$ with $s^2 \ge m_0^2\nu_0^2 + 8m_0\mu\rho$ and check on

$$\Big[\max\Big(L_x, \frac{2\rho a - \nu_0}{\mu}\Big),\ \min\Big(X_a, \frac{2\rho b - \nu_0}{\mu}, \frac{-m_0\nu_0 + s}{2m_0\mu}\Big)\Big].$$

The first two bounds express $a \le y \le b$. The reciprocal cap gives $m_0\mu x^2+m_0\nu_0 x \le 2\rho$, hence $(2m_0\mu x+m_0\nu_0)^2 \le m_0^2\nu_0^2+8m_0\mu\rho \le s^2$, which proves the last upper bound. If $\mu = 0$, the stationary row is needed only when $2\rho a \le \nu_0 \le 2\rho b$. Its interval is $[L_x, X_a]$ when $\nu_0 \le 0$, and $[L_x, \min(X_a, 2\rho/(m_0\nu_0))]$ when $\nu_0 > 0$. These alternatives cover every feasible maximiser, including all zero coefficients and interval endpoints.
For each row it remains to check $\log x + \sum_{j=-2}^{2} d_j x^j \ge 0$ on its positive interval. Its derivative has denominator x^3 and numerator

$$-2d_{-2} - d_{-1}x + x^2 + d_1x^3 + 2d_2x^4.$$

This polynomial is nonzero because its x^2 coefficient is one. A continuous differentiable function on a compact interval has a minimum at an endpoint or an interior critical point, so the two endpoints and the finitely many real roots in the interval suffice. Rational root isolation, for example by Sturm sequences [1], and the outward logarithm bound (15) certify these values. This proves the implication from the finite scalar checks to (20) throughout its stated domain.
Some boxes are impossible before a scalar witness is needed. Besides the probability tests, if $\operatorname{tr} C \le T_0$ and $\operatorname{tr}(AC) \ge h > 0$, then $\lambda_{\max}(A) \ge h/T_0$. Let $z_0 = T_0/h$. If $z_0 \le U/8$ and $K_* z_0((U - z_0)/7)^7 \le 1$, inverse-eigenvalue AM–GM contradicts $\det A < K_*$. This is the contact/determinant empty-box test.

The tree. The complete tree on (18) has 17 347 nodes and 8674 terminal boxes. Of these, 931 violate the necessary inequalities and 7743 have scalar determinant witnesses. The exact replay checks 38 704 critical intervals; the smallest logarithmic margin is $6181717/500000000000 > 0$. Together with the two outer m-ranges, this closes `0019` for all admissible chains.

Conclusion of the case. Let a chain as in Theorem 1 lie over `0019` and suppose $\Delta = \det L < 972$. In the contact coordinates above, $e < 1137$ and (16) give $\det K < 10233/512$; the two outer ranges of m are excluded by AM–GM, so the parameters of the chain lie in the box (18) and satisfy (19); by Computational result 11, every point of (18) satisfying the necessary probability and matrix inequalities lies in a terminal box of the tree, which is either empty or carries a scalar witness (20) proving $\log\det A \ge \log K_*$, contradicting $\det A < K_*$. Hence $\Delta \ge 972$.

4 A two-vector flag bound

This section proves the flag bound on which the case `0017` rests.

Lemma 12 (Two successive projections). *If s and u are the squared lengths of two successive shortest projected vectors, every nonzero vector in the next projection has squared length at least $2s/3$. Consequently, for any integral rank-two sublattice Z of N^*,*

$$\Delta \ge \frac{4e}{9\det(Z^{\mathsf{T}}XZ)}. \tag{21}$$

Here the columns of the integral matrix Z are a basis of that sublattice.

Proof. Scale $s = 1$ and suppose a further projected vector has squared length $x < 2/3$. With first phase $r \in [0, 1/2]$, shortness gives $3/4 \le u \le x/(1-r^2)$, hence $r > 1/3$. The first normal and two adjacent lifts form a triangle of lengths u, $x+ur^2$, $x+u(1-r)^2$, all less than one. Their complementary square roots must sum to at most one by Lemma 4. Increasing x to $2/3$ and u to $2/[3(1-r^2)]$ strictly decreases the sum to

$$2\sqrt{\frac{1-3r^2}{3(1-r^2)}} + \sqrt{\frac{3r-1}{3(1+r)}} \ge 1.$$

For the last inequality let $y = \sqrt{(3r-1)/(3(1+r))} \in [0, 1/3]$. The square of the first radical minus $(1-y)^2/4$ equals $y(1-3y)(1+y)/(2(1-3y^2)) \ge 0$. This contradiction proves the projection bound. Take a Gauss-reduced basis of the rank-two lattice, with squared orthogonal lengths c, d. Apply the bound first to the two chain normals and a lift of the first basis vector, giving $c \ge 2v/3$; then to the first normal and lifts of both reduced basis vectors, giving $d \ge 2u/3$. The relevant first choices remain shortest in these sublattices. Thus $\det(Z^{\mathsf{T}} X Z) = cd \ge 4uv/9$, which with (1) proves (21). □

5 Case `0017`: the kernel and the anisotropic plane

This section proves Theorem 1 for `0017`. The form is $P = A_P$, the matrix of Appendix A (minimum 4, determinant 768, 99 contact pairs); it is equivalent to P_K (Appendix A). The section metric S and $R = S^{-1}$ are the physical coordinates of (7), in which $\det S = e/768$ and $\operatorname{tr} S^{-1} \le 9$.

Computational result 13 (The kernel geometry and complete five-parameter cover). The integral kernel, projected cosets and contact decomposition specified below exist for the printed form. Their exact basis and contact identities give the stated tensor moments, kernel cone inequalities, and projected rank-four cone vertices. Every vector used to invoke a low-rank minimum bound has the stated integral lift or doubling property. For the closed five-parameter root at the end of this section, the 3 010 595-node tree has 1 505 298 terminal regions. Every terminal region either violates a displayed necessary inequality or has a rigorous lower bound $\operatorname{tr} S^{-1} > 9$ from the displayed Schur estimates. Verification includes complete cone-vertex enumeration and rational evaluation of every branch and split, with an empty final frontier. The fixed reference checks include the identity between the physical basis Gram matrix and the printed marker, whose determinant is 768, and the integral rank-two dual plane. The rank-three and rank-four Hermite inequalities are cited where used.

The plane, the kernel, and the flag bound. The three shortest reference dual pairs span an integral plane of determinant $1/3$. Its perpendicular kernel has rank seven and determinant 256. In the associated orthogonal decomposition write the physical section metric as

$$S = \begin{pmatrix} H & U \\ U^{\mathsf{T}} & C \end{pmatrix}, \qquad H = \begin{pmatrix} A & V \\ V^{\mathsf{T}} & B \end{pmatrix}, \qquad \dim A = 3, \quad \dim B = 4, \quad \dim C = 2.$$

Let $k = \det H, g = \det S$, and let $\lambda \le \Lambda$ be the eigenvalues of C. The block determinant identity and Lemma 12 give

$$\Delta \ge 1024\,\frac{g^2}{k}. \tag{22}$$

Marking gives $\operatorname{tr} S^{-1} \le 9$, $g \ge 1$, and $g < 379/256$. Lemma 7, checked at $9/20$ and $11/4$, gives $(9/20)I \prec S \prec (11/4)I$. A counterexample to (22) has

$$\frac{256}{243} < k < \frac{9475}{1296}.$$

The upper bound follows because the rank-two Schur complement is greater than $(9/20)I$.

Reference contacts and projected minima. Write $e_2(B) = ((\operatorname{tr} B)^2 - \operatorname{tr} B^2)/2$ for the sum of the pairwise products of a symmetric matrix's eigenvalues. The 99 section contacts have squared component lengths and counts

(A,B,C)	$(4,0,0)$	$(0,4,0)$	$(2,2,0)$	$(2,0,2)$	$(0,2,2)$
pairs	3	12	48	12	24.

The pure rank-three and rank-four lattices have determinant 64. Projection of the mixed contacts extends each by four cosets, so each projected lattice has reference determinant four. Its nonzero cosets have integral lifts with plane squared reference length two, and doubling every projected vector lies in the pure lattice. Opposite lifts are both integral, so averaging them removes every coupling to the plane without modifying the unknown metric. If q_i are the three unit plane directions, their costs $c_i = q_i^{\mathsf T} C q_i$ satisfy

$$\sum_i q_i q_i^{\mathsf T} = \frac{3}{2} I, \qquad \sum_i c_i = \frac{3}{2}(\lambda + \Lambda), \qquad \min_i c_i \le \frac{3\lambda + \Lambda}{4}.$$

The projected directional lower bound is therefore $m_0 = \max(1/2,\ 2 - \Lambda)$. The first term uses doubling and minimum four; the second uses the two opposite lifts. All integrality and coset assertions are checked from the reference basis.
Let $a = \operatorname{tr} A - 3$, $b = \operatorname{tr} B - 4$, $T = a + b$, $d_0 = m_0 - 1$, and $c = \lambda + \Lambda - 2$. The pure contacts give $a, b \ge 0$. Three integral rank-three sections and eighteen rank-three sections involving the rank-four block give

$$\begin{aligned}
&a + \frac{3c}{2} \ge 0, && b + 2c \ge \Lambda - \lambda,\\
&a \ge \sqrt{\frac{18}{\lambda + \Lambda}} - 3, && a \ge 2 - \frac{3\lambda + \Lambda}{2},\\
&b \ge \frac{8}{\sqrt{3\lambda + \Lambda}} - 4, && b \ge 4 - 3\lambda - \Lambda,\\
&a \ge 3d_0, && b \ge 4d_0.
\end{aligned} \tag{23}$$

For example the three complementary rank-three section determinants give $e_2(A) \ge \sum_i 1/c_i \ge 6/(\lambda + \Lambda)$, and $\operatorname{tr} A \ge \sqrt{3 e_2(A)}$ proves the first square-root bound. The

rank-four sections give $e_2(B) \geq 24/(3\lambda + \Lambda)$ and $\operatorname{tr} B \geq \sqrt{(8/3)e_2(B)}$. The linear bounds sum the projected directions belonging to the smallest plane cost. The second inequality in the first row follows from the three nonnegative phase slacks, as quantified below. The classical values $\gamma_3^3 = 2$ and $\gamma_4^4 = 4$ [11,2] give the low-rank bounds used here: minimum a_0 implies determinant at least $a_0^3/2$ in rank three and $a_0^4/4$ in rank four.

Keeping kernel trace and determinant together. For fixed trace $7 + T$ and determinant k, the comparison spectrum is $(t, \ell, \dots, \ell)$, where

$$t + 6\ell = 7 + T, \qquad t\ell^6 = k, \qquad t \geq \ell > 0.$$

It bounds the actual largest eigenvalue by t, the squared deformation norm by $(t - 1)^2 + 6(\ell - 1)^2$, and the inverse trace below by $1/t + 6/\ell$. For each eigenvalue x of H, AM–GM on the other six gives $k \leq x[(7 + T - x)/6]^6$. This right side decreases for $x \geq (7 + T)/7$, so $x \leq t$. The inverse-trace bound follows by inverting the decreasing dependence on U in Lemma 8. For the squared norm, $t = \ell$ is the scalar case by AM–GM. Otherwise interpolate $-\log x$ at ℓ with its tangent and at t. Lemma 6 gives $-\log x \geq q_2 x^2 + q_1 x + q_0$ on $0 < x \leq t$, with $q_2 > 0$. Sum over the eigenvalues of H and compare with $(t, \ell, \dots, \ell)$, which has the same trace and determinant. It follows that $\operatorname{tr} H^2 \leq t^2 + 6\ell^2$, proving the norm bound.
The kernel contact identities also give $\operatorname{tr}((H - I)^2) \leq 2T^2$ and $H \succeq (1 - T/2)I$. Define

$$R_* = \min\{(t - 1)^2 + 6(\ell - 1)^2,\ 2T^2\}, \qquad m = \max\left(\frac{9}{20},\ 1 - \frac{T}{2}\right),$$

$$t_* = \min\left(t,\ \frac{11}{4}\right).$$

Then $mI \preceq H \preceq t_* I$ and $R_* \geq a^2/3 + b^2/4$. Sum (13) over the nine eigenvalues of S, using $S \prec (11/4)I$, $\operatorname{tr} S^{-1} \leq 9$, and $g < 379/256$. This gives

$$T + \lambda + \Lambda \leq 2 + 3\log\left(\frac{379}{256}\right), \qquad T < \frac{23}{10}. \tag{24}$$

The last bound uses $\lambda + \Lambda > 9/10$ and $\log(379/256) < 2/5$.

The coupling bound. Let $r_A^2 = \operatorname{tr}((A - I)^2)$, $r_B^2 = \operatorname{tr}((B - I)^2)$, and $r_C^2 = \operatorname{tr}((C - I)^2)$. Opposite mixed contacts give the squared inequalities

$$2\|U_A\|_F^2 \leq r_A^2 + \frac{a^2}{3} + 2ac + 3r_C^2,$$
$$6\|U_B\|_F^2 \leq r_B^2 + \frac{b^2}{2} + 3bc + \frac{3c^2}{2} + 3r_C^2.$$

For the rank-four block, the three projected orthonormal frames have fourth moment $r_B^2 + b^2/2$; sum the squares of their individual paired-contact slacks. For the rank-three block, write the diagonal entries of $A - I$ as x_i and the three plane excesses

as $k_i = c_i - 1$. The difference between the asserted upper bound and the sum of squared paired slacks is $\sum_i (x_i + 2k_i)^2 - (\sum_i (x_i + 2k_i))^2/3 \geq 0$. These identities establish the displayed inequalities for arbitrary couplings. The pure and projected ternary contacts imply $x_i \geq 0$, $\sum x_i = a$, $x_i \leq a - 2d_0$, and $|A_{jk}| \leq (a - x_i - 2d_0)/2$. Their capped-simplex vertices give

$$r_A^2 \leq \begin{cases} 2a^2 - 4ad_0 + 6d_0^2, & d_0 \leq 0, \\ 5a^2 - 28ad_0 + 42d_0^2, & 3d_0 \leq a \leq 4d_0,\ d_0 \geq 0, \\ 2a^2 - 10ad_0 + 18d_0^2, & a \geq 4d_0 \geq 0. \end{cases}$$

The rank-four projected cone and its section minors give

$$r_B^2 \leq \min\left\{ \frac{5}{2}b^2 - 18d_0 b + 36d_0^2,\ (4+b)^2 - 4 - 2b - \frac{48}{3\lambda + \Lambda} \right\}.$$

Adding these inequalities gives

$$\|U\|_F^2 \leq \frac{r_A^2}{2} + \frac{r_B^2}{6} + \frac{a^2}{6} + \frac{b^2}{12} + \frac{c(T+a)}{2} + \frac{c^2}{4} + 2r_C^2.$$

Here $ac + bc/2 = c(T+a)/2$ because $T = a + b$; the same signed c and the same trace parameters are retained. Because $r_A^2 + r_B^2 \leq R_*$, maximising the first two terms allocates available variance first to A, retaining $r_B^2 \geq b^2/4$. Together with their separate caps this gives one upper bound on $\|U\|_F^2$.
Two further bounds are retained when stronger. If M_x is the maximum of $\sum x_i^2$ at the ternary capped-simplex vertices, then

$$\|U_A\|_F^2 \leq \frac{a^2}{2} + \frac{M_x}{2} + 2ac + \frac{3c^2}{4} + \frac{3r_C^2}{2} + \sqrt{\frac{3}{2}}\,(\Lambda - \lambda)\sqrt{M_x - \frac{a^2}{3}}.$$

This follows by expanding the same three paired slacks and applying Cauchy–Schwarz to their centred diagonal and plane terms. The weaker bound $2(a + 3c/2)^2$ is also valid by squaring the total nonnegative slack. For U_B, let $d_1 = b + 2c$, $\delta = \Lambda - \lambda$. Its three phase slacks have sum $3d_1$ and square sum $3d_1^2 + 6\delta^2$, whence $d_1 \geq \delta$. Their largest entry is at least $d_1 + \delta$ when $d_1 \geq 2\delta$, and at least $(3d_1 + \sqrt{12\delta^2 - 3d_1^2})/2$ otherwise, by the capped-simplex square-sum argument. Any two phase projectors sum to at least $I/2$, giving

$$\|U_B\|_F^2 \leq \begin{cases} d_1^2 - d_1\delta + \dfrac{5}{2}\delta^2, & d_1 \geq 2\delta, \\ \left(\dfrac{3}{4}\right) d_1^2 + \dfrac{3}{2}\delta^2 - \dfrac{3}{4}d_1\sqrt{12\delta^2 - 3d_1^2}, & \delta \leq d_1 \leq 2\delta. \end{cases}$$

Let P_U be the least of the proved total coupling upper bounds.

Component determinants. The minimum bounds of the projected lattices imply $\det A \geq \frac{1}{2}m_0^3$ and $\det B \geq m_0^4$. A sharper ternary lower bound is

$\det A \geq \max(0, F(a, d_0))$, with

$$F(a,d) = \begin{cases} (1+a)(1-d^2) - \dfrac{(1-d)(a-2d)^2}{2}, & d \leq 0, \\ (1+4d-a)[(1+a-2d)^2 - (a-3d)^2], & 0 \leq d,\ 3d \leq a \leq 4d, \\ (1+a-2d)(1+2d) - \dfrac{(a-2d)^2}{4} & \\ \qquad - \dfrac{(1+2d)(a-4d)^2}{4}, & a \geq 4d \geq 0. \end{cases}$$

We prove the bound by expanding the determinant in the actual matrix entries. Write the diagonal of $A - I$ as (x, y, z), where $x, y, z \geq 0$, $x + y + z = a$, and each is at most $h = a - 2d$. Each off-diagonal absolute value is bounded by half of h minus the opposite diagonal entry. Replacing the three square terms by these upper bounds and the off-diagonal product by its negative absolute value lowers the determinant. Expansion gives the lower bound

$$F_-(a,d) + \frac{3-a+5d}{2}(xy+xz+yz) + \frac{xyz}{2},$$

where $F_-(a,d) = (1+a)(1-d^2) - (1-d)(a-2d)^2/2$. For $d \geq 0$, the coefficient $3 - a + 5d$ is positive because $a \leq T < 23/10$. For $-1 \leq d \leq 0$, either $F_- \leq 0$ and positivity of A suffices, or $F_- > 0$ implies $3 - a + 5d \geq 0$. Indeed, putting $v = a - 3 - 5d \geq 0$ gives

$$F_- = (1-d)\big[-\frac{1-d^2}{2} - 2(1+d)v - \frac{v^2}{2}\big] \leq 0,$$

which rules out $3 - a + 5d < 0$ in the positive case. The capped-simplex inequalities

$$xy+xz+yz \geq h(a-h), \qquad xy+xz+yz \geq 2ah - 3h^2, \qquad xyz \geq h^2(a-2h)$$

now give, respectively, the last and middle formulas for F; nonnegativity of the two symmetric sums gives the first. The first inequality follows from the square-sum mass-transfer bound. The second is the sum of $(h-x)(h-y) \geq 0$ over the three pairs. For the third, add $h(h-x)(h-z)$, $h(h-y)(h-z)$ and $z(h-x)(h-y)$ and expand. For the rank-four block, if $w = b - 4d_0 \in [0, 2m_0]$, the projected D_4 cone gives $\det B \geq (m_0+w)^2(m_0 - w/2)^2$. Its trace-one vertices have spectra $(3/2, -1/2, 0, 0)$ or $(1, 1, -1/2, -1/2)$. After scaling and adding $m_0 I$, both are positive semidefinite on the stated range; their determinant difference is $w^3(2m_0 - w)/4 \geq 0$. Concavity proves the assertion. The exact finite cone-vertex verification is included in the data. Let the resulting determinant lower bounds be A_*, B_*. Jacobi's identities give a kernel inverse-trace lower bound by minimising

$$\max\Big\{3\Big(\frac{B_*}{k}\Big)^{1/3},\ \frac{6}{k(\lambda+\Lambda)y}\Big\} + 4y^{1/4}, \qquad y \geq \frac{A_*}{k}.$$

For constants $K_0, V_0 > 0$, the minimum of $\max(K_0, V_0/y) + 4y^{1/4}$ on $y \geq y_0$ occurs at $\max\{y_0, \min(V_0^{4/5}, V_0/K_0)\}$, as direct differentiation on the two branches shows. Take the larger of this bound and $1/t + 6/\ell$, and denote it by I_H.

The minimised Jacobi bound also has a purely rational witness form. Let $q = y^{1/4} > 0$, and write its cost as $\max(k_J, v_J/q^4) + 4q$, with $k_J \geq 0$. If $q \geq q_0 \geq 0$, the following four lower bounds require only the displayed checks:

$$\begin{array}{r|l} k_J + 4q_0 & \text{always}, \\ \dfrac{v_J}{q_0^4} + 4q_0 & q_0 > 0, \quad v_J \leq q_0^5, \\ 5r & r \geq 0, \quad r^5 \leq v_J, \\ k_J + 4s & s > 0, \quad s^5 \leq v_J, \quad k_J s^4 \leq v_J. \end{array}$$

The derivative of $v_J/q^4 + 4q$ has the sign of $q^5 - v_J$. The third bound follows by dividing

$$4q^5 - 5rq^4 + r^5 = (q - r)^2(4q^3 + 3rq^2 + 2r^2q + r^3) \geq 0$$

by q^4. For the fourth, use $k_J + 4q$ when $q \geq s$; when $q \leq s$, the derivative bound gives $v_J/q^4 + 4q \geq v_J/s^4 + 4s \geq k_J + 4s$. These rational inequalities establish all four branches.

The final Schur estimate. Let $C_0 = C - U^{\mathsf{T}} H^{-1} U$. Its actual determinant is bounded above by

$$K_0 = \min\left\{ \frac{\frac{379}{256}}{k},\ \sqrt{\frac{\frac{243}{256}}{k}},\ \lambda\Lambda \right\}.$$

If its eigenvalues are $x \leq y$, then $x \leq \lambda$, $y \leq \Lambda$. Increase the pair continuously toward (λ, Λ) until its product is K_0. Both inverse-trace lower costs below decrease, so this enlargement is safe. Let $y = K_0/x$. The admissible interval also uses

$$x \leq \sqrt{K_0}, \qquad x + y \geq \lambda + \Lambda - \frac{P_U}{m}, \qquad C_0 \succeq rI + \left(\frac{r}{t_*}\right)(C - rI), \quad r = \frac{9}{20}.$$

The last inequality follows from the Schur complement of $S - rI$ and $H^{-1} \preceq (1 - r/t_*)(H - rI)^{-1}$. Block inversion and Frobenius Cauchy–Schwarz give

$$\operatorname{tr} S^{-1} \geq I_H + \max\{f_1(x), f_2(x)\},$$

where

$$f_1(x) = \frac{1 + \frac{\lambda}{t_*}}{x} + \left(1 + \frac{\Lambda}{t_*}\right)\frac{x}{K_0} - \frac{2}{t_*},$$

$$f_2(x) = \frac{1 + \frac{\zeta_1^2}{P_U}}{x} + \left(\frac{1}{K_0} + \frac{\zeta_2^2}{P_U}\right)x + \frac{2\zeta_1\zeta_2}{P_U},$$

$$\zeta_1 = \lambda - \sqrt{K_0}, \qquad \zeta_2 = \frac{\Lambda}{\sqrt{K_0}} - 1.$$

The coupling contribution satisfies

$$J = \operatorname{tr}(U^{\mathsf{T}} H^{-2} U C_0^{-1}) \geq \frac{\left[\operatorname{tr}((C - C_0) C_0^{-1/2})\right]^2}{P_U}.$$

The classical trace rearrangement inequality [10], in dimension two, bounds the trace inside the square below by $(\lambda - x)/\sqrt{x} + (\Lambda - y)/\sqrt{y} \geq 0$, proving f_2. The bound $H^{-2} \succeq H^{-1}/t_*$ gives f_1. If $P_U = 0$, use $U = 0$, $C_0 = C$ instead of division. The number ζ_1 may be negative; its sign is retained. On a terminal box the three bounds are evaluated as follows, and only so: every quantity, K_0 with its term $\lambda\Lambda$ included, is an interval over the box, so that at every point of the box the enlargement to the product K_0 exists and f_2 holds with the pointwise values; the coefficients A, B, C of $f_j(x) = A/x + Bx + C$ are intervals (the square of an interval containing 0 has lower endpoint 0; the sign of ζ_1 is carried by the interval product $2\zeta_1\zeta_2$), the admissible interval of x encloses the enlarged pair of every point, and the minimum is taken with the lower endpoints of A, B, C. A pointwise evaluation of f_2 at box endpoints would not be a lower bound, since $\zeta_1/\sqrt{x} + \zeta_2\sqrt{x}$ is not monotone in (λ, Λ, K_0); the leaf predicate of the checker (`PK\_trace\_five\_scalar\_interval.py`) is the interval evaluation.

Each function is $A/x + Bx + C$ with $A, B > 0$. Its minimum on a positive interval is an endpoint or $2\sqrt{AB} + C$, since its derivative is $B - A/x^2$. The bare inverse cost $f_0(x) = 1/x + x/K_0$ is valid as well: the omitted coupling contribution is nonnegative. It suffices to take the maximum of the three separate minima, since $\min_x \max_j f_j(x) \geq \max_j \min_x f_j(x)$. The same inverse-cost and coupling inequalities hold before saturating the product; saturation only lowers these costs. A lower bound greater than nine contradicts marking.

One vertex argument for both ternary bounds. The same capped-simplex calculation gives

$$M_x \leq \begin{cases} a^2, & d_0 \leq 0, \\ 3a^2 - 16ad_0 + 24d_0^2, & 0 \leq 3d_0 \leq a \leq 4d_0, \\ a^2 - 4ad_0 + 8d_0^2, & a \geq 4d_0 \geq 0. \end{cases}$$

Indeed, transfer mass between two coordinates until one reaches zero or the cap $a - 2d_0$. This increases their square sum and preserves their sum. The resulting lists are $(a, 0, 0)$, $(a - 2d_0, a - 2d_0, 4d_0 - a)$, and $(a - 2d_0, 2d_0, 0)$ in the three displayed regimes, respectively.

Here is a common justification for maximising these bounds and the bounds for r_A^2 over a parameter rectangle. A quadratic $q(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$ is convex when $A, C \geq 0$ and $B^2 \leq 4AC$. To verify this directly, its homogeneous quadratic part is nonnegative: if $A > 0$, multiply it by $4A$ to obtain $(2Ax + By)^2 + (4AC - B^2)y^2$; if $A = 0$, then $B = 0$. For two points z_0, z_1 and $0 \leq t \leq 1$,

$$(1 - t)q(z_0) + tq(z_1) - q((1 - t)z_0 + tz_1) = t(1 - t)q_2(z_0 - z_1) \geq 0,$$

where q_2 denotes the homogeneous quadratic part. This proves the segment bound, including equality and degenerate segments. The six coefficient triples (A, B, C) needed

here are

$$(2,-4,6),\ (5,-28,42),\ (2,-10,18),\qquad (1,0,0),\ (3,-16,24),\ (1,-4,8).$$

Each satisfies the stated inequalities. Thus positive semidefiniteness suffices for both families.

Choose a containing rectangle with $a_- \geq 0$, intersect it with $a \geq 3d_0$, and divide it by $d_0 = 0$ and $a = 4d_0$. On each resulting closed polygon use its corresponding polynomial. Every point of a polygon lies on a segment joining two boundary points; each boundary point lies on an edge segment between vertices. Two applications of the segment bound therefore reduce the maximum to vertex values. A segment or a single-point polygon is covered by the same argument. The vertices are the feasible rectangle corners and the intersections of its sides with $d_0 = 0$, $a = 3d_0$, or $a = 4d_0$; the intersection $(0,0)$ is included when present. All coordinates and all polynomial evaluations are rational.

An equivalent finite verification divides the a interval at the abscissae of these vertices. On each subinterval, two affine functions bound d_0; nonnegative combinations of the polygon inequalities verify those bounds. Interpolate first along each affine boundary and then vertically between them. The four endpoint values bound the whole slice. Any extra endpoint on an edge is bounded by that edge's vertex values. A rational separating combination rejects an empty polygon. Consequently this construction introduces no assumption about an unexamined interior maximum.

The complete cover. Use the five parameters $(k, T, \theta, \lambda, \Lambda)$, where $T = a + b > 0$ and $\theta = a/T$. Thus $0 \leq \theta \leq 1$, $a = T\theta$ and $b = T(1-\theta)$ exactly, including the faces $a = 0$ or $b = 0$. Positivity of T follows from $k > 1$ and AM–GM on the seven kernel eigenvalues. The closed root is

$$\left[\frac{256}{243},\ \frac{9475}{1296}\right] \times \left[0,\ \frac{23}{10}\right] \times [0,\ 1] \times \left[\frac{9}{20},\ \frac{11}{4}\right]^2.$$

The order $\lambda \leq \Lambda$, (23), (24), and inverse AM–GM contract the box before evaluating the Schur bound. Every expression is evaluated outward with rational arithmetic. The preceding common vertex argument certifies both ternary piecewise quadratic maxima on the clipped rectangles. Radical endpoints are checked by integer roots, and the marked logarithm uses (15). An additional necessary contraction is

$$\frac{49}{7+T} + \frac{1}{\lambda} + \frac{1}{\Lambda} \leq 9.$$

Block inversion bounds $\operatorname{tr} S^{-1}$ below by $\operatorname{tr} H^{-1} + \operatorname{tr} C^{-1}$, and Cauchy–Schwarz gives the first term. The ambient Schur floor and the trace-loss bound above further contract the same two Schur eigenvalues. Each contraction intersects the current intervals with a proved necessary bound, so repeating it preserves every actual candidate. When a box crosses a ternary branch boundary, all intersecting closed branches are retained; none is selected from a midpoint. Either the universal inverse-trace estimate or the stronger comparison-spectrum and Jacobi estimates may certify an early terminal contradiction.

The completed partition has $3\,010\,595$ nodes and $1\,505\,298$ terminal regions, with zero pending regions. Its required finite checks cover every contraction, branch, early terminal rule and split, together with 62 arithmetic controls. Induction on the finite tree therefore places each possible actual candidate in a retained child until a proved necessary-inequality or inverse-trace contradiction excludes it. This proves `0017` conditional only on the fixed reference checks and the finite cover stated in Computational result 13, besides the attributed classical rank-three and rank-four inequalities. The Lean theorem performs this full deduction from the stated finite certificate facts; the finite payload itself is checked by the exact replay named in Section 6, outside Lean.

Conclusion of the case. Let a chain as in Theorem 1 lie over `0017` and suppose $\Delta = \det L < 972$. In the physical coordinates of (7), the flag bound (22) and the marking give $256/243 < k < 9475/1296$, and the necessary inequalities (23)–(24) place $(k, T, \theta, \lambda, \Lambda)$ in the closed root above; by Computational result 13, every point of the root lies in a terminal region of the tree, which either violates a displayed necessary inequality or carries a proved lower bound $\operatorname{tr} S^{-1} > 9$, contradicting the marking. Hence $\Delta \geq 972$.

6 What is verified, and where

Theorem 1 is the conjunction of the two conclusions above. In the style of the paragraph "What is imported, precisely" of [4, Section 6.2], we list the finite checks it uses, the records that certify it, and the hypotheses it rests on.

The finite checks. Two Computational results and the checks of the printed forms enter.

- Computational result 11 (`0019`): the simplex contact data (B, z, the 240 remaining minimal vectors in $\mathbf{1}^{\perp}$) and the complete scalar cover of the box (18): $17\,347$ nodes, 8674 terminal boxes (931 empty, 7743 with scalar witnesses), $38\,704$ critical intervals replayed exactly, smallest logarithmic margin $6181717/500000000000$.
- Computational result 13 (`0017`): the kernel geometry (the integral plane of determinant $1/3$, the kernel of rank seven and determinant 256, the contact decomposition and the projected cosets) and the complete five-parameter cover: $3\,010\,595$ nodes, $1\,505\,298$ terminal regions, 62 arithmetic controls, empty final frontier.
- The printed reference forms (Appendix A): each printed A_P is positive definite, has minimum s_P, its minimum-vector pairs have the stated number, their tensors span the 45-dimensional space of symmetric matrices, and the normalised determinant is the displayed $\det P$. For the two forms of this paper ($s_P = 6$, 129 pairs, $\det P = 5120/9$; $s_P = 4$, 99 pairs, $\det P = 768$) these statements are checked exactly in Appendix A.

The rules under which these checks are accepted as proofs (matrix and polynomial identities compared coefficient by coefficient, exact short-vector enumeration by a rational LDL^{T} decomposition, linear or scalar terminal witnesses checked by rational interval arithmetic, polynomial signs or Bernstein coefficients, and the three obligations

of a finite covering tree: the root contains every counterexample, each terminal witness excludes its whole closed box, the boxes cover the root) are those of the γ_{11} paper [5, Section 7 and Appendix D.7], whose verifier conventions this paper follows.

The verification records. The entry points of the two local proofs in the certificate archive described in [4, Section 9] are the following records.

form	final record	what it certifies
`0019`	`simplex_chain_transfer_verified.json`	covariance reduction and continuous scalar witnesses; complete scalar-cover replay archived
`0017`	`PK_trace_five_scalar_cover_verified.json`	fixed-coordinate transport, projected minima, all necessary contractions and the full Schur-cover deduction; finite reference and five-parameter cover checks archived

The `0019` record certifies a full rational cover, and both checks bind their projected sections and contact coordinates to the printed markers. The records and their checkers are in the certificate archive `hermite11_local_certificates.zip` of the Zenodo record doi:10.5281/zenodo.23979316. Its `case_map.json` identifies each principal record, its size, the underlying checker sources and recorded result; `certificate_manifest.json` binds the stored files and compressed logical streams. The common archive identifiers and verification procedure are given in [4, Section 9]. The Lean development of the series [4, Section 8] proves the analytic implications of the ten local cases, these two among them, from the stated finite certificate facts; the two covering trees of this paper are checked by the exact replays recorded above, outside Lean.

The hypotheses. Theorem 1 uses the following inputs.

1. The admissible-chain conditions of Definition 2, including the marking and the exact cutoffs.
2. The lemmata of Section 2.2 (Lemmata 4–10 and the enclosure (15)), proved in [5, Appendix D] (Lemmata 22, 23, 26, 27, 34, 38 and 40 there, and the paragraph on logarithms of Appendix D.7).
3. For `0017` only: the classical values $\gamma_3^3 = 2$ and $\gamma_4^4 = 4$ [11,2], used as "minimum a_0 implies determinant at least $a_0^3/2$ in rank three and $a_0^4/4$ in rank four".
4. Computational results 11 and 13, the checks of the printed forms (Appendix A), and the verification records above.

Lean correspondence. The local analytic statements are `0017`: `Hermite11.case0017_from_finite_input`; `0019`: `Hermite11.case0019_from_finite_input`. Their finite hypotheses, integral-equivalence wrappers and saved reference records are listed in `case_map.json`. Large cover payloads remain external exact checks.

7 Use of computational tools

Refer to Section 10 of [4].

References

1. Basu, S., Pollack, R., Roy, M.-F.: Algorithms in Real Algebraic Geometry, 2nd edn. Algorithms and Computation in Mathematics, vol. 10. Springer, Berlin (2006). https://doi.org/10.1007/3-540-33099-2
2. Conway, J.H., Sloane, N.J.A.: Sphere Packings, Lattices and Groups, 3rd edn. Springer, New York (1999). https://doi.org/10.1007/978-1-4757-6568-7
3. Dutour Sikirić, M., van Woerden, W.: The lattice packing problem in dimension 9 by Voronoi's algorithm. arXiv:2508.20719 (August 2025; v2, 2026). Data: https://doi.org/10.5281/zenodo.15707640
4. Géraud-Stewart, R., Naccache, D.: The Hermite constants in dimensions 10, 11 and 12. Posted concurrently to this work (2026)
5. Géraud-Stewart, R., Naccache, D.: A focus on Hermite constant 11. Posted concurrently to this work (2026)
6. Géraud-Stewart, R., Naccache, D.: γ_{11}: contact geometry. Posted concurrently to this work (2026)
7. Géraud-Stewart, R., Naccache, D.: γ_{11}: a finite spectral relaxation. Posted concurrently to this work (2026)
8. Géraud-Stewart, R., Naccache, D.: γ_{11}: one selected contact, two linked blocks. Posted concurrently to this work (2026)
9. Géraud-Stewart, R., Naccache, D.: γ_{11}: unequal triangles and adapted references. Posted concurrently to this work (2026)
10. Horn, R.A., Johnson, C.R.: Matrix Analysis, 2nd edn. Cambridge University Press, Cambridge (2013). https://doi.org/10.1017/CBO9781139020411
11. Martinet, J.: Perfect Lattices in Euclidean Spaces. Grundlehren der mathematischen Wissenschaften, vol. 327. Springer, Berlin (2003). https://doi.org/10.1007/978-3-662-05167-2
12. Nebe, G., Sloane, N.J.A.: A catalogue of lattices. Entries KAPPA9.2, KAPPA11, K12, http://www.math.rwth-aachen.de/~Gabriele.Nebe/LATTICES/

A The two forms, in the bases of the certificates

For each of the two forms we print the integer matrix A_P in the basis used by its certificate, with $P = (4/s_P)A_P$ the form scaled to minimum 4; the identities of Sections 3 and 5 (the simplex matrix B with columns $e_1, \dots, e_8$ and $(2 \cdot \mathbf{1}_8, 3)^{\mathsf{T}}$; the three shortest dual pairs and the kernel) refer to these bases.

`0019:030884313`. $s_P = 6$, $\det P = 5120/9$, 129 contact pairs.

$$A_P = \begin{pmatrix} 6 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & -17 \\ 3 & 6 & 3 & 3 & 3 & 3 & 3 & 3 & -17 \\ 3 & 3 & 6 & 3 & 3 & 3 & 3 & 3 & -17 \\ 3 & 3 & 3 & 6 & 3 & 3 & 3 & 3 & -17 \\ 3 & 3 & 3 & 3 & 6 & 3 & 3 & 3 & -17 \\ 3 & 3 & 3 & 3 & 3 & 6 & 3 & 3 & -17 \\ 3 & 3 & 3 & 3 & 3 & 3 & 6 & 3 & -17 \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & 6 & -17 \\ -17 & -17 & -17 & -17 & -17 & -17 & -17 & -17 & 86 \end{pmatrix}.$$

`0017:001746552`. $s_P = 4$, $\det P = 768$, 99 contact pairs.

$$A_P = \begin{pmatrix} 4 & 0 & -2 & -2 & 0 & -2 & -2 & -2 & 1 \\ 0 & 4 & 2 & 0 & -2 & 2 & 2 & 2 & -3 \\ -2 & 2 & 4 & 2 & 0 & 2 & 2 & 2 & -3 \\ -2 & 0 & 2 & 4 & 0 & 1 & 1 & 1 & -2 \\ 0 & -2 & 0 & 0 & 4 & -1 & -1 & -1 & 0 \\ -2 & 2 & 2 & 1 & -1 & 4 & 2 & 2 & -3 \\ -2 & 2 & 2 & 1 & -1 & 2 & 4 & 2 & -3 \\ -2 & 2 & 2 & 1 & -1 & 2 & 2 & 4 & -3 \\ 1 & -3 & -3 & -2 & 0 & -3 & -3 & -3 & 6 \end{pmatrix}.$$

What was checked. For each of the two matrices we checked, in exact rational arithmetic, that it is symmetric and positive definite, that $\det A_P$ is 21870 (so $\det P = (2/3)^9 \cdot 21870 = 5120/9$) and 768, that no nonzero integer vector has $A_P[z] < s_P$ (exact enumeration), that the number of minimal vectors is $258 = 2 \cdot 129$ and $198 = 2 \cdot 99$, and that the cc^{T} over the minimal vectors span Sym_9, so that both forms are perfect. These are the assertions on the printed reference forms listed in Section 6. For `0019` we also checked $B^{\mathsf{T}}PB = 2(I + \mathbf{1}\mathbf{1}^{\mathsf{T}})$, $B^{\mathsf{T}}z = \mathbf{1}$, $\det B = 3$, that the nine columns of B are minimal vectors of P, and that the other 240 minimal vectors are perpendicular to $\mathbf{1}$ in contact coordinates; for `0017`, that the three shortest dual pairs have X-length $2/3$, span an integral plane of determinant $1/3$, and that the sublattice of $\mathbb{Z}^9$ orthogonal to that plane has rank seven and determinant 256 in the metric P.

The relation to P_S and P_K. The root paper prints these two forms in other bases [4, Appendix B]: as

$$P_S = \begin{pmatrix} 2G_{E_8} & \frac{2}{3}f_2 \\ \frac{2}{3}f_2^{\mathsf{T}} & 4 \end{pmatrix},$$

with G_{E_8} the Gram matrix of the simple roots of E_8 and $\det P_S = 5120/9$, and as the matrix P_K of KAPPA9.2, with $\det P_K = 768$; the root paper's Computational result 9 states that `0019:030884313` and `0017:001746552`, scaled to minimum 4, are equivalent to P_S and P_K. Here A_P is the integral representative in the certificate basis; it is the record of the data set itself, entry by entry (the script `hermite11\_printed\_forms\_check.py` of the Zenodo record [4, Section 9] checks the eleven printed forms of the series against the rows of their keys in the certificate database). The change of basis and scaling are given by the following unimodular matrices

$$U_S = \begin{pmatrix} -1 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & -1 & 2 & 0 & 0 & 0 & 0 & -1 & 0 \\ -2 & -1 & 2 & 0 & 0 & 0 & -1 & 1 & 0 \\ -2 & -1 & 2 & 0 & 0 & -1 & 1 & 0 & 0 \\ -2 & -1 & 2 & 0 & -1 & 1 & 0 & 0 & 0 \\ -2 & -1 & 2 & -1 & 1 & 0 & 0 & 0 & 0 \\ -2 & -2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -3 & -2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$U_K = \begin{pmatrix} -2 & -2 & 0 & 1 & 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 1 & 1 & 1 & -1 \\ -1 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ -2 & -2 & 1 & 1 & -1 & 2 & 1 & 1 & 0 \end{pmatrix},$$

for which $U_S^{\mathsf{T}}\,(\frac{2}{3}A_P)\,U_S = P_S$ with the A_P of `0019` and $\det U_S = -1$, and $U_K^{\mathsf{T}} A_P U_K = P_K$ with the A_P of `0017` and $\det U_K = 1$; the columns of U_S and U_K are minimal vectors of the respective A_P, and the two identities were checked exactly. Since a marking is carried along by a change of basis ([4, after Proposition 7]), Theorem 1 may be read with either basis.