γ_{11}: A Finite Spectral Relaxation

THE MACADAMIA 0008

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Abstract. The root paper of this series reduces $\gamma_{11} = 4/\sqrt[11]{972}$ to a statement about chains of lattices $L \supset M \supset N$ of ranks 11, 10 and 9, cut along shortest dual vectors, whose rank-9 member lies over one of 4 487 813 perfect forms. Ten of these forms, the *macadamias*, lie under an actual lattice of rank 10 of small determinant and need a proof at rank 11, where the bound 972 is attained. This paper settles the form `0008:022143116`, which lies under a section of K_{11} with 276 minimal vectors: every admissible chain whose rank-9 member is admissibly marked by this form has $\det L \geq 972$. In coordinates aligned with the form, the section metric is $S = I + E$ with three blocks of size three. Two sufficient flag tests come from short dual covectors and integral triangles; local reductions, proved from rational comparisons and Bernstein or Sturm certificates, remove explicit regions of the parameters; every remaining real spectrum is mapped, with explicit allowances, into a finite relaxation on 81 grid nodes, which a tree of 400 nodes with 1 091 rational certificates, replayed in two exact arithmetic implementations, excludes. The proof is computer-assisted, with explicit finite checks stated as one Computational result.

Keywords: Hermite constant · Lattice packings · Perfect forms · Coxeter–Todd lattice · Spectral relaxation · Bernstein certificates · Verified computation

1 Introduction

This paper belongs to a series of seven papers, made public concurrently, whose aim is to establish the Hermite constants γ_{10}, γ_{11} and γ_{12}. The root paper [5] recalls the problem, proves γ_{10} in full, derives γ_{12} from γ_{11} and reduces γ_{11} to a statement about chains of lattices $L \supset M \supset N$ of ranks 11, 10 and 9 whose rank-9 member lies, in Voronoi's sense [15,12], over one of 4 487 813 perfect forms of rank 9; the γ_{11} paper [6] excludes all of these forms but ten by exact certificates and assembles the proof of $\gamma_{11} = 4/\sqrt[11]{972}$ from the root paper, the certificates and five companion papers, of which this is one (the others are [7,8,9,10]).

The ten forms that resist, the *macadamias* of [5], are those that lie under an actual lattice of rank 10 with minimum 4 and determinant below $d_{\mathrm{cut}} = 4 \cdot 972^{9/11}$, so that no exclusion at rank 10 can exist for them; for these the inequality $\det L \geq 972$ must

be proved at the level of the rank-11 lattice, where it is attained by K_{11}, the section of the Coxeter–Todd lattice [3] listed as KAPPA11 in [13] (see [2, Ch. 4]). This paper settles the fourth macadamia of [5, Table 4], the form `0008:022143116` of the classification of Dutour Sikirić and van Woerden [4], which lies under a section of K_{11} with 276 minimal vectors, determinant 972 and dual minimum $8/9$.

Theorem 1 (informal; Theorem 13 below). *Assume Computational result 14 and the reference data in Appendix A. Every admissible chain of Definition 2 marked by* `0008:022143116` *has* $\det L \geq 972$.

The idea. The proof works in coordinates aligned with the form, in which the rank-9 section metric is $S = I + E$ with E cut into three blocks of size three, and all interblock couplings retained. Two sufficient *flag tests* are derived from three short dual covectors and from twelve integral triangles: each says that $\det L \geq 972$ as soon as a certain logarithmic expression in $\det S$ and in the partial inverse traces of S is nonnegative. A counterexample has both flags negative, and the contact inequalities of the form confine its parameters $(a, b, \tau, \lambda_-, \lambda_+)$ to an explicit box. *Local reductions*, proved from finitely many rational comparisons and univariate, bivariate and four-variable polynomial certificates (Bernstein expansions and Sturm sequences), remove explicit regions of that box: a small radius, a small ℓ, a small η strip (seven polynomial charts), a large a, and three trace regions. Every remaining real spectrum is then mapped, with explicit allowances, into a *finite relaxation*: measures on a grid of 81 rational nodes, three block-square variables, and necessary linear rows. A tree of 400 nodes, with 155 terminal exclusions, 154 splits and 91 contractions alone (234 contractions in all), whose 1 091 rational certificates were replayed in two exact arithmetic implementations (FLINT and Python's `Fraction`), excludes the relaxation with an empty final frontier. Figure 1 shows the outline.

Organisation. Section 2 recalls what is taken from the root paper [5] and from the γ_{11} paper [6]: the chain, the constants, the hypotheses, the marking, and the shared lemmata on determinants, spectra and rational certificates. Section 3 introduces the aligned coordinates and the single Computational result on which the proof rests. Sections 4 to 17 contain the proof in fourteen steps: the flag tests and the containing domain, the contact information retained by the relaxation, the linked-F determinant gain, the compression branches, the A–C coupling, the positive contact moment and the block shifts, the trace and large-a reductions, the radius and ℓ reductions, the small-η strip, the coupling allowance and the block kernel, the weighted inverse loss, the seven polynomial charts, the finite spectral grid, and the exact separation. Section 18 states what is verified and where, in the style of the root paper's paragraph "What is imported, precisely".

2 The local setting and common tools

2.1 Notation and admissible chains

We use the common coordinates of [6, Appendix D]. The determinant of a lattice is the determinant of its Gram matrix, and its minimum is the least nonzero squared

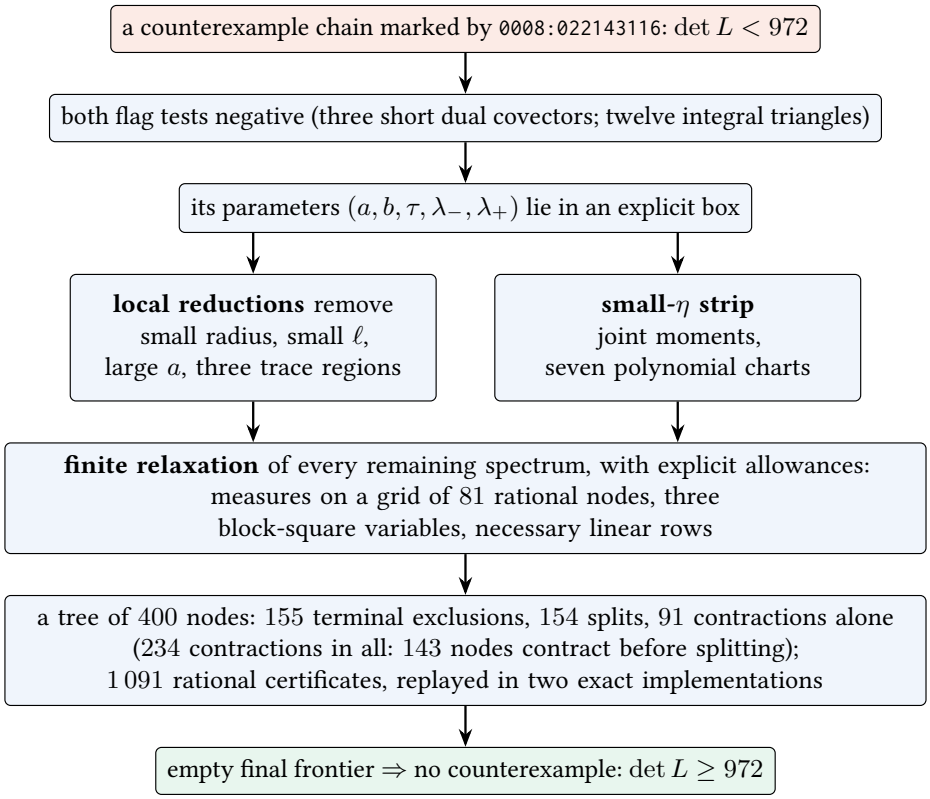


Fig. 1. The proof for 0008: from a hypothetical counterexample to the empty frontier of the finite relaxation. The local reductions and the small-η strip remove explicit regions of the box; they are proved from finitely many rational comparisons and univariate, bivariate and four-variable polynomial certificates (Bernstein expansions and Sturm sequences). Everything that remains is excluded by the finite relaxation. This is an outline: the fourteen steps of Sections 4–17 feed these boxes.

length. Its dual is $L^* = \{y \in \operatorname{span} L : y \cdot x \in \mathbb{Z} \text{ for all } x \in L\}$. A sublattice $M \subset L$ is *primitive* if $M = L \cap \operatorname{span} M$; a primitive vector generates a primitive rank-one sublattice. Write $Q[x] = x^{\mathsf{T}} Q x$, with T denoting transpose, and $\operatorname{tr} A$ for the sum of the diagonal entries. For symmetric matrices, $A \succeq 0$ means $A[x] \geq 0$ for every x; $A \succ 0$ means $A[x] > 0$ for $x \neq 0$, and $A \preceq B$ means $B - A \succeq 0$. The nested sections form a flag of ranks 11, 10 and 9; its notation is

$$\begin{gathered} e = \det N, \qquad d = \det M = \frac{e}{u}, \qquad \Delta = \det L = \frac{e}{uv}, \\ u = \min M^*, \qquad v = \min L^*. \end{gathered} \tag{1}$$

The cutoffs are the positive numbers $d_{\mathrm{cut}}^{11} = 4^{11} 972^9$ and $e_{\mathrm{cut}}^{55} = 2^{246} 3^{197}$.

Definition 2 (admissible residual chain). *A chain $L \supset M \supset N$ is admissible for P if L has rank eleven and minimum at least four, $M = L \cap n_{11}^{\perp}$ and $N = M \cap n_{10}^{\perp}$ for shortest nonzero vectors $n_{11} \in L^*$ and $n_{10} \in M^*$, its rank-nine Gram matrix is marked by P as defined below, and*

$$\begin{aligned} 768 \le d < d_{\mathrm{cut}} &= 4\,972^{9/11}, \\ e < e_{\mathrm{cut}} &= \Big(\frac{4096}{3}\Big)^{1/10} d_{\mathrm{cut}}^{9/10} < 1137. \end{aligned} \tag{2}$$

Shortest dual vectors are primitive, so the determinant identities above follow by an adapted basis. Rounding a lift of n_{10} across the last cut gives $v \le u + v/4$, hence $v \le 4u/3$ and $\Delta \ge 3d/(4u)$. The global reduction [5, Proposition 15] supplies such a chain from a hypothetical counterexample.

Definition 3 (marking). *Let Q be the Gram matrix of N in an integral basis, and $X = Q^{-1}$. A perfect form P of minimum four marks Q if $\min Q \ge 4$ and there are weights satisfying*

$$\begin{gathered} X = \sum_{c \in \mathcal{C}(P)} w_c\, cc^{\mathsf{T}}, \qquad w_c \ge 0, \qquad \sum_c w_c \le \frac{9}{4}, \qquad w_c \le \frac{1}{4}, \\ \operatorname{tr}(PX) \le 9, \qquad \det P \le e. \end{gathered} \tag{3}$$

Here $\mathcal{C}(P)$ contains one vector from each opposite pair of minimum vectors of P.

Perfect means that the matrices cc^{T} span the symmetric matrices. A marking is transported by an integral change of basis. Its existence for arbitrary Q is proved in [5, Proposition 7]; here the marker is fixed to the printed form.

Gluing coordinates. Vectors $\beta, \alpha \in \mathbb{R}^9$ and a scalar $\vartheta \in \mathbb{R}$ give the dual squared lengths at $(z, k, k') \in \mathbb{Z}^9 \times \mathbb{Z}^2$ as

$$X[z] + u(\beta^{\mathsf{T}} z + k)^2 + v(\alpha^{\mathsf{T}} z + \vartheta k + k')^2,$$

and the primal squared lengths as

$$Q[x - k\beta - k'(\alpha - \vartheta\beta)] + \frac{(k - \vartheta k')^2}{u} + \frac{k'^2}{v}.$$

Write $\|t\| = \min_{j \in \mathbb{Z}} |t - j|$. The first shortest-normal condition implies

$$X[z] + u\,\|\beta^{\mathsf{T}} z\|^2 \ge u \qquad (0 \ne z \in \mathbb{Z}^9).$$

The second condition applies to every nonzero vector of M^* and every lift to L^*.

Physical coordinates. Choose C_P with $P = C_P^{\mathsf{T}} C_P$ and write

$$Q = C_P^{\mathsf{T}} S C_P, \qquad R = S^{-1}. \tag{4}$$

The columns of C_P realise the reference lattice in Euclidean space. Thus S measures squared lengths in these fixed coordinates, with reference metric I, and R is its inverse: $\det S = e/\det P$, $\operatorname{tr} R = \operatorname{tr}(PX) \le 9$; integral vectors c, z become $C_P c, C_P^{-\mathsf{T}} z$. We write $|x| = \|x\|$ for Euclidean length and $\|E\|_F^2 = \operatorname{tr}(E^{\mathsf{T}}E)$ for the squared Frobenius norm. Eigenvalues are listed in decreasing order. The notation $I_n, e_i, \mathbf{1}$ denotes the identity, coordinate vectors and all-one vector; Sym_n denotes the symmetric $n \times n$ matrices. We use $(x)_+ = \max(x, 0)$; subscripts $-$ and $+$ on a box coordinate denote its lower and upper endpoints. Schur complements, cofactors and adjugates have the conventions and block identities of [6, Appendix D.2]. Exact LDL^{T} factorisation and bounded short-vector enumeration are described in [6, Section 5.1].

2.2 Statements used here

The following statements are shared by the companion papers of the series; their proofs are given in Appendix D of [6], under the numbers given in brackets. In Lemmata 4 and 5 the chain is admissible for P, with e, X and Δ as in (1).

Lemma 4 (one section vector controls both extensions; [6, Lemma 23]). *For every $0 \neq z \in \mathbb{Z}^9$,*

$$\Delta \ge \frac{e}{2X[z]^2}. \tag{5}$$

If three nonzero section covectors sum to zero and all have squared X-length at most x, then

$$\Delta \ge \frac{16e}{27x^2}. \tag{6}$$

Lemma 5 (a weighted section triangle; [6, Lemma 24]). *Let $z_1, z_2, z_3 \in \mathbb{Z}^9 \setminus \{0\}$ satisfy $z_1 + z_2 + z_3 = 0$, and let $q_i = X[z_i]$ and $Q_\triangle = q_1 + q_2 + q_3$. Then*

$$\Delta \ge \frac{841\, e}{162\, Q_\triangle^2}. \tag{7}$$

Lemma 6 (inverse-trace eigenvalue bound; [6, Lemma 27]). *Let $B \succ 0$ have size $n \ge 2$, $\operatorname{tr} B^{-1} \le n$, and $\det B < \rho$. Define*

$$F_n(t) = t\Big(\frac{n-1}{n-\frac{1}{t}}\Big)^{n-1} \qquad \Big(t > \frac{1}{n}\Big).$$

Every eigenvalue λ of B satisfies $F_n(\lambda) < \rho$. In particular, if $H > 1$ and $F_n(H) \ge \rho$, then $B \prec HI$.

Lemma 7 (a shared D_n squared-norm bound; [6, Lemma 28]). *Let E be a symmetric $n \times n$ matrix, $n \ge 4$, with $E_{ii} + E_{jj} \ge 2|E_{ij}|$ for $i \neq j$. Then $\|E\|_F^2 \le (n+1)(\operatorname{tr} E)^2/2$.*

Lemma 8 (a shared D_n determinant bound; [6, Lemma 32]). *Let E be symmetric of size $n \ge 7$, $E_{ii} + E_{jj} \ge 2|E_{ij}|$ for $i \neq j$, and $0 \le s = \operatorname{tr} E \le 1$. If $1 + s - (n-1)s^2/4 > 0$, then*

$$I + E \succ 0, \qquad \det(I+E) \ge 1 + s - \frac{n-1}{4}s^2.$$

In particular this applies for $n = 7, 8$ with $s \le 1$, and for $n = 10$ with $s \le 3/10$.

Section 12 uses one step of the proof of Lemma 8 ([6, proof of Lemma 32]), the *logarithmic determinant averaging*: if $B = \sum_a w_a B_a$ with $B_a \succ 0$, $w_a \ge 0$ and $\sum_a w_a = 1$, then $B \succ 0$ and $\sum_a w_a \log\det B_a \le \log\det B$ (apply $\log x \le x - 1$ to the eigenvalues of $B^{-1/2} B_a B^{-1/2}$ to obtain $\log\det B_a \le \log\det B + \operatorname{tr}(B^{-1} B_a) - n$, multiply by w_a and sum: the trace terms cancel).

Lemma 9 (determinant from trace and inverse trace; [6, Lemma 34]). *Let $n \ge 3$, $B \succ 0$, $\operatorname{tr} B = \sigma$, and $\operatorname{tr} B^{-1} = U$. Then $\sigma U \ge n^2$. If equality holds, $B = (\sigma/n) I$. Otherwise define*

$$t = \frac{U\sigma - n(n-2) + \sqrt{(U\sigma - n^2)(U\sigma - (n-2)^2)}}{2U}, \qquad \ell = \frac{\sigma - t}{n-1}. \tag{8}$$

Then $\det B \ge t\ell^{n-1}$. This lower bound increases with σ at fixed U, and decreases with U at fixed σ.

Definition 10 (contact cone; [6, Definition 35]). *For a reference lattice of minimum four with contact vectors $\mathcal{C}$ in physical coordinates, its* contact cone *consists of the symmetric matrices E satisfying $E[c] \ge 0$ for every $c \in \mathcal{C}$. Every minimum-four metric $I + E$ lies in this cone.*

Positive identities such as $K(\operatorname{tr} E)^2 - \operatorname{tr}(E^2) = \sum_{c,c'} a_{cc'} E[c] E[c']$ with $a_{cc'} \ge 0$, or their cubic analogues, prove norm bounds for *every* contact-cone matrix; they are checked by comparing polynomial coefficients. The identities of Section 5 are of this kind.

Lemma 11 (concentrating bounded mass; [6, Lemma 40]). *Let $0 \le x_i \le r$, $\sum_i x_i \le m$, and $r > 0$. If f is convex and nondecreasing on $[0, r]$, with $f(0) = 0$, then*

$$\sum_i f(x_i) \le j f(r) + f(m - jr), \qquad j = \lfloor \frac{m}{r} \rfloor.$$

In particular, define $\mathcal{C}_r(m) = jr^2 + (m - jr)^2$; then $\sum_i x_i^2 \le \mathcal{C}_r(m)$. For zero capacity use $\mathcal{C}_0(m) = 0$.

Lemma 12 (rational linear certificates; [6, Lemma 44]). *Suppose a vector z satisfies $Az \le b$ and coordinate bounds $\ell \le z \le h$, all rational. For any rational $\lambda \ge 0$, let $a = A^{\mathsf{T}}\lambda$. If*

$$\sum_j \min(a_j l_j, a_j h_j) > \lambda^{\mathsf{T}} b, \tag{9}$$

there is no such z. More generally, for a linear objective $q^{\mathsf{T}} z$ and $r = q - A^{\mathsf{T}}\lambda$,

$$q^{\mathsf{T}} z \le \lambda^{\mathsf{T}} b + \sum_j \max(r_j l_j, r_j h_j). \tag{10}$$

The rules of a finite proof. The certificate formats that every Computational result of the series must satisfy are those of [6] (its Section 7, "The verifier", Appendix B, "The layout of the certificates", and the paragraph "Products, squares, logarithms and Bernstein's rule" of its Appendix D.7); those used here are: a polynomial on $[0,1]$ is nonnegative if all its Bernstein coefficients are nonnegative, strictly positive if they are all positive, and the expansion is checked against the target polynomial coefficient by coefficient (degree elevation permitted); a polynomial on a product of intervals is treated by a tensor Bernstein expansion, or by an expansion in some of the variables whose univariate coefficient polynomials are certified positive; Sturm's theorem [1] checks a saved univariate sign certificate by its rational signed remainder identities and endpoint signs; logarithms are enclosed by the rational series

$$\log r = 2\sum_{j=0}^{m-1}\frac{z^{2j+1}}{2j+1} + \mathcal{R}_m, \qquad z=\frac{r-1}{r+1}, \qquad |\mathcal{R}_m| \le \frac{2|z|^{2m+1}}{(2m+1)(1-z^2)}, \tag{11}$$

and square roots by squaring rational endpoints; a linear or scalar terminal witness is checked by Lemma 12, a rational Farkas certificate [14]; a matrix identity is tested entry by entry, including zero entries, and a positive contact identity must have every coefficient nonnegative and every factor a stated contact slack or a square. A finite covering tree stores at each node a closed product of rational intervals, a split (two children whose union is the parent, sharing the cut), a contraction (a necessary bound whose intersection contains every counterexample in the box) or a terminal witness; the verifier rejects an unknown address, a repeated node, a wrong parent box, a gap between children, an unsupported contraction or an invalid witness, and at the end the pending set must be empty. Spectral grids are handled differently from parameter boxes: the interpolation proof of Section 16 first maps every real spectrum into the finite relaxation with explicit error allowances, and the rational certificate excludes that relaxation.

3 The form `0008:022143116`, the aligned coordinates and the finite checks

Appendix A prints the integral Gram matrix A_P of the form, with $P = (4/s_P)A_P$, $s_P = 4$, $\det P = 864$ and 90 contact pairs. The theorem of this paper is the following.

Theorem 13. *Assume Computational result 14 and the reference data of the form (Appendix A). Let $L \supset M \supset N$ be a chain as in Definition 2 whose rank-9 member N has a Gram matrix admissibly marked by* `0008:022143116`. *Then $\Delta = \det L \ge 972$.*

The proof occupies Sections 4 to 17 and is assembled in Section 18. It rests on one Computational result.

Computational result 14 (the local geometry and complete spectral cover for `0008`). The aligned coordinates, integral covectors and twelve triangles below have the stated basis, incidence and length identities. The finite ray identities identify the pure, linked and mixed contacts used below. Further nonnegative coefficient identities

establish (18) on the complete 45-dimensional contact cone. The coupling bounds and positive shifts are derived from the contact inequalities below. For the six local reductions proved in Sections 10 and 11, the scalar inputs are the rational endpoint comparisons (31) and the finite polynomial certificates specified there. The $\eta \le 1/30$ exclusion is derived from the additional finite inputs in Sections 12 and 15; the improved ℓ reduction uses it explicitly. All region boundaries are included. On the remaining root (17), the 400-node final tree has $1\,091$ rational certificates, each satisfying (62) for its stated exclusion or outward contraction, and an empty final frontier. The rows must be exactly the necessary rows derived below, including all interpolation allowances and denominator signs.

The finite coefficient identities, coefficient signs, and final tree are computational inputs.

The aligned coordinates. The reference determinant is $864 = \det P = e/\det S$, in the normalisation (4); it agrees with the root paper's Table 4, which gives $\det M = 972$ and $u = 8/9$ for the section of K_{11} lying over the form, since the rank-9 section of that M by a shortest dual vector has determinant $ud = (8/9)\cdot 972 = 864$ ([5, Fact 4], Appendix A). In aligned physical coordinates write

$$S = I + E, \qquad E = \begin{pmatrix} A & U_{AB} & U_{AC} \\ U_{AB}^{\mathsf T} & B & U_{BC} \\ U_{AC}^{\mathsf T} & U_{BC}^{\mathsf T} & C \end{pmatrix}, \qquad F = C + \frac{A}{3}, \tag{12}$$

with all blocks of size three. No coupling is set to zero. Let Π_A, Π_B, Π_C be the orthogonal coordinate projections, and define

$$\begin{gathered} a = \operatorname{tr} A, \quad b = \operatorname{tr} B, \quad c = \operatorname{tr} C, \quad \tau = c + \frac{a}{3}, \\ t = \frac{2a}{3} + b + \tau, \quad \ell = \frac{4a}{9} + b + \frac{5\tau}{3}, \quad \eta = b + \tau, \\ r_k = \operatorname{tr}((\Pi_k E \Pi_k)^2), \quad v_k = \operatorname{tr}(\Pi_k E^2), \quad r^2 = v_A + v_B + v_C, \quad q = \frac{r^2}{2} + \frac{2v_C}{3}, \\ \iota_k = \operatorname{tr}(\Pi_k S^{-1}), \qquad \delta = 9 - \sum_k \iota_k \ge 0. \end{gathered} \tag{13}$$

Thus $r = \|E\|_F$ and $r_k = \|\Pi_k E \Pi_k\|_F^2$.

4 Two sufficient flag tests and the containing domain

The three integral covectors in the C direction have average squared dual length $2\iota_C/9$, so the shortest of them, z say, has $X[z] \le 2\iota_C/9$. By (5) and $e = 864 \det S$,

$$\Delta \ge \frac{e}{2X[z]^2} \ge \frac{864 \det S}{2\left(\frac{2\iota_C}{9}\right)^2} = \frac{8748 \det S}{\iota_C^2} = 972\,\frac{\det S}{\left(\frac{\iota_C}{3}\right)^2},$$

so Lemma 4 proves $\Delta \geq 972$ whenever

$$\Phi_C = \log\det S - 2\log\Big(\frac{\iota_C}{3}\Big) \geq 0. \tag{14}$$

A second test uses twelve integral short-shell triangles. Their exact incidence identity gives their average $Q_\triangle$ as $\iota_A/3 + 4\iota_C/9$, so the triangle with the least $Q_\triangle = s_\triangle$ has $s_\triangle \leq \iota_A/3 + 4\iota_C/9$, and the weighted triangle bound (7) of Lemma 5 with $841 \cdot 864/162 = 13456/3$ gives

$$\Delta \geq \frac{841\, e}{162\, s_\triangle^2} = \frac{13456}{3}\,\frac{\det S}{s_\triangle^2}, \qquad s_\triangle \leq \frac{\iota_A}{3} + \frac{4\iota_C}{9}.$$

Since $972 \cdot 3/13456 = (27/58)^2$ and $27(\iota_A/3 + 4\iota_C/9) = 9\iota_A + 12\iota_C$, the right side is at least 972 as soon as $\det S \geq (27 s_\triangle/58)^2$, and thus it suffices that

$$\Phi_W = \log\det S - 2\log\big(\frac{9\iota_A + 12\iota_C}{58}\big) \geq 0. \tag{15}$$

The finite covector and triangle lists are checked in integral coordinates. Their normalization is $e = 864\det S$; for example $1944(2/9)^2/864 = 1/9$, which verifies (14). Every point in the region passes at least one of the two tests.
The local reductions remove the union of the following four regions:

$$r \leq \frac{43}{100}, \qquad \ell \leq \frac{32}{125}, \qquad \eta \leq \frac{1}{30}, \qquad a \geq \frac{15}{16}. \tag{16}$$

They also remove each of $t \geq 11/15$, $3t + 4\delta \geq 637/288$, and $9\tau + (45/2)\delta \geq 8a + 5b$. Sections 10 and 11 prove these six non-η reductions from the stated finite scalar inputs and the contact bounds developed next. More precisely, the improved $\ell \leq 32/125$ argument covers $\eta \geq 1/30$; its complementary part uses the separate $\eta \leq 1/30$ exclusion. The radius argument only needs the unconditional auxiliary bound $\ell > 1/4$. Every argument uses the full matrix S, including its interblock couplings. Pure and linked contacts give $a, b, \tau \geq 0$. Since $\det S = e/864 < 1137/864 = 379/288$ by (2), Lemma 6 at $8/15$ and $113/50$ gives $(8/15)I \prec S \prec (113/50)I$. Writing $\lambda_- = \lambda_{\min}(E)$, $\lambda_+ = \lambda_{\max}(E)$, every remaining counterexample has

$$(a, b, \tau, \lambda_-, \lambda_+) \in \Big[0, \frac{15}{16}\Big] \times \Big[0, \frac{11}{15}\Big]^2 \times \Big[-\frac{7}{15}, 0\Big] \times \Big[0, \frac{63}{50}\Big]. \tag{17}$$

If $\lambda_- \geq 0$, then $\det S \geq 1$ and $\iota_C \leq 3$, so the first flag already succeeds.

5 The contact information retained by the relaxation

The exact homogeneous contact identities give

$$\begin{aligned} r^2 &\leq 6t^2, \qquad r^2 \leq \frac{45\ell^2}{8}, \qquad E \succeq -\frac{24}{25}\ell I, \\ q &\leq \frac{51\ell^2}{16} - \frac{73}{112}\ell\eta, \qquad q \leq \frac{51\ell^2}{16} - \frac{7}{5}\eta^2. \end{aligned} \tag{18}$$

Here are the precise computational identities needed for these assertions. Let $f_i = E[v_i] \geq 0$ be the reference-contact slacks in the aligned coordinates. For $\nu = 1, 2, 3, 4$, let $P_\nu(E) = \sum_{i,j,k} w_{ijk}^{(\nu)} f_i f_j f_k$, with the finite rational weights $w_{ijk}^{(\nu)} \geq 0$. The required identities are

$$t(6t^2 - r^2) = P_1(E), \qquad \ell\left(\frac{45\ell^2}{8} - r^2\right) = P_2(E),$$
$$\ell\left(\frac{51\ell^2}{16} - q - \frac{73}{112}\ell\eta\right) = P_3(E), \qquad \ell\left(\frac{51\ell^2}{16} - q - \kappa\eta^2\right) = P_4(E),$$

where $\kappa = 87893197/59234544 > 7/5$. The spectral bound uses fixed positive semidefinite matrices M_i and the linear identity

$$E + \frac{24}{25}\ell I = \sum_i f_i M_i.$$

Equality of every polynomial coefficient, the signs of the weights, and positivity of the fixed matrices are the computational obligations in Computational result 14. The identities hold in the full matrix, including all interblock couplings.

For completeness, division by t or ℓ needs no additional rank computation. The nonnegative parameters a, b, τ imply $t, \ell \geq 0$. If either vanishes, all three parameters vanish. The independently proved positive shift (20) then has $s_A = s_B = s_C = 0$, so $E \succeq 0$. Since $\operatorname{tr} E = t = 0$, every eigenvalue is zero and $E = 0$. Otherwise $t, \ell > 0$, and division in the four displayed cubic identities proves the scalar bounds. The replacement of κ by $7/5$ uses $\eta^2 \geq 0$. The matrix identity proves the spectral bound directly. Set $k_0 = \min(\tau, a/3 + 5b/9)$. The linked ternary cone gives $\|F\|_F^2 \leq 2\tau^2 + 3k_0^2$. With centered blocks $A_c = A - aI/3$, $B_c = B - bI/3$, $F_c = F - \tau I/3$, the explicit identity

$$\begin{aligned}\frac{5}{6}r^2 + \frac{\Lambda_0}{3} - q = \frac{3}{8}(2\tau^2 + 3k_0^2 - \|F\|_F^2) + \frac{8}{27}\|A_c + \frac{3F_c}{8}\|_F^2 \\ + \frac{1}{3}\|B_c\|_F^2 + \frac{2}{3}\|U_{AB}\|_F^2,\end{aligned}$$
$$\Lambda_0 = \frac{53\tau^2}{24} + \frac{27k_0^2}{8} - \frac{8a^2}{27} - \frac{2a\tau}{9} - \frac{b^2}{3}$$

gives $2v_C - r^2 \leq \Lambda_0$. To check the remaining scalar bound directly, let $h_0 = a/3 + 5b/9$. Then $k_0^2 \leq \tau h_0$ and

$$\frac{159}{200}\ell^2 - \Lambda_0 = \frac{27}{8}(\tau h_0 - k_0^2) + \frac{34}{75}a^2 + \frac{677}{600}b^2 + \frac{53}{75}ab + \frac{11}{40}a\tau + \frac{31}{40}b\tau \geq 0.$$

Combining this with (18) gives

$$v_C \leq \frac{2151\ell^2}{1000} - \frac{219}{560}\ell\eta. \tag{19}$$

Let $a_0 = \min(a, b/3)$, $b_0 = \min(b, a/3)$, $M_0 = \min(a, (a+b+3\tau)/2)$, and $D_0 = M_0^2 + a_0^2 + (a - M_0 + a_0)^2$. The pure D_6 and linked ternary contact cones give

$$r_A \le \min\left\{2a^2 + 3a_0^2, \frac{3D_0 + a^2}{2}\right\}, \qquad r_B \le 2b^2 + 3b_0^2,$$
$$r_A + r_B + 2\|U_{AB}\|_F^2 \le \frac{7(a+b)^2}{2}, \qquad \frac{(\operatorname{tr} \Pi_k E)^2}{3} \le r_k \le v_k.$$

The separate-block bounds have a direct proof. Pure pair contacts give $-a \le A_{ii} \le a$ and $-b \le B_{ii} \le b$. Summing the cross inequalities $A_{ii} + B_{jj} \ge 0$ over j improves the floors to $A_{ii} \ge -a_0$ and $B_{ii} \ge -b_0$. The averaged mixed contacts give $a + b - 2A_{ii} + 3F_{ii} \ge 0$; since $F_{ii} \le \tau$, we obtain $A_{ii} \le M_0$. Concentration of these bounded diagonal entries gives $\sum_i A_{ii}^2 \le D_0$. For either ternary block X, pair contacts also give

$$4\sum_{i<j} X_{ij}^2 \le (\operatorname{tr} X)^2 + \sum_i X_{ii}^2, \qquad \|X\|_F^2 \le \frac{1}{2}\Big((\operatorname{tr} X)^2 + 3\sum_i X_{ii}^2\Big).$$

Using the weaker cap $A_{ii} \le a$ gives $\sum_i A_{ii}^2 \le a^2 + 2a_0^2$; the same argument applies to B. These prove both displayed separate-block bounds and $\sum_{i<j} B_{ij}^2 \le (b^2 + b_0^2)/2$. The combined A, B bound is Lemma 7 with $n = 6$, applied to the full block including U_{AB}. Finally, the bounds on r_k follow by centering each block and retaining the nonnegative coupling squares in v_k.

Lemma 15 (ternary spectral bounds from a diagonal floor). *Let G be a real symmetric ternary matrix with trace θ, $2|G_{ij}| \le G_{ii} + G_{jj}$ for $i \ne j$, and $G_{ii} \ge -\varrho$. Let $R_G = \sqrt{3\theta^2 + 6\varrho^2}$. Then*

$$\frac{\theta - R_G}{2} I \preceq G \preceq \frac{\theta + R_G}{2} I.$$

Proof. Write $d_i = G_{ii}$. Pair contacts give $d_i \le \theta$. The following two-product identity controls every linear function of such a diagonal. If $c_1 \le c_2 \le c_3$ and $\sum_i d_i = \theta$, then

$$-\varrho c_1 + \varrho c_2 + \theta c_3 - \sum_i d_i c_i = (d_1 + \varrho)(c_2 - c_1) + (\theta - d_3)(c_3 - c_2) \ge 0.$$

Reordering the coefficients shows that every linear function is at most its largest value on the six permutations of $(-\varrho, \varrho, \theta)$.
For an arbitrary vector v, let $x_i = |v_i|$, $S_x = x_1 + x_2 + x_3$, and $D_x = \sum_i d_i x_i^2$. The pair contacts give

$$2D_x - S_x \sum_i d_i x_i \le G[v] \le S_x \sum_i d_i x_i.$$

Both bounds are linear in (d_1, d_2, d_3). At $(-\varrho, \varrho, \theta)$ the upper expression is

$$Q(x, y, z) = (x + y + z)(-\varrho x + \varrho y + \theta z),$$

and the lower expression is $Q(-x,-y,z)$. Permuting the diagonal only permutes the three arguments. Thus it remains to bound Q for arbitrary real arguments.
Let $p = x+y+z$, $n = x^2+y^2+z^2$ and $w = (2px-n, 2py-n, 2pz-n)$. Direct expansion gives $\|w\|^2 = 3n^2$. Cauchy–Schwarz with $(-\varrho, \varrho, \theta)$ therefore gives

$$(2Q(x,y,z)-\theta n)^2 = \big((-\varrho,\varrho,\theta)\cdot w\big)^2 \le (2\varrho^2+\theta^2)3n^2 = R_G^2 n^2.$$

Since $R_G, n \ge 0$, this is exactly $(\theta - R_G)n/2 \le Q \le (\theta + R_G)n/2$. Apply the linear-functional bound to the upper envelope and to the negative of the lower envelope. Their six endpoint values satisfy these same bounds, proving the assertion for every v. □

Applying Lemma 15 to A and B supplies their square-root endpoints. With $R_a = \sqrt{3a^2+6a_0^2}$ and $R_b = \sqrt{3b^2+6b_0^2}$, the block spectral endpoints are

$$\begin{aligned}
\alpha_A &= \max\!\left(\frac{a-R_a}{2}, -b-3\tau\right), & \beta_A &= \min\!\left(\frac{a+R_a}{2}, a+2b+3\tau\right),\\
\alpha_B &= \frac{b-R_b}{2}, & \beta_B &= \frac{b+R_b}{2},\\
\alpha_C &= -\tau - \frac{\beta_A}{3}, & \beta_C &= 2\tau - \frac{\alpha_A}{3}.
\end{aligned}$$

The last row follows from $-\tau I \preceq F \preceq 2\tau I$; the additional A bounds follow from the positive linked shift. Paired contacts further give, for $Z_0 = a+3b+3\tau$,

$$\|U_{AC}\|_F^2 \le A_C := \frac{Z_0^2}{18} + \frac{Z_0\tau}{\sqrt{3}} + \frac{21\tau^2}{4}, \qquad \|U_{BC}\|_F^2 \le \frac{Z_0^2}{12} + \frac{b^2+b_0^2}{2}.$$

All three coupling energies are retained through

$$2\|U_{AB}\|_F^2 = v_A + v_B - v_C - r_A - r_B + r_C$$

and its cyclic versions. Finally, for any $\alpha_0 > 0$, the contact-moment identity gives

$$E + \operatorname{diag}(s_A I_3, s_B I_3, s_C I_3) \succeq 0, \qquad \begin{cases} s_A = b + (3+\alpha_0)\tau, \\ s_B = \dfrac{a}{4} + \dfrac{7b}{4} + \dfrac{3\tau}{4}, \\ s_C = \dfrac{a}{3} + \dfrac{2b}{3} + \left(1 + \dfrac{3}{4\alpha_0}\right)\tau. \end{cases} \tag{20}$$

The bounds $\|U_{AC}\|_F^2 \le A_C$ and (20) are proved in Sections 8 and 9.

6 The linked-F determinant gain

The following estimate retains the actual correlation between the A and C blocks in the small-η strip. Let

$$H = A + (b+3\tau)I, \qquad u_H = \operatorname{tr} H, \qquad e_H = \frac{1}{2}[(\operatorname{tr} H)^2 - \operatorname{tr}(H^2)],$$
$$p_H = \det H, \qquad \gamma = 1 + \frac{b}{3}.$$

For each $i \in \{1,2,3\}$ let $j < k$ be the other two indices, and let $\sigma \in \{-1,1\}$. Define

$$\lambda_{i,\sigma} = \frac{1}{2}\Big(\frac{u_H}{2} - H_{ii} + \sigma H_{jk}\Big), \qquad \mu_{i,\sigma} = \frac{1}{4}(\tau - F_{ii} + 2\sigma F_{jk}).$$

The contact identities give $\lambda_{i,\sigma} \geq 0$; the pair contacts of F give $\mu_{i,\sigma} \geq 0$. Writing $R_{i,\sigma} = (e_j + \sigma e_k)(e_j + \sigma e_k)^{\mathsf{T}}$, entrywise expansion gives $H = \sum \lambda_{i,\sigma} R_{i,\sigma}$. In particular $0 \preceq H \preceq u_H I$. Let $V_{i,\sigma}$ have diagonal -1 at i, diagonal 1 at j,k, entries σ at $(j,k),(k,j)$ and all other entries zero. Then $F = \sum \mu_{i,\sigma} V_{i,\sigma}$ and $\sum \mu_{i,\sigma} = \tau$.
Here is the precise additional finite moment certificate. With v,p,q ranging over these six indices, set

$$c_{vpq} = 3\,\mathrm{tr}(R_p R_q V_v) + 2\big(\mathrm{tr}\,R_p\ \mathrm{tr}\,R_q - \mathrm{tr}(R_p R_q)\big).$$

The certificate checks the signs $c_{vpq} \geq 0$ and the polynomial identities

$$\begin{aligned} \mathrm{tr}(HF) &= 4\sum_p \lambda_p \mu_p, \\ 3\,\mathrm{tr}(H^2 F) + 4\tau e_H &= \sum_{v,p,q} c_{vpq}\mu_v \lambda_p \lambda_q. \end{aligned} \tag{21}$$

The formulas specify finitely many rational coefficients; symmetry in p,q reduces the 216 ordered sign checks to 126. Their implication is immediate: both expressions on the left are nonnegative for the same actual H, F.

Lemma 16 (the linked determinant gain). *Suppose the stated root and pair contacts hold, together with (21) and its nonnegative coefficient signs. If $u_H/3 < \gamma$, let*

$$c_H = \gamma^3 - \frac{\gamma^2 u_H}{3} + \frac{\gamma e_H}{9}, \qquad n_H = 4\gamma^2 - \gamma u_H + \frac{2e_H}{27}.$$

Then

$$(\gamma^2 + 3\tau^2)\Big(c_H - \frac{p_H}{27}\Big) + \tau\gamma^2 n_H \leq \gamma^2 \det(I + C). \tag{22}$$

Proof. Set $C_* := \gamma I - H/3$ and $D_F := F + \tau I$. Their sum is the actual block metric:

$$C_* + D_F = I + C.$$

The root decomposition gives $0 \prec C_* \preceq \gamma I$. The pair contacts imply $-\tau I \preceq F \preceq 2\tau I$, so $D_F \succeq 0$. They also give $\mathrm{tr}(F^2) \leq 5\tau^2$, as shown in the separate-block estimates. Consequently $\mathrm{tr}\,D_F = 4\tau$ and

$$e_2(D_F) = \frac{1}{2}[(\mathrm{tr}\,D_F)^2 - \mathrm{tr}(D_F^2)] \geq 3\tau^2.$$

The ternary determinant and adjugate identities give

$$\begin{aligned} d_* := \det C_* &= c_H - \frac{p_H}{27}, \\ \mathrm{adj}(C_*) &= \Big(\gamma^2 - \frac{\gamma u_H}{3} + \frac{e_H}{9}\Big)I + \Big(\frac{\gamma}{3} - \frac{u_H}{9}\Big)H + \frac{H^2}{9}. \end{aligned}$$

Substitution yields the exact remainder

$$\operatorname{tr}(\operatorname{adj}(C_*)D_F) - \tau n_H = \Big(\frac{\gamma}{3} - \frac{u_H}{9}\Big)\operatorname{tr}(HF) + \frac{3\operatorname{tr}(H^2F) + 4\tau e_H}{27} \geq 0.$$

Each eigenvalue of C_* is at most γ and their product is $d_* > 0$. Thus every eigenvalue is at least d_*/γ^2. Since $D_F \succeq 0$, its adjugate is positive semidefinite even when D_F is singular: its eigenvalues are the pairwise products of the nonnegative eigenvalues of D_F. It follows that

$$\operatorname{tr}(\operatorname{adj}(D_F)C_*) \geq \frac{d_*}{\gamma^2}\operatorname{tr}(\operatorname{adj}(D_F)) = \frac{d_*}{\gamma^2}e_2(D_F) \geq \frac{3\tau^2 d_*}{\gamma^2}.$$

Finally the ternary polynomial identity

$$\det(C_* + D_F) = d_* + \operatorname{tr}(\operatorname{adj}(C_*)D_F) + \operatorname{tr}(\operatorname{adj}(D_F)C_*) + \det D_F$$

and $\det D_F \geq 0$ prove (22) after multiplication by γ^2. □

The B block has a similarly direct determinant bound. Its pair contacts give eigenvalues of $I + B$ in $[\ell_B, u_B] = [1-b, 1+2b]$ and sum $3+b$. For $0 \leq b \leq 1$, any three numbers x, y, z in this interval satisfy

$$\begin{aligned}(u_B - \ell_B)&[xyz - \ell_B u_B(x + y + z - \ell_B - u_B)] \\ &= u_B(x - \ell_B)(y - \ell_B)(z - \ell_B) + \ell_B(u_B - x)(u_B - y)(u_B - z) \geq 0.\end{aligned}$$

If $b = 0$, all three numbers equal one; otherwise divide by $u_B - \ell_B > 0$. Since $x + y + z - \ell_B - u_B = 1$, this proves

$$\det(I + B) \geq (1 - b)(1 + 2b). \tag{23}$$

7 A shared argument for eight compression branches

Write $T = a + b$. The following two bounds retain the same A block:

$$\begin{aligned} v_C \leq J_0 = \frac{T^2}{4} - \frac{Ta}{3} + \frac{T\tau}{2} + \frac{4D_0}{9} - \frac{5a\tau}{3} \\ + \frac{10M_0\tau}{3} + \frac{45\tau^2}{4} + \frac{b^2 + b_0^2}{2} + A_C, \end{aligned} \tag{24}$$

$$\begin{aligned} r_A + 6\|U_{BC}\|_F^2 \leq K_0 = 3D_0 + \frac{3T^2}{2} - 2Ta + 3T\tau - 6a\tau \\ + 12M_0\tau + \frac{27\tau^2}{2} + 3(b^2 + b_0^2). \end{aligned} \tag{25}$$

Here is their derivation. Let $x_i = A_{ii}$, $f_i = F_{ii}$, $d_i = T - 2x_i + 3f_i$, and let A_o, B_o be the sums of squared upper off-diagonal entries of A, B. Fix i, and let $j < k$ be the other two indices. For $s \in \{\pm 1\}^3$, let $p(s) = s_1 s_2 s_3$ and $z_i(s) = e_j - p(s)e_k$. The finite contact identities identify the following sixteen physical vectors for this i:

$$v_{i,s}^{\pm} = 2^{-1/2}(\pm z_i(s),\ s,\ \sqrt{3}e_i).$$

Each has reference squared length four, so $E[v_{i,s}^{\pm}] \geq 0$. Their average cancels the A–B and A–C terms and gives

$$g_i(s) := \frac{1}{2}\big(E[v_{i,s}^{+}]+E[v_{i,s}^{-}]\big) = \frac{d_i}{2}+\sum_{h<\ell} B_{h\ell}s_h s_\ell+\sqrt{3}\sum_h (U_{BC})_{hi}s_h - A_{jk}p(s) \geq 0.$$

Every nonconstant product of distinct sign coordinates has average zero on this cube. Expanding therefore gives both identities

$$\frac{1}{8}\sum_s g_i(s) = \frac{d_i}{2}, \qquad \frac{1}{8}\sum_s g_i(s)g_i(-s) = \frac{d_i^2}{4} + B_o - A_{jk}^2 - 3\|U_{BC}[:,i]\|^2 =: q_i.$$

Thus $d_i \geq 0$ and $q_i \geq 0$. Since $\sum_i d_i = Z_0 = a+3b+3\tau$, $\sum_i d_i^2 \leq Z_0^2$. Summing $q_i \geq 0$ and discarding $A_o \geq 0$ gives $\|U_{BC}\|_F^2 \leq Z_0^2/12 + B_o$, which proves the stated B–C bound after using $B_o \leq (b^2+b_0^2)/2$. In particular, the pure cross-contact bound $x_i \geq -b/3$ gives $f_i \geq -a/3 - 5b/9$, the diagonal floor used in the linked-moment estimate. The three residuals satisfy

$$r_A + 6\|U_{BC}\|_F^2 + 2\sum_i q_i = \sum_i x_i^2 + \frac{1}{2}\sum_i d_i^2 + 6B_o.$$

The diagonal bounds $-a_0 \leq x_i \leq M_0$ and $\sum x_i = a$ give $\sum x_i^2 \leq D_0$: apply Lemma 11 to $x_i + a_0$, whose sum $a+3a_0$ lies between M_0+a_0 and $2(M_0+a_0)$, and expand. Also $B_o \leq (b^2+b_0^2)/2$. No vertex maximization is needed for the terms involving F. Its pair contacts $2|F_{ij}| \leq f_i + f_j$ give

$$-\tau \leq f_i \leq \tau, \quad \sum_i f_i^2 \leq 3\tau^2, \quad \sum_{i<j} F_{ij}^2 \leq \frac{1}{4}\sum_{i<j}(f_i+f_j)^2 = \frac{1}{4}\Big(\tau^2+\sum_i f_i^2\Big) \leq \tau^2.$$

Since $x_i \leq M_0$ and $\sum_i(\tau - f_i) = 2\tau$,

$$\sum_i x_i f_i = a\tau - \sum_i x_i(\tau - f_i) \geq a\tau - 2M_0\tau.$$

Expanding $\sum d_i^2$ now bounds its F-dependent contribution by $-6a\tau + 12M_0\tau + 27\tau^2/2$, proving K_0. For J_0, expand $r_C = \|F - A/3\|_F^2$ and the same paired inequality. The F-dependent terms are

$$\frac{7}{4}\sum f_i^2 + 2\sum_{i<j} F_{ij}^2 - \frac{5}{3}x^{\mathsf{T}}f - \frac{1}{9}A_o - \frac{4}{3}\sum_{i<j} A_{ij}F_{ij}.$$

For each $i<j$, the square $(A_{ij}/3+2F_{ij})^2 \geq 0$ bounds the last two terms by $4\sum_{i<j} F_{ij}^2$, increasing the off-diagonal coefficient to six. The preceding three bounds then give $45\tau^2/4 - 5a\tau/3 + 10M_0\tau/3$. Adding the bound for U_{AC} proves J_0.

Each bound is the minimum of its eight quadratic substitutions obtained by replacing M_0, a_0, b_0 independently by their two defining candidates. Indeed $0 \leq a_0 \leq a$ and $(a+a_0)/2 \leq M_0 \leq a$. For $d, e \geq 0$,

$$\begin{aligned} &D_0(M_0+d, a_0+e) - D_0(M_0, a_0) \\ &\qquad = 2(2M_0 - a - a_0)d + 2(a - M_0 + 2a_0)e + 2(d^2 - de + e^2) \geq 0. \end{aligned}$$

All other occurrences have nonnegative coefficients. Thus every substitution is an upper bound, and one attains the three actual minima simultaneously. On a parameter box $|z - z_*| \le w$, a quadratic is replaced by its valid affine upper bound

$$z^{\mathsf{T}} H z \le 2(Hz_*)^{\mathsf{T}} z - z_*^{\mathsf{T}} H z_* + w^{\mathsf{T}} |H| w. \tag{26}$$

This keeps the first-moment correlations and bounds only the quadratic remainder.

8 The full A–C coupling bound

Write $U = U_{AC}$, $S_U = (U + U^{\mathsf{T}})/2$, and $W_U = (U - U^{\mathsf{T}})/2$. Keep the contact vectors and sign polynomials of the preceding section. Their half-difference is

$$c_i(s) := \frac{1}{2}\big(E[v_{i,s}^+] - E[v_{i,s}^-]\big) = z_i(s)^{\mathsf{T}} U_{AB} s + \sqrt{3}\, z_i(s)^{\mathsf{T}} U e_i.$$

Both contact slacks are nonnegative, so $|c_i(s)| \le g_i(s)$. The ordinary and parity-weighted sign averages give, for $j < k$ complementary to i,

$$\frac{1}{8}\sum_s c_i(s) = \sqrt{3}\, U_{ji}, \qquad \frac{1}{8}\sum_s p(s) c_i(s) = -\sqrt{3}\, U_{ki}.$$

All terms from U_{AB} have zero average in both identities. Since $|p(s)| = 1$ and $\frac{1}{8}\sum_s g_i(s) = d_i/2$, we obtain

$$|U_{hi}| \le \frac{d_i}{2\sqrt{3}} \qquad (h \ne i). \tag{27}$$

The linked contacts give $F \pm (2/\sqrt{3}) S_U$ in the ternary contact cone. Here is a direct consequence of such a pair of cone relations. If $F \pm Y$ lie in that cone and $\tau = \operatorname{tr} F$, their diagonal bounds imply $|Y_{ii}| \le \tau$. Their pair inequalities and $2|Y_{ij}| \le |F_{ij} + Y_{ij}| + |F_{ij} - Y_{ij}|$ imply $2|Y_{ij}| \le F_{ii} + F_{jj}$. Hence

$$\sum_{i<j} |Y_{ij}| \le \tau, \qquad \sum_i Y_{ii}^2 \le 3\tau^2, \qquad \sum_{i<j} Y_{ij}^2 \le \tau^2.$$

This proves the required bounds. Applied to $Y = (2/\sqrt{3}) S_U$, they give

$$\|S_U\|_F^2 \le \frac{15}{4}\tau^2, \qquad \sum_{i<j} (S_U)_{ij}^2 \le \frac{3}{4}\tau^2. \tag{28}$$

Because $U_{ij} = (S_U)_{ij} + (W_U)_{ij}$ and $U_{ji} = (S_U)_{ij} - (W_U)_{ij}$, (27) gives

$$|(W_U)_{ij}| \le \frac{\min(d_i, d_j)}{2\sqrt{3}} + |(S_U)_{ij}|.$$

For three nonnegative numbers with sum Z_0, one has $\sum_{i<j} \min(d_i, d_j)^2 \le Z_0^2/3$. Indeed, after ordering them as $d_1 \le d_2 \le d_3$, the difference is the sum of two nonnegative products:

$$Z_0^2 - 3(2d_1^2 + d_2^2) = (d_2 - d_1)(d_2 + 5d_1) + (d_3 - d_2)(2d_1 + 3d_2 + d_3).$$

Squaring the entry bounds and applying Cauchy–Schwarz to the three cross terms now gives

$$\|W_U\|_F \le \frac{Z_0}{\sqrt{18}} + \sqrt{\frac{3}{2}}\,\tau.$$

The symmetric and skew-symmetric parts are orthogonal in the Frobenius inner product. Combining this inequality with (28) proves

$$\|U_{AC}\|_F^2 = \|S_U\|_F^2 + \|W_U\|_F^2 \le \frac{Z_0^2}{18} + \frac{Z_0\tau}{\sqrt{3}} + \frac{21}{4}\tau^2 = A_C.$$

A rational upper replacement for $1/\sqrt{3}$ preserves the bound because $Z_0, \tau \ge 0$.

9 A positive contact moment and the block shifts

We give the full positivity argument for (20). For a symmetric ternary matrix M, the six vectors $e_j \pm e_k$ ($j < k$ complementary to i) give the identity

$$G = \sum_{i=1}^{3} \sum_{\varepsilon=\pm1} \frac{\frac{\operatorname{tr} G}{2} - G_{ii} + \varepsilon G_{jk}}{2} (e_j + \varepsilon e_k)(e_j + \varepsilon e_k)^{\mathsf{T}}.$$

Thus $|G_{jk}| \le \operatorname{tr} G/2 - G_{ii}$ for every complementary pair implies $0 \preceq G \preceq (\operatorname{tr} G)I$: the coefficients are nonnegative, their sum is $\operatorname{tr} G/2$, and each root has squared length two. Reversing all off-diagonal signs preserves these coefficient inequalities and hence positivity.

Let $m_i = d_i/2$ and define

$$H_A = A + \operatorname{diag}\big(b + \frac{3}{2}(\tau - F_{ii})\big), \qquad J_F = F + \operatorname{diag}\big(\frac{\tau - F_{ii}}{2}\big).$$

The parity average of g_i is $-A_{jk}$, so its nonnegativity gives $|A_{jk}| \le m_i$. Since

$$\operatorname{tr} H_A = Z_0, \qquad \frac{\operatorname{tr} H_A}{2} - (H_A)_{ii} = m_i, \qquad \operatorname{tr} J_F = 2\tau,$$

H_A satisfies the six coefficient inequalities. The pair contacts $2|F_{jk}| \le \tau - F_{ii}$ show that J_F does too. Consequently both are positive. This also proves the spectral bounds used above: $-\tau \le F_{ii} \le \tau$ bounds the diagonal correction in J_F between zero and τI, and the one in H_A between bI and $(b + 3\tau)I$. Hence

$$-\tau I \preceq F \preceq 2\tau I, \qquad -(b + 3\tau)I \preceq A \preceq (a + 2b + 3\tau)I.$$

For any square matrix X, write $\operatorname{off} X = X - \operatorname{diag}(X_{ii})$, and let $\widehat{U} = \sqrt{3}\,U_{AC}$. The mixed contacts give the positive matrix

$$N = \begin{pmatrix} H_A & 2U_{AB} & \operatorname{off} \widehat{U} \\ 2U_{AB}^{\mathsf{T}} & \big(\frac{Z_0}{2}\big)I + 3\operatorname{off} B & \sqrt{3}\,U_{BC} \\ (\operatorname{off} \widehat{U})^{\mathsf{T}} & \sqrt{3}\,U_{BC}^{\mathsf{T}} & \operatorname{diag}(m_i) \end{pmatrix} \succeq 0. \tag{29}$$

Here is a direct verification, including the normalization. Set $\sigma^{\pm}_{i,s} = E[v^{\pm}_{i,s}] = g_i(s) \pm c_i(s)$ and $w^{\pm}_{i,s} = (\pm z_i(s), s, e_i)$. Then

$$N = \frac{1}{16} \sum_{i,s,\pm} \sigma^{\pm}_{i,s} w^{\pm}_{i,s} (w^{\pm}_{i,s})^{\mathsf{T}}. \tag{30}$$

Every summand is positive semidefinite. To verify the identity, evaluate both sides on $(x, y, z) \in (\mathbb{R}^3)^3$. For a fixed i, let $P_i(s) = x_j - p(s)x_k$, $Q_i(s) = z_i + s^{\mathsf{T}}y$, and let $\langle\cdot\rangle_s$ denote the average over the eight signs. The opposite pair contributes $\langle g_i(P_i^2 + Q_i^2) + 2c_iP_iQ_i\rangle_s$. Expansion gives

$$\begin{aligned}
\langle g_i P_i^2\rangle_s &= m_i(x_j^2 + x_k^2) + 2A_{jk}x_jx_k,\\
\langle g_i Q_i^2\rangle_s &= m_i(z_i^2 + \|y\|^2) + 2\sum_{h<\ell} B_{h\ell}y_hy_\ell + 2\sqrt{3}\, z_i \sum_h (U_{BC})_{hi}y_h,\\
\langle c_iP_iQ_i\rangle_s &= \sqrt{3}\, z_i\big((U_{AC})_{ji}x_j + (U_{AC})_{ki}x_k\big)\\
&\qquad + x_j\sum_h (U_{AB})_{jh}y_h + x_k\sum_h (U_{AB})_{kh}y_h.
\end{aligned}$$

Each equality uses only that a product of sign coordinates has average zero unless every exponent is even. Summing over i and using $(H_A)_{jj} = \sum_{i\neq j} m_i$ and $\sum_i m_i = Z_0/2$ gives precisely (29).
Define the two further corrections

$$\overline{H}_A = 2\,\mathrm{diag}((H_A)_{ii}) - H_A, \qquad B_* = \mathrm{diag}(B_{ii} + b) - \frac{1}{2}\,\mathrm{off}\, B.$$

The sign reversal already proves $\overline{H}_A \succeq 0$. For B_*, its trace is $4b$, and the same coefficient criterion reduces to $|B_{jk}|/2 \le b - B_{ii}$. The pair contacts give the stronger inequality $|B_{jk}|/2 \le (b - B_{ii})/4$, with $b - B_{ii} \ge 0$. Thus $B_* \succeq 0$ as well. The diagonal entries of J_F are nonnegative because $J_F \succeq 0$.
Let $\alpha, \beta \ge 0$ and $\alpha\beta \ge 9/4$, and take

$$s_A = b + (3+\alpha)\tau, \qquad s_B = \frac{a}{4} + \frac{7b}{4} + \frac{3\tau}{4}, \qquad s_C = \frac{a}{3} + \frac{2b}{3} + \left(1 + \frac{\beta}{3}\right)\tau.$$

The linked cone bound $|Y_{ii}| \le \tau$ used above gives $|\widehat{U}_{ii}| \le 3\tau/2$. Therefore

$$\Xi = \begin{pmatrix} \alpha\tau I & 0 & \mathrm{diag}(\widehat{U}_{ii}) \\ 0 & 0 & 0 \\ \mathrm{diag}(\widehat{U}_{ii}) & 0 & \beta\tau I \end{pmatrix} \succeq 0.$$

Indeed it is the direct sum of three two-dimensional forms with nonnegative diagonals and determinant $\alpha\beta\tau^2 - \widehat{U}_{ii}^2 \ge 0$, and three zero coordinates. The two-dimensional criterion follows by completing a square when a diagonal is positive; if it vanishes, the determinant condition forces the off-diagonal entry to vanish. This includes $\tau = 0$.

Finally let

$$D = \operatorname{diag}(I, I, \sqrt{3}I), \qquad D_1 = \operatorname{diag}\Big(I, \frac{I}{2}, I\Big), \qquad D_2 = \operatorname{diag}\Big(0, \frac{I}{2}, I\Big).$$

Substitution of the displayed block entries gives the exact identity

$$D\big(E + \operatorname{diag}(s_A I, s_B I, s_C I)\big)D = D_1 N D_1 + D_2 N D_2 \\ + \operatorname{diag}\big(3\operatorname{diag}((J_F)_{ii}),\, B_*,\, \overline{H}_A + 3J_F\big) + \Xi.$$

Its left side has diagonal blocks $A + s_A I$, $B + s_B I$ and $3(C + s_C I)$ and off-diagonal blocks U_{AB}, $\sqrt{3}U_{AC}$ and $\sqrt{3}U_{BC}$. On the right, the off-diagonal blocks are U_{AB} (from $D_1 N D_1$), off $\widehat{U} + \operatorname{diag}(\widehat{U}_{ii}) = \sqrt{3}U_{AC}$ and $\sqrt{3}U_{BC}$ (half from each of $D_1 N D_1$, $D_2 N D_2$), and the diagonal blocks are as follows. In the first, $(H_A)_{ii} - A_{ii} + 3(J_F)_{ii} = b + \frac{3}{2}(\tau - F_{ii}) + \frac{3}{2}(\tau + F_{ii}) = b + 3\tau$; in the third, $\overline{H}_A = -A + \operatorname{diag}(2A_{ii} + b + \frac{3}{2}(\tau - F_{ii}))$ and $3J_F = 3F + \operatorname{diag}(\frac{3}{2}(\tau - F_{ii}))$, so the diagonal correction of $2\operatorname{diag}(m_i) + \overline{H}_A + 3J_F = \operatorname{diag}(d_i) + \overline{H}_A + 3J_F$ is $d_i + 2A_{ii} - 3F_{ii} + b + 3\tau = T + b + 3\tau = a + 2b + 3\tau$, and $3C = 3F - A$. Hence

$$H_A + 3\operatorname{diag}((J_F)_{ii}) + \alpha\tau I = A + (b + 3\tau + \alpha\tau)I = A + s_A I,$$
$$\frac{1}{2}\Big(\Big(\frac{Z_0}{2}\Big)I + 3\operatorname{off} B\Big) + B_* = B + \Big(\frac{Z_0}{4} + b\Big)I = B + s_B I,$$
$$\operatorname{diag}(d_i) + \overline{H}_A + 3J_F + \beta\tau I = -A + 3F + (a + 2b + 3\tau + \beta\tau)I = 3(C + s_C I).$$

Every term on the right is positive. Since D is invertible, the desired shift is positive. Taking $\alpha = \alpha_0 > 0$ and $\beta = 9/(4\alpha_0)$ proves (20) for every choice used by the relaxation.

10 The three trace reductions and the large-a region

Throughout the local reductions, suppose that S is a counterexample: $S \succ 0$, $\operatorname{tr} S^{-1} \le 9$, $\det S < 379/288$, and both flags (14) and (15) are negative. The contact bounds already proved give $a, b, \tau \ge 0$. We first give the exact finite scalar inputs and then prove their consequences for every such matrix.

The trace cutoff requires two rational numbers d_0, d_1 satisfying

$$0 < d_0 < d_1, \qquad \frac{379}{288} \le d_1 d_0^8, \qquad 9 \le \frac{1}{d_1} + \frac{8}{d_0}, \qquad d_1 + 8d_0 \le 9 + \frac{11}{15}. \tag{31}$$

These are five rational comparisons, counting positivity and separation separately. If $\operatorname{tr} S \ge d_1 + 8d_0$, Lemma 9, including its monotonicities, gives $\det S \ge d_1 d_0^8 \ge 379/288$, a contradiction. Hence

$$t < \frac{11}{15} < \frac{3}{4}. \tag{32}$$

A single further polynomial certificate gives a determinant bound that retains the inverse-trace deficit. Define

$$P_{\mathrm{tr}}(X) = 8^8 X^9 - (9X - 1)^8 - \frac{27}{7}(X - 1)^2(9X - 1)^7.$$

The finite input is a rational polynomial Q_{tr}, the coefficient identity $P_{\mathrm{tr}} = (X-1)^2 Q_{\mathrm{tr}}$, and a positive Bernstein expansion of Q_{tr} on $[1, 23/10]$. The identity forces Q_{tr} to have degree seven. As explained in Section 2.2, equality of coefficients and positivity of finitely many Bernstein coefficients prove positivity on the whole interval.

Lemma 17 (trace compensation). *With these finite data, every positive* 9×9 *matrix* S *with* $\operatorname{tr} S^{-1} \le 9$ *and* $\operatorname{tr} S \le 9 + 3/4$ *satisfies*

$$\det S \ge 1 + \frac{3t + 4\delta}{7}, \qquad t = \operatorname{tr} S - 9, \quad \delta = 9 - \operatorname{tr} S^{-1}. \tag{33}$$

Proof. Cauchy–Schwarz gives $(\operatorname{tr} S)(\operatorname{tr} S^{-1}) \ge 81$, so $t \ge 0$. For $x \in [1, 23/10]$, let

$$d(x) = \frac{8x}{9x-1}, \qquad T(x) = \frac{9(x-1)^2}{9x-1}, \qquad D(x) = x\, d(x)^8.$$

Then $0 < d(x) \le x$, $x + 8d(x) = 9 + T(x)$, and $1/x + 8/d(x) = 9$. The continuous function T has $T(1) = 0$ and $T(23/10) = 1521/1970 > 3/4$. The intermediate value theorem therefore provides x with $T(x) = t$. Lemma 9 gives $\det S \ge D(x)$; its scalar equality case covers $x = 1$. The polynomial identity implies

$$(9x-1)^8 \Big(D(x) - 1 - \frac{3}{7} T(x) \Big) = (x-1)^2 Q_{\mathrm{tr}}(x) \ge 0.$$

Thus initially $\det S \ge 1 + 3t/7$.
To retain δ, write $z = \delta/9$ and apply this initial bound to $(1-z)S$. Since $0 < \operatorname{tr} S^{-1} \le 9$, we have $0 \le z < 1$; the scaled matrix has inverse trace nine and trace at most $9 + 3/4$. Moreover Cauchy–Schwarz gives $(9+t)z \le t$. Consequently

$$\det S \ge f_t(z), \qquad f_t(y) = \frac{1 + \frac{3}{7}\{t - (9+t)y\}}{(1-y)^9}.$$

For $0 \le y \le z$, differentiation gives

$$f_t'(y) = \frac{\frac{36}{7} + \frac{24}{7}\{t - (9+t)y\}}{(1-y)^{10}} \ge \frac{36}{7}.$$

Integration from zero to z proves $f_t(z) \ge 1 + 3t/7 + 36z/7 = 1 + (3t + 4\delta)/7$. □

Combining (33) with $\det S < 379/288$ immediately gives the second trace reduction

$$3t + 4\delta < \frac{637}{288}. \tag{34}$$

For the third, set $h_{AB} = a + b \ge 0$ and $v_0 = (6 + 3h_{AB})/(6 + h_{AB})$. Each coordinate satisfies $S_{ii}(S^{-1})_{ii} \ge 1$. Summing over the six A, B coordinates and applying Cauchy–Schwarz yields

$$36 \le (6 + h_{AB})(\iota_A + \iota_B), \qquad \iota_C \le 3v_0 - \delta.$$

Also $\operatorname{tr} S \le 39/4$ and $(\operatorname{tr} S)(9-\delta) \ge 81$ imply $0 \le \delta \le 9/13 < 3/2$. The exact identity

$$3(6+h_{AB})^2\Big(1+\frac{2h_{AB}}{3}-v_0^2\Big) = 2h_{AB}^3$$

gives $v_0 \ge 1$ and $v_0^2 \le 1 + 2h_{AB}/3$. Since $0 < \iota_C/3 \le v_0 - \delta/3$,

$$\Big(\frac{\iota_C}{3}\Big)^2 \le \Big(v_0 - \frac{\delta}{3}\Big)^2 \le 1 + \frac{2h_{AB}}{3} - \frac{\delta}{2}.$$

Here the last step uses $-2v_0\delta/3 \le -2\delta/3$ and $\delta^2/9 \le \delta/6$. If $9\tau + (45/2)\delta \ge 8a + 5b$, then

$$\frac{3t}{7} - \frac{2h_{AB}}{3} + \frac{15\delta}{14} = \frac{9\tau + \frac{45}{2}\delta - 8a - 5b}{21} \ge 0.$$

Trace compensation now implies $(\iota_C/3)^2 \le \det S$, so $\Phi_C \ge 0$, a contradiction. Therefore

$$9\tau + \frac{45}{2}\delta < 8a + 5b. \tag{35}$$

The large-a exclusion uses the same compensation and a single bivariate polynomial. Suppose $a \ge 15/16$. By (32) and $\tau \ge 0$, the pair (a,b) lies in the closed triangle $a \ge 15/16$, $b \ge 0$, $2a/3 + b \le 3/4$. Every point of this triangle has the form

$$a = \frac{15+3x}{16}, \qquad b = \frac{(1-x)y}{8}, \qquad 0 \le x, y \le 1.$$

Indeed $x = (16a-15)/3 \in [0,1]$; if $x < 1$, take $y = 8b/(1-x)$, while at $x = 1$ the triangle forces $b = 0$ and one may take $y = 0$. Define

$$P_a(a,b) = 3364(7+2a+3b)\{(3+a)(3+b)\}^2 - 5103(21+8a+11b+4ab)^2.$$

The finite input is a nonnegative tensor Bernstein expansion of $P_a((15+3x)/16, (1-x)y/8)$ on $[0,1]^2$. Its basis functions are nonnegative, so it proves $P_a(a,b) \ge 0$ on the entire triangle. Clearing the positive denominators gives

$$\Big\{\frac{27}{58}\Big(4 - \frac{1}{3+a} - \frac{4}{3+b}\Big)\Big\}^2 \le \frac{7+2a+3b}{7}. \tag{36}$$

The three-coordinate version of the preceding Cauchy–Schwarz argument gives $\iota_A \ge 9/(3+a)$ and $\iota_B \ge 9/(3+b)$. Thus the positive weighted inverse quantity in the second flag satisfies

$$0 < \frac{9\iota_A + 12\iota_C}{58} \le \frac{27}{58}\Big(4 - \frac{1}{3+a} - \frac{4}{3+b}\Big).$$

On the other hand, trace compensation and $\tau, \delta \ge 0$ give $\det S \ge (7+2a+3b)/7$. Squaring the last inverse bound and using (36) proves $\Phi_W \ge 0$, a contradiction. This excludes the whole region $a \ge 15/16$.

11 One local flag estimate for the radius and ℓ reductions

The two remaining reductions use a sharper one-sided logarithm remainder. Define, for $0 \le s < 1$,

$$K_{\mathrm{loc}}(s) = \frac{1}{2} + \frac{s}{3} + \frac{s^2}{4(1-s)}, \qquad L_q(s) = 1 - \frac{s}{3} - \frac{s^2}{6}, \qquad L_C(s) = \frac{2s + s^2}{9}.$$

Both L_q, L_C are nonnegative on $[0, 1]$.

Lemma 18 (a common local flag bound). *If $S = I + E \succ 0$ and $E \succeq -sI$ for $0 \le s < 1$, then*

$$\Phi_C \ge \ell - K_{\mathrm{loc}}(s) r^2 - \frac{2v_C}{3(1-s)}, \tag{37}$$

$$(1-s)\Phi_C \ge (1-s)\ell - L_q(s) q - L_C(s) v_C. \tag{38}$$

In particular, if $v_C \le d$, then

$$(1-s)\Phi_C \ge (1-s)\ell - \Big(\frac{1}{2} - \frac{s}{6} - \frac{s^2}{12}\Big) r^2 - \frac{2}{3} d. \tag{39}$$

Proof. For every $x \ge -s$,

$$\log(1+x) \ge x - K_{\mathrm{loc}}(s) x^2. \tag{40}$$

For $x \ge 0$, differentiate $\log(1+x) - x + x^2/2$ to obtain $x^2/(1+x) \ge 0$. For $-s \le x \le 0$, the function

$$g_s(x) = \log(1+x) - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4(1-s)}$$

has derivative $x^3(x+s)/\{(1+x)(1-s)\} \le 0$ and $g_s(0) = 0$. It follows that $g_s(x) \ge 0$. The inequalities $x^3 \ge -sx^2$ and $x^4 \le s^2 x^2$ then give (40). Summing over the eigenvalues of E gives $\log\det S \ge t - K_{\mathrm{loc}}(s) r^2$.

Let x_i be those eigenvalues and p_i the diagonal entries of Π_C in the same orthonormal eigenbasis. Then $0 \le p_i \le 1$, $\sum_i p_i = 3$, $\sum_i p_i x_i = c$, and $\sum_i p_i x_i^2 = v_C$. The exact identity

$$\frac{1}{1+x} = 1 - x + \frac{x^2}{1+x} \le 1 - x + \frac{x^2}{1-s}$$

therefore proves $\iota_C \le 3 - c + v_C/(1-s)$. Apply $\log y \le y - 1$ to the positive number $y = \iota_C/3$ and use $t + 2c/3 = \ell$ to obtain (37). Multiplication by $1 - s > 0$, followed by $q = r^2/2 + 2v_C/3$, gives (38). Alternatively, $(1-s)K_{\mathrm{loc}}(s) = 1/2 - s/6 - s^2/12$ gives (39). No upper spectral endpoint is used. □

Let $k_* = 24/25$. The contact bound (18) permits $s = k_* \ell$ whenever $k_* \ell < 1$. For a rational $m \ge 2151/1000$, (19) implies

$$v_C \le m\ell^2 - \frac{3}{5}\,\frac{73}{112}\,\ell\eta.$$

Combining this with the first bound for q in (18) and (38) shows that, if $\eta \ge \eta_0$, then

$$(1 - k_*\ell)\Phi_C \ge \ell\, R_{m,\eta_0}(\ell), \tag{41}$$

where the following is a cubic polynomial in X:

$$R_{m,\eta_0}(X) = 1 - k_*X - X\Big\{\frac{51}{16}L_q(k_*X) + mL_C(k_*X)\Big\} + \frac{73}{112}\eta_0\Big\{L_q(k_*X) + \frac{3}{5}L_C(k_*X)\Big\}.$$

The replacement of η by η_0 is valid because its coefficient is nonnegative. The two finite scalar inputs are positive Bernstein expansions of this polynomial for the following parameters:

interval $[0, L]$	η_0	m	consequence
$[0, 1/4]$	0	$9/4$	$\ell > 1/4$
$[0, 32/125]$	$1/30$	$2151/1000$	$\ell > 32/125$ when $\eta \ge 1/30$

For both rows $k_*L < 1$. Their polynomial positivity makes the right side of (41) nonnegative, including $\ell = 0$ without division by ℓ, contradicting $\Phi_C < 0$. The first row is unconditional because $\eta \ge 0$. The second row, together with the separate exclusion of $\eta \le 1/30$, removes all of $\ell \le 32/125$.

It remains to exclude $r \le 43/100$. The case $r = 0$ gives $E = 0$ and $\Phi_C = 0$, so assume $r > 0$. The first row of the table allows us to assume $\ell > 1/4$. From (19) and $\ell, \eta \ge 0$ we have

$$v_C \le \frac{9}{4}\ell^2, \qquad v_C \le r^2, \qquad E \succeq -\min\{r, k_*\ell\}I.$$

The middle inequality follows from $0 \le p_i \le 1$; the spectral bound by r follows from $|x_i|^2 \le \sum_j x_j^2 = r^2$. Define two quadratics in their second argument:

$$\mathcal{N}_1(r, x) = -\frac{r^2}{2} + \Big(1 + \frac{k_*r^2}{6}\Big)x + \Big(-k_* - \frac{3}{2} + \frac{k_*^2r^2}{12}\Big)x^2,$$
$$\mathcal{N}_2(r, x) = -\frac{7r^2}{6} + \Big(1 + \frac{k_*r^2}{6}\Big)x + \Big(-k_* + \frac{k_*^2r^2}{12}\Big)x^2.$$

Using (39), the three choices below bound $(1 - s)\Phi_C$ from below. Each has $0 \le s < 1$.

range of ℓ	s	d	lower bound
$[1/4, 2r/3]$	$k_*\ell$	$9\ell^2/4$	$\mathcal{N}_1(r, \ell)$
$[2r/3, 25r/24]$	$k_*\ell$	r^2	$\mathcal{N}_2(r, \ell)$
$[25r/24, \infty)$	r	r^2	$\mathcal{N}_2(r, 25r/24)$

For the last row, the right side of (39) is affine increasing in ℓ, with slope $1 - r > 0$, and equals the displayed value at $\ell = 25r/24$. Since $r \le 43/100 < 1$, both quadratics are concave in x. Their values on the first two intervals are therefore at least the smaller of the endpoint values. The common endpoint satisfies $\mathcal{N}_1(r, 2r/3) = \mathcal{N}_2(r, 2r/3)$.

Only three univariate polynomial certificates are needed. Define

$$R_{\mathrm{quarter}}(X) = \frac{1}{4} - \frac{k_* + \frac{3}{2}}{16} + \Big(-\frac{1}{2} + \frac{k_*}{24} + \frac{k_*^2}{192}\Big)X^2,$$
$$R_{\mathrm{middle}}(X) = \frac{2}{3} - \Big(\frac{4k_*}{9} + \frac{7}{6}\Big)X + \frac{k_*}{9}X^2 + \frac{k_*^2}{27}X^3,$$
$$R_{\mathrm{right}}(X) = \frac{25}{24} - \Big(\frac{25}{24} + \frac{7}{6}\Big)X + \frac{1}{6}X^2 + \frac{1}{12}X^3.$$

Direct expansion gives the polynomial identities

$$R_{\mathrm{quarter}}(r) = \mathcal{N}_1\Big(r, \frac{1}{4}\Big),$$
$$rR_{\mathrm{middle}}(r) = \mathcal{N}_1\Big(r, 2\frac{r}{3}\Big) = \mathcal{N}_2\Big(r, 2\frac{r}{3}\Big),$$
$$rR_{\mathrm{right}}(r) = \mathcal{N}_2\Big(r, 25\frac{r}{24}\Big).$$

In the third identity, $\mathcal{N}_2(r, 25r/24)$ is also the right side of (39) with $s = r$, $d = r^2$ and $\ell = 25r/24$, because $k_* \cdot 25/24 = 1$. The finite inputs are positive Bernstein expansions of these three polynomials on $[0, 43/100]$. These imply nonnegative endpoint values, so each row in the preceding table gives $\Phi_C \geq 0$, the desired contradiction. Empty intervals need no check, and shared endpoints belong to both applicable estimates. Thus the whole closed radius region is excluded.

For these six reductions, the scalar data consist of (31), one degree-seven compensation certificate, one bivariate large-a certificate, the two instances of the same ℓ cubic, and the three radius polynomials. These are finite coefficient and sign checks. Their conversion into statements about all actual matrices is exactly the trace and flag argument above; no additional continuous exclusion premise is required. The separate η exclusion remains necessary both for the full improved ℓ region and for the final containing domain.

12 The small-η strip: joint moments

We now give the analytic implications of the finite certificates for $0 \leq \eta \leq 1/30$. They use the quarter and large-a reductions, which are independent of this strip. Hence a remaining counterexample satisfies

$$b, \tau \geq 0, \qquad b + \tau \leq \frac{1}{30}, \qquad \frac{9}{16} - \frac{9}{4}b - \frac{15}{4}\tau < a < \frac{15}{16}. \tag{42}$$

Retain $H = A + (b + 3\tau)I$, $u_H = \operatorname{tr} H$, $e_H = e_2(H)$ and $p_H = \det H$ from the linked-gain argument. Define

$$L_H = \frac{2}{3}b + 3\tau, \qquad d_H = \frac{7}{5}\tau, \qquad v_H = \sqrt{u_H^2 - 3e_H},$$
$$\alpha = 1 - b - 3\tau, \qquad \beta = 1 - b, \qquad \gamma = 1 + \frac{b}{3}.$$

Thus $u_H = a + 3b + 9\tau$, $I + A = \alpha I + H$ and $C_* := \gamma I - H/3 \preceq I + C$. We keep v_H distinct from the projected moments v_A, v_B, v_C and from the radius r.
Here are the finite data behind the joint moments. Using H_A from (29), for $i < j$ with complementary index k let

$$t_{ij} = |A_{ij}|, \qquad u_{ij} = \frac{\operatorname{tr} H_A}{2} - (H_A)_{kk} - t_{ij}, \qquad \zeta_i = \frac{3}{2}(\tau + F_{ii}).$$

All nine coordinates are nonnegative. The diagonal entries of H are $h_i = \zeta_i + \sum_{j \neq i}(t_{ij} + u_{ij})$, where the pair subscripts are unordered, and $H_{ij} = \varepsilon_{ij} t_{ij}$ for suitable $\varepsilon_{ij} \in \{-1, 1\}$. The already proved root and pair contacts give

$$\sum_i \zeta_i = 6\tau, \qquad \zeta_i \leq 3\tau, \qquad h_i \geq 3\tau, \qquad h_i + h_j - 2t_{ij} \geq 6\tau. \tag{43}$$

For the last two inequalities use $A_{ii} \geq -b/3$ and $A_{ii} + A_{jj} \geq 2|A_{ij}|$. These follow from the pure cross contacts and the pair contacts, respectively. Thus the nine coordinates lie in an explicitly specified rational polyhedral cone.
For each of the eight choices of the three signs, the finite ray certificate supplies matrices V_i, R_j such that every point of this cone has a decomposition

$$H = \sum_i \mu_i V_i + \sum_j \nu_j R_j, \qquad \mu_i, \nu_j \geq 0, \qquad \sum_i \mu_i = \tau. \tag{44}$$

Coverage has a finite certificate, as follows. Start with the coordinate rays of the nonnegative orthant and impose the linear inequalities in (43) one at a time. For a cut $f(x) \geq 0$, retain each ray r_i with $f(r_i) \geq 0$ and include every crossing ray $f(r_i)r_j - f(r_j)r_i$ with $f(r_i) > 0 > f(r_j)$. A shorter new list is allowed only with explicit nonnegative recombinations of all these retained and crossing rays.
To prove one cut preserves coverage, write $x = \sum_i w_i r_i$, $w_i \geq 0$, and let $A_* = \sum_{f(r_i)>0} w_i f(r_i)$ and $B_* = -\sum_{f(r_j)<0} w_j f(r_j)$. The cut says $0 \leq B_* \leq A_*$. If $A_* = 0$, every negative-ray weight is zero. Otherwise the exact identity

$$x = \left(1 - \frac{B_*}{A_*}\right) \sum_{f(r_i)>0} w_i r_i + \sum_{f(r_i)=0} w_i r_i + \sum_{\substack{f(r_i)>0 \\ f(r_j)<0}} \frac{w_i w_j}{A_*} \{f(r_i)r_j - f(r_j)r_i\}$$

expresses x as a nonnegative combination of the new rays. Induction proves the entire cone is covered. The fixed ray identities, signs and pruning recombinations therefore certify coverage. Both the vertex list and the direction list are required to be nonempty. The additional checks on these finite lists are

$$\tau(V_i) = 1, \quad \tau(R_j) = 0, \quad \operatorname{tr} V_i \leq 18, \quad \operatorname{tr} R_j = 2, \quad V_i - \frac{7}{5} I \succ 0, \quad R_j \succeq 0.$$

Here τ on a ray means $(\zeta_1 + \zeta_2 + \zeta_3)/6$ in its nine-coordinate representation. Positivity of the fixed rational matrices is supplied by finite weighted-square factorizations. Equation (44) immediately proves $H \succeq d_H I$, including $\tau = 0$.

Only four further rational inequalities per pair of a vertex and a ray are needed. Set

$$\mathcal{N}_{ij} = V_i + \frac{18 - \operatorname{tr} V_i}{2} R_j, \qquad \mathcal{M}_{ij} = \mathcal{N}_{ij} - \frac{7}{5} I.$$

The checks are

$$\begin{aligned} e_2(\mathcal{N}_{ij}) \geq \frac{153}{2}, \quad \operatorname{tr}\mathcal{N}_{ij}\ \operatorname{tr} R_j - \operatorname{tr}(\mathcal{N}_{ij} R_j) \geq 12, \\ \det \mathcal{M}_{ij} \geq \frac{17}{125}, \qquad\qquad \operatorname{tr}(\operatorname{adj}(\mathcal{M}_{ij}) R_j) \geq \frac{23}{25}. \end{aligned} \tag{45}$$

They imply, whenever $u_H \geq 18\tau$,

$$e_H \geq 6\tau u_H - \frac{63}{2}\tau^2, \qquad p_H \geq \frac{7}{5}\tau e_H - \frac{3}{2}\tau^2 u_H - \frac{27}{5}\tau^3. \tag{46}$$

We give the transfer argument explicitly. First let $\tau = 1$ and recall $u_H \geq 18$. Normalize the ray coefficients in (44) to probabilities q_j and leave the vertex probabilities μ_i unchanged. If their total ray mass is zero, choose any probability vector q_j. Taking traces shows that

$$H = \sum_{i,j} \mu_i q_j \Big(V_i + \frac{u_H - \operatorname{tr} V_i}{2} R_j \Big).$$

Indeed $\sum_i \mu_i (u_H - \operatorname{tr} V_i)/2 = \sum_j \nu_j$; this also proves the identity when the latter sum vanishes. All matrices in this convex combination have trace u_H. Write $z = u_H - 18 \geq 0$. For each endpoint,

$$e_2\Big(\mathcal{N}_{ij} + \frac{zR_j}{2}\Big) = e_2(\mathcal{N}_{ij}) + \frac{z}{2}\{\operatorname{tr}\mathcal{N}_{ij}\ \operatorname{tr} R_j - \operatorname{tr}(\mathcal{N}_{ij} R_j)\} + \frac{z^2}{4} e_2(R_j) \geq \frac{153}{2} + 6z.$$

At fixed trace u_H, $e_2(X) = (u_H^2 - \|X\|_F^2)/2$ is concave, because the squared norm is convex. The bound therefore survives this convex combination. Also

$$\begin{aligned} \det\Big(\mathcal{M}_{ij} + \frac{zR_j}{2}\Big) = \det \mathcal{M}_{ij} + \frac{z}{2} \operatorname{tr}(\operatorname{adj}(\mathcal{M}_{ij}) R_j) \\ + \frac{z^2}{4} \operatorname{tr}(\operatorname{adj}(R_j) \mathcal{M}_{ij}) + \frac{z^3}{8} \det R_j \geq \frac{17}{125} + \frac{23}{50} z. \end{aligned}$$

The last two terms are nonnegative: $\mathcal{M}_{ij} \succ 0$, $R_j \succeq 0$, and the adjugate of a positive semidefinite ternary matrix is positive semidefinite. The logarithmic determinant averaging argument in the proof of Lemma 8 transfers this common positive lower bound to $H - 7I/5$. Thus

$$e_2(H) \geq 6u_H - \frac{63}{2}, \qquad \det\Big(H - \frac{7I}{5}\Big) \geq \frac{23}{50} u_H - \frac{1018}{125}.$$

For $\tau > 0$, apply these statements to H/τ and multiply by τ^2 and τ^3, respectively. Expanding $\det(H - d_H I) = p_H - d_H e_H + d_H^2 u_H - d_H^3$ proves (46). For $\tau = 0$, both inequalities follow from $H \succeq 0$ without division.

There is also a useful bound from the diagonal alone:

$$e_H \geq \frac{3}{2} L_H (u_H - 2L_H). \tag{47}$$

To see this, the root cone for H gives $|H_{jk}| \leq u_H/2 - H_{ii}$, while the pure cross contacts give $H_{ii} \geq L_H$. Let $x_i = H_{ii} - L_H$ and $c_* = u_H/2 - L_H$. Then $0 \leq x_i \leq c_*$ and $\sum x_i = u_H - 3L_H$. For three numbers in $[0, c_*]$ with sum A_*,

$$\sum x_i^2 \leq c_*^2 + (A_* - c_*)^2.$$

For $c_* > 0$ this follows by expanding $\prod_i (c_* - x_i) + \prod_i x_i \geq 0$, which gives $\sum_{i<j} x_i x_j \geq c_*(A_* - c_*)$; the case $c_* = 0$ is immediate. Consequently $\sum H_{ii}^2 \leq u_H^2/2 - u_H L_H + 2L_H^2$. Together with the three off-diagonal bounds, this gives $\operatorname{tr} H^2 \leq u_H^2 - 3u_H L_H + 6L_H^2$, proving (47).

The domain (42) implies $u_H \geq 21\tau$ and $16u_H \geq 95L_H$ by substitution. Equations (46) and (47) therefore imply

$$0 \leq v_H \leq u_H - \frac{169\tau}{20}, \qquad v_H \leq u_H - \frac{7L_H}{4}. \tag{48}$$

For precision, the differences between the squared right sides and the available upper bounds for v_H^2 are, respectively,

$$\frac{\tau}{400}\{440(u_H - 21\tau) + \tau\}, \qquad \frac{L_H}{16}(16u_H - 95L_H),$$

which are nonnegative; the right sides in (48) are themselves nonnegative. If $\eta \leq 1/40$ and $a \geq 21/40$, then $u_H \geq 10L_H$, and the same calculation, with difference $L_H(u_H - 10L_H)/2$, gives $v_H \leq u_H - 2L_H$.

We shall retain the same cubic determinant throughout the strip. For the three eigenvalues x, y, z of H, elementary expansion gives

$$4v_H^2 - (3x - u_H)^2 = 3(y - z)^2,$$

and hence $x \leq (u_H + 2v_H)/3$. A second expansion gives

$$(u_H - v_H)^2(u_H + 2v_H) - 27xyz = \{2v_H - (3x - u_H)\}(3x - u_H + v_H)^2 \geq 0.$$

Thus, with $p_+ = (u_H - v_H)^2(u_H + 2v_H)/27$,

$$p_H \leq p_+ \leq \frac{u_H^3}{27}. \tag{49}$$

The last inequality uses $0 \leq v_H \leq u_H$ and $u_H^3 - 27p_+ = v_H^2(3u_H - 2v_H) \geq 0$. The lower endpoint p_0 will be either

$$p_0^{\mathrm{basic}} = d_H e_H - d_H^2 u_H + d_H^3, \qquad p_0^{\mathrm{joint}} = \frac{7}{5}\tau e_H - \frac{3}{2}\tau^2 u_H - \frac{27}{5}\tau^3.$$

The first follows from $\det(H - d_H I) \geq 0$ and the second from (46). Both refer to the same p_H.

13 A single coupling allowance and the full block kernel

Let $\omega = \|U_{AB}\|_F^2 + \|U_{AC}\|_F^2 + \|U_{BC}\|_F^2$. The additional contact certificate is the finite polynomial identity

$$(a+b+\tau)\{a^2 - r_A - 6\omega + \Psi(a,b,\tau)\} = \sum_{i,j,k} c_{ijk} f_i f_j f_k, \qquad c_{ijk} \geq 0,$$

where the f_i are the contact slacks and

$$\Psi(a,b,\tau) = \frac{176080}{27951}ab + \frac{5061213}{745360}a\tau + \frac{1680066}{46585}b^2 + \frac{40332203}{745360}b\tau + \frac{7927677}{149072}\tau^2.$$

Coefficient comparison gives $\Psi \leq 7a(b+\tau) + 37b^2 + 55b\tau + 54\tau^2$. Since $a > 0$ on (42), division by $a+b+\tau$ is legitimate. Substituting

$$e_H = \frac{a^2 - r_A}{2} + 2a(b+3\tau) + 3(b+3\tau)^2$$

then proves the common allowance

$$0 \leq \omega \leq w_* := \frac{e_H}{3} + \frac{ab}{2} - \frac{5a\tau}{6} + \frac{31b^2}{6} + \frac{19b\tau}{6}. \tag{50}$$

In particular, $w_* \geq 0$ at every actual metric.
Let K_{AB} be the leading 6×6 block of S. For $1/4 \leq \theta \leq 1$, the positive moment matrix (29), evaluated at $(x, y/(2\theta), 0)$ and multiplied by θ, gives

$$0 \leq \theta H_A[x] + \frac{\operatorname{tr} H_A}{8\theta}\|y\|^2 + \frac{3}{4\theta}\operatorname{off}(B)[y] + 2x^{\mathsf{T}} U_{AB} y.$$

Here $\operatorname{tr} H_A = a + 3\eta$ and $H = H_A + \operatorname{diag}(\zeta_i)$ with $\zeta_i \geq 0$. The pair contacts of B imply

$$\operatorname{diag}(B)[y] + c\,\operatorname{off}(B)[y] \geq -b\|y\|^2 \qquad (|c| \leq 2).$$

Indeed, sum the three nonnegative two-coordinate forms $(B_{ii} + B_{jj})(y_i^2 + y_j^2) + 2cB_{ij}y_iy_j$; $|cB_{ij}| \leq B_{ii} + B_{jj}$ makes each nonnegative. Since $|1 - 3/(4\theta)| \leq 2$, subtracting the positive moment above from $K_{AB}[x,y]$ and using $H \succeq d_H I$ proves

$$K_{AB} \succeq \min\Big\{\alpha + (1-\theta)d_H,\ 1 - b - \frac{a+3\eta}{8\theta}\Big\} I. \tag{51}$$

Define the following scalar expressions, all evaluated at the same (a, b, τ, v_H):

$$\begin{aligned}
d_A &= \alpha^3 + \alpha^2 u_H + \alpha e_H, & n_A &= 3\alpha^2 + 2\alpha u_H + e_H,\\
c_H &= \gamma^3 - \frac{\gamma^2 u_H}{3} + \frac{\gamma e_H}{9}, & n_C &= 3\gamma^2 - \frac{2\gamma u_H}{3} + \frac{e_H}{9},\\
n_H &= 4\gamma^2 - \gamma u_H + \frac{2e_H}{27}, & \ell_C &= \gamma - \frac{u_H + 2v_H}{9},\\
B_{\det} &= (1-b)(1+2b),
\end{aligned}$$

and

$$\mathcal{P}(\pi) = \frac{B_{\det}}{\gamma^2}(d_A + \pi)\{(\gamma^2 + 3\tau^2)\Big(c_H - \frac{\pi}{27}\Big) + \tau\gamma^2 n_H\}.$$

The variable π in $\mathcal{P}$ is a scalar placeholder for p_H. The domain gives $\alpha \geq 9/10$, $\beta \geq 29/30$, $\gamma \geq 1$ and $u_H/3 < \gamma$. Hence $C_* \succ 0$. The spectral bound just proved gives $C_* \succeq \ell_C I$, and the linked gain and the B determinant bound give

$$\mathsf{P} := \det(I + A)\det(I + B)\det(I + C) \geq \mathcal{P}(p_H). \tag{52}$$

The identities

$$\det(I + A) = d_A + p_H, \qquad \operatorname{tr}(I + A)^{-1} = \frac{n_A}{d_A + p_H},$$
$$\det C_* = c_H - \frac{p_H}{27}, \qquad \operatorname{tr} C_*^{-1} = \frac{n_C}{c_H - \frac{p_H}{27}}$$

follow by expanding the ternary determinant and its three principal minors. They are valid at repeated eigenvalues.
Choose $h_K > 0$ below both arms in (51) and suppose

$$\ell_C > 0, \qquad J := h_K\ell_C - w_* > 0, \qquad h_K\ell_C \leq \alpha\beta. \tag{53}$$

Then two Schur steps give

$$g := \det S \geq \mathsf{P}\,\frac{J}{h_K\ell_C}, \qquad \iota_C \leq \operatorname{tr}(I + C)^{-1} + \frac{w_*}{\ell_C J}. \tag{54}$$

Here is a proof of both bounds. For a positive block matrix with diagonal blocks at least a_*I, b_*I and squared coupling norm q_*, its Schur complement is $B - N$, where $N \succeq 0$ and $\operatorname{tr} N \leq q_*/a_*$. The eigenvalues of $B^{-1/2}NB^{-1/2}$ have sum at most $q_*/(a_*b_*) < 1$. The inequality $\prod(1 - x_i) \geq 1 - \sum x_i$ proves a determinant factor at least $1 - q_*/(a_*b_*)$. Apply this first with $q_* = \|U_{AB}\|_F^2 =: v_*$ and $a_*b_* = \alpha\beta$, then with $q_* = \|U_{AC}\|_F^2 + \|U_{BC}\|_F^2 =: z_*$ and $a_*b_* = h_K\ell_C$. The identity

$$\left(1 - \frac{v_*}{k_1}\right)\left(1 - \frac{z_*}{k_2}\right) - \left\{1 - \frac{v_* + z_*}{k_2}\right\} = \frac{v_*(k_1 - k_2 + z_*)}{k_1k_2} \geq 0$$

for $k_1 = \alpha\beta \geq k_2 = h_K\ell_C$ proves the first bound using $v_* + z_* \leq w_*$. For the inverse bound, the same whitening and $\sum x_i/(1 - x_i) \leq (\sum x_i)/(1 - \sum x_i)$ give

$$\operatorname{tr}(B - N)^{-1} - \operatorname{tr} B^{-1} \leq \frac{t_*}{\ell_C(\ell_C - t_*)}, \qquad t_* := \operatorname{tr} N \leq \frac{w_*}{h_K}.$$

The right side increases with $t_* < \ell_C$, yielding the second bound. Thus both losses use the same w_*.
The finite product-derivative guards will include

$$n_H > 0, \qquad 27c_H - d_A - \frac{2u_H^3}{27} > 0, \qquad 0 \leq p_0 < \frac{u_H^3}{27}. \tag{55}$$

For $\pi \in [p_0, p_+]$, differentiation of $\mathcal{P}$ gives

$$\mathcal{P}'(\pi) = \frac{B_{\det}}{27\gamma^2}\{(\gamma^2+3\tau^2)(27c_H - d_A - 2\pi) + 27\tau\gamma^2 n_H\} > 0.$$

Consequently (52) and (54) remain valid after replacing $\mathcal{P}(p_H)$ by $\mathcal{P}(p_0)$. An auxiliary chart point with $p_0 > p_+$ cannot represent an actual metric and requires no interpolation interval.

14 Weighted inverse loss and one scalar endpoint

We record the inverse estimate which makes the weighted flag effective. Let $m_\sharp = 113/50$, so the spectral bounds already proved give $I + A, I + B \preceq m_\sharp I$. Suppose $K_{AB} \preceq L_w \operatorname{diag}(I, 4I)$ with $L_w \geq 1$. Then

$$\frac{\iota_A}{3} + \frac{4\iota_C}{9} \leq 4 - \frac{\operatorname{tr}(I+A)^{-1} + 4\operatorname{tr}(I+B)^{-1}}{9} - \frac{\left(\frac{\mathsf{P}}{g}\right)^{1/3} - 1}{3L_w}. \tag{56}$$

To prove it, let $\varrho_1, \varrho_2 \geq 1$ be the cube roots of the determinant losses in the two successive Schur steps:

$$\varrho_1^3 = \frac{\det(I+A)\det(I+B)}{\det K_{AB}}, \qquad \varrho_2^3 = \frac{\det K_{AB}\det(I+C)}{g}.$$

Fischer's inequality [11], itself the positive Schur comparison, gives $\varrho_1, \varrho_2 \geq 1$ and $\varrho_1\varrho_2 = (\mathsf{P}/g)^{1/3}$. The positive inverse increments at the first step are $D_A = (K_{AB}^{-1})_{AA} - (I+A)^{-1}$ and $D_B = (K_{AB}^{-1})_{BB} - (I+B)^{-1}$. To check the trace identity, let $Y = (K_{AB}^{-1})_{BB}$. The block inverse formula gives $Y^{-1} = I + B - U_{AB}^{\mathsf{T}}(I+A)^{-1}U_{AB}$ and $D_A = (I+A)^{-1}U_{AB}YU_{AB}^{\mathsf{T}}(I+A)^{-1}$. Taking cyclic traces yields

$$\operatorname{tr}((I+A)D_A) = \operatorname{tr}((I+B)D_B) = \operatorname{tr}((I+B)Y) - 3 \geq 3(\varrho_1 - 1).$$

For the inequality, the positive matrix $(I+B)^{1/2}Y(I+B)^{1/2}$ has determinant ϱ_1^3 by the Schur determinant formula. AM–GM makes its trace at least $3\varrho_1$, proving the displayed lower bound. At the second step, $D_K = (S^{-1})_{AB,AB} - K_{AB}^{-1} \succeq 0$ similarly satisfies $\operatorname{tr}(K_{AB}D_K) \geq 3(\varrho_2 - 1)$. The matrix ceilings therefore give

$$\operatorname{tr} D_A + 4\operatorname{tr} D_B \geq 3\,\frac{5}{m_\sharp}(\varrho_1 - 1), \qquad \operatorname{tr}(\operatorname{diag}(I, 4I)D_K) \geq \frac{3}{L_w}(\varrho_2 - 1).$$

AM–GM and marking give $g \geq 1$, while AM–GM on the nine eigenvalues of the three diagonal blocks gives $\mathsf{P} \leq (13/12)^9$ because $\operatorname{tr} S \leq 39/4$. Hence $\varrho_1\varrho_2 \leq 2197/1728 < 5/m_\sharp \leq 5L_w/m_\sharp$. The identity

$$3\,\frac{5}{m_\sharp}(\varrho_1 - 1) + \frac{3}{L_w}(\varrho_2 - 1) - \frac{3}{L_w}(\varrho_1\varrho_2 - 1) = 3(\varrho_1 - 1)\Big(\frac{5}{m_\sharp} - \frac{\varrho_2}{L_w}\Big) \geq 0$$

shows that the total weighted inverse increment is at least $3\{(\mathsf{P}/g)^{1/3}-1\}/L_w$. Finally marking gives $\iota_A/3+4\iota_C/9 \le 4-(\iota_A+4\iota_B)/9$; substitution proves (56).
The required weighted ceiling is rational. Set

$$A_{\max}=\alpha+\frac{u_H+2v_H}{3},\qquad D_w=4A_{\max}-1-2b,$$
$$L_n=A_{\max}D_w+w_*,\qquad L_w=\frac{L_n}{D_w}.$$

When $D_w>0$ and $A_{\max}\ge 1$, Young's inequality gives

$$2x^{\mathsf{T}}U_{AB}y\le \frac{w_*}{D_w}\|x\|^2+D_w\|y\|^2.$$

Combine this with $I+A\preceq A_{\max}I$ and $I+B\preceq(1+2b)I$ to obtain $K_{AB}\preceq L_w\,\mathrm{diag}(I,4I)$ and $L_w\ge 1$.
For any positive lower product $\mathsf{P}_0\le\mathsf{P}$ and inverse lower bounds $i_A\le\mathrm{tr}(I+A)^{-1}$, $i_B\le\mathrm{tr}(I+B)^{-1}$, define $W_0=4-(i_A+4i_B)/9$. Since also $g\le\mathsf{P}$, (56) implies

$$\frac{\iota_A}{3}+\frac{4\iota_C}{9}\le W_0-\frac{\max\left\{1,\left(\frac{\mathsf{P}_0}{g}\right)^{1/3}\right\}-1}{3L_w}. \tag{57}$$

The maximum is necessary if $\mathsf{P}_0<g$. For fixed P_0,W_0,L_w, the gap between $\sqrt{g}$ and $27/58$ times the right side is nondecreasing for $g\ge 1$. Indeed, apart from constants it is the maximum of $\sqrt{g}+9/(58L_w)$ and $\sqrt{g}+q_*g^{-1/3}$, where

$$0<q_*:=\frac{9}{58L_w}\mathsf{P}_0^{1/3}\le\frac{9}{58}\left(\frac{13}{12}\right)^3<1.$$

The first function increases; the derivative of the second is $g^{-1/2}/2-q_*g^{-4/3}/3\ge g^{-1/2}/6>0$. It therefore suffices to check this gap at any lower determinant endpoint $g_0\ge 1$.
In particular take $g_0=\mathsf{P}_0J/(h_K\ell_C)$ and $x_*=w_*/(h_K\ell_C)\in[0,1)$. The polynomial identity

$$1-(1-x_*)\Big(1+\frac{x_*}{3}\Big)^3=\frac{x_*^2(x_*^2+8x_*+18)}{27}\ge 0$$

proves $(1-x_*)^{-1/3}\ge 1+x_*/3$. Thus a sufficient endpoint check is

$$g_0\ge\Big(\frac{27}{58}S_{\mathrm{lin}}\Big)^2,\qquad S_{\mathrm{lin}}=W_0-\frac{w_*}{9L_wh_K\ell_C}. \tag{58}$$

This remains sufficient if $S_{\mathrm{lin}}<0$, since $\sqrt{g_0}\ge|27S_{\mathrm{lin}}/58|\ge 27S_{\mathrm{lin}}/58$. The monotonic gap then proves (15) at the actual metric.
For five of the charts below, both P_0 and i_A retain the same variable $\pi\in[p_0,p_+]$:

$$\mathsf{P}_0=\mathcal{P}(\pi),\qquad i_A=\frac{n_A}{d_A+\pi},\qquad i_B=\frac{9}{3+b},\qquad g_0(\pi)=\mathcal{P}(\pi)\frac{J}{h_K\ell_C}.$$

A single derivative comparison justifies using p_0 in both places. Let

$$R_{27} = (\gamma^2 + 3\tau^2)(27c_H - d_A - 2p_+) + 27\tau\gamma^2 n_H.$$

Then

$$g_0'(\pi) \geq \frac{B_{\det} J R_{27}}{27\gamma^2 h_K \ell_C}, \qquad S_{\mathrm{lin}}'(\pi) = \frac{n_A}{9(d_A + \pi)^2} \geq 0.$$

For every π in the whole interval, $S_{\mathrm{lin}}(\pi) \leq 5/2$. Indeed,

$$(3 + a)n_A - 9(d_A + p_+) = \frac{2}{3} v_H^2 (3 + a - v_H) \geq 0,$$

using $3+a = 3\alpha+u_H$ and $v_H \leq u_H$. Hence $S_{\mathrm{lin}}(\pi) \leq 4-1/(3+a)-4/(3+b) \leq 5/2$: it is enough to use $3+a \leq 33/8, 3+b \leq 91/30$ and $8/33+120/91 > 3/2$. Consequently the finite guard

$$3364 B_{\det} J R_{27} (d_A + p_0)^2 - 10935 n_A \gamma^2 h_K \ell_C \geq 0 \tag{59}$$

makes $g_0(\pi) - (27S_{\mathrm{lin}}(\pi)/58)^2$ nondecreasing. This follows by differentiating, replacing S_{lin} by $5/2$ and $d_A + \pi$ by its positive lower bound $d_A + p_0$, then clearing positive denominators. Checking (58) at p_0 therefore establishes it at the actual p_H.

15 Seven polynomial charts and their finite checks

For the charts define

$$a_0 = \frac{36}{67} - \frac{9}{4}b - \frac{15}{4}\tau, \qquad a_t = \frac{441}{800} - \frac{9}{4}b - \frac{15}{4}\tau,$$
$$a_1 = \frac{9}{8} - \frac{3}{2}\eta, \qquad s_a = 21\tau - 3b, \qquad b_*(\eta) = \frac{33}{34}\eta - \frac{24}{1139}.$$

Table 1 specifies the ranges and the variance ceiling V_*. Here $u_H = a + 3b + 9\tau$ and $L_H = 2b/3 + 3\tau$ as before.

Table 1. The seven polynomial charts of the strip $\eta \leq 1/30$: the ranges of η, b, a and the variance ceiling V_*.

chart	η	b	a	V_*
$\mathcal{L}$	$[0, 1/40]$	$[0, \eta]$	$[a_0, 21/40]$	$u_H - 7L_H/4$
$\mathcal{H}$	$[0, 1/40]$	$[0, \eta]$	$[21/40, a_1]$	$u_H - 2L_H$
$\mathcal{J}_L$	$[1/40, 3/100]$	$[0, b_*]$	$[a_0, s_a]$	$u_H - 8\tau$
$\mathcal{J}_H$	$[1/40, 3/100]$	$[0, b_*]$	$[s_a, a_1]$	$u_H - 7b/12 - 53\tau/8$
$\mathcal{J}_B$	$[1/40, 3/100]$	$[b_*, \eta]$	$[a_0, a_1]$	$u_H - 7b/12 - 53\tau/8$
$\mathcal{T}_L$	$[3/100, 1/30]$	$[0, 96\eta/131]$	$[a_t, 15/16]$	$u_H - 169\tau/20$
$\mathcal{T}_H$	$[3/100, 1/30]$	$[96\eta/131, \eta]$	$[a_t, 15/16]$	$u_H - 7L_H/4$

Every row is parametrized by a closed unit four-cube: interpolate η by y, then b by z, then a by u, and set $v_H = V_* v$, where $u, y, z, v \in [0, 1]$. Always set $\tau = \eta - b$ and $e_H = (u_H^2 - v_H^2)/3$. These substitutions are polynomial.
These cubes cover every actual point in (42). That domain gives $a > a_0, a_t$ and $a < a_1$. If $\eta \le 1/40$, split at $a = 21/40$ and use the corresponding variance bounds. If $1/40 \le \eta \le 3/100$, then $0 < b_* < \eta$ and $s_a - a_0 = (51/2)(b_* - b)$. Split first at $b = b_*$ and, below it, at $a = s_a$. The bound $v_H \le u_H - 8\tau$ follows from (48); averaging it with $v_H \le u_H - 7L_H/4$ gives the ceiling in $\mathcal{J}_H, \mathcal{J}_B$. For $3/100 \le \eta \le 1/30$, split at $b = 96\eta/131$ and use the two bounds in (48). In any interval containing an actual coordinate, affine interpolation has a parameter in $[0, 1]$; if its width is zero, use parameter zero. This covers degenerate intervals and the boundaries between charts. The choices of θ, h_K, p_0 are those of Table 2.

Table 2. The choices of θ (the kernel parameter of (51)), h_K and p_0 on each chart.

chart	θ	h_K	p_0
$\mathcal{L}$	$4/5$	$1 - (25a + 235b + 83\tau)/160$	p_0^{basic}
$\mathcal{H}$	1	$1 - a/8 - (11b + 3\tau)/8$	p_0^{basic}
$\mathcal{J}_L$	1	α	p_0^{joint}
$\mathcal{J}_H, \mathcal{J}_B$	1	$1 - a/8 - (11b + 3\tau)/8$	p_0^{joint}
$\mathcal{T}_L, \mathcal{T}_H$	$3/4$	$1599/1600 - a/6 - 3b/2 - \tau/2$	p_0^{joint}

The two kernel-arm comparisons in (51) are checked as polynomials. For $\mathcal{L}$ the second margin is $\tau/20$. For $\mathcal{H}$ its second margin is zero, and only $\alpha - h_K \ge 0$ is needed. For $\mathcal{J}_L$ the comparison is $s_a - a \ge 0$, while for $\mathcal{J}_H, \mathcal{J}_B$ it is $a - s_a \ge 0$. For the trace charts the second margin is $1/1600$, and the first is

$$\alpha + \frac{7\tau}{20} - h_K = \frac{a - a_t}{6} + \frac{111}{40}\left(\frac{1}{30} - \eta\right) + \frac{29}{10}b \ge 0.$$

All these comparisons concern the full kernel.
Here are the main polynomial tests. Write

$$q_0 = d_A + p_0, \qquad q_C = c_H - \frac{p_+}{27}, \qquad b_+ = 3 + b,$$
$$\mathcal{K} = q_0\Big\{(\gamma^2 + 3\tau^2)\Big(c_H - \frac{p_0}{27}\Big) + \tau\gamma^2 n_H\Big\}B_{\text{det}}.$$

On $\mathcal{L}$, inverse order, $C_* \preceq I + C$, and (54) give

$$\iota_C \le \frac{n_C}{q_C} + \frac{w_*}{\ell_C J}, \qquad g \ge \frac{\mathcal{K}J}{\gamma^2 h_K \ell_C}.$$

The strictly positive polynomial

$$9\mathcal{K}q_C^2\ell_C J^3 - \gamma^2 h_K(n_C\ell_C J + w_* q_C)^2 > 0 \tag{60}$$

therefore proves $9g > \iota_C^2$, and hence the first flag succeeds. The denominators q_C, ℓ_C, J, h_K are checked positive.

For every other chart let $q_A = d_A + p_+$ on $\mathcal{H}$, and $q_A = q_0$ on the five joint or trace charts. Set $W_n = 36q_A(2+b) - n_A b_+$. The common strictly positive main polynomial is

$$3364\mathcal{K}JL_n^2 h_K \ell_C q_A^2 b_+^2 - 9\gamma^2 (L_n h_K \ell_C W_n - w_* D_w q_A b_+)^2 > 0. \tag{61}$$

Clearing positive denominators makes this exactly (58) at $g_0 = \mathcal{K}J/(\gamma^2 h_K \ell_C)$ and $W_0 = W_n/(9q_A b_+)$. On $\mathcal{H}$, n_A/q_A is a valid inverse lower bound because $p_H \le p_+$. Together with the independent product lower endpoint p_0 this suffices directly. On the other five charts, also check (59); the common-cubic argument then transfers the same test to p_H.

We specify all side checks to make the computational boundary explicit. Every check below means its polynomial after the four-cube substitution, with positive Bernstein coefficients unless a weak exception is listed. On every chart check

$$J, \quad h_K, \quad \ell_C, \quad \alpha\beta - h_K \ell_C, \quad n_H, \quad 27c_H - d_A - \frac{2u_H^3}{27}, \quad \nu_0, \quad \frac{u_H^3}{27} - p_0,$$

where $p_0 = \tau\nu_0$ and

$$\nu_0 = \begin{cases} \dfrac{7e_H}{5} - \dfrac{49\tau u_H}{25} + \dfrac{343\tau^2}{125}, & \mathcal{L}, \mathcal{H}, \\ \dfrac{7e_H}{5} - \dfrac{3\tau u_H}{2} - \dfrac{27\tau^2}{5}, & \text{all other charts.} \end{cases}$$

Check $16u_H - 95L_H$ except on $\mathcal{H}$, where it is replaced by $u_H - 10L_H$. Check $u_H - 18\tau$ on the three joint charts and $u_H - 21\tau$ and V_* on both trace charts. The kernel checks are the explicit margins described above: both nontrivial arms on $\mathcal{L}$ and the trace charts, the first arm on $\mathcal{H}$, and the signed comparison $s_a - a$ or $a - s_a$ on the joint charts. On $\mathcal{L}$ check q_C. On each other chart check

$$q_A, \quad D_w, \quad L_n, \quad A_{\max} - 1, \quad \mathcal{K}J - \gamma^2 h_K \ell_C,$$

so in particular $g_0 > 1$ as required by the monotone-gap argument. Finally the five joint and trace charts require the polynomial in (59) to be positive.

The following checks permit zero Bernstein coefficients: ν_0 on $\mathcal{L}, \mathcal{H}$; the second kernel margin on $\mathcal{L}$; the first kernel margin and $u_H - 10L_H$ on $\mathcal{H}$; the signed kernel comparison on all three joint charts; and the first kernel margin on both trace charts. This distinction includes the zero-τ and sector-boundary cases. All other listed side checks and both main comparisons are strict. Every required denominator is consequently positive at an actual point.

A strictly positive polynomial on a four-cube may be certified by a positive tensor Bernstein expansion, or by an expansion in the first three variables whose univariate coefficient polynomials are certified positive on $[0,1]$. The basis functions in the first three variables are nonnegative and sum to one. Thus the second format needs only univariate sign certificates, whether Bernstein expansions or exact Sturm checks. Every coefficient identity must be checked; degree elevation is permitted.

For $\mathcal{J}_H$, 18 exceptional univariate rows use saved Sturm sequences. Their original integer coefficient lists have been checked in Lean: the positive derivative scaling, signed remainder identities, nonzero terminal constant and endpoint signs prove strict positivity on $[0, 1]$. Their binomial factors and positive row normalizations are checked as well. One mixed-row theorem accepts these Sturm certificates alongside Bernstein certificates. No replacement witnesses are needed. The coefficient identities assembling the complete tensor expansions, and the remaining ray, moment, contact-polynomial and seven-chart payloads, remain explicit finite verification obligations.
Conditional on precisely these finite checks, every actual point of $\eta \le 1/30$ lies in one of the seven charts and makes one of the two flags nonnegative. This proves the strip exclusion, supplies the complementary part of the improved ℓ reduction, and justifies removing the local regions before the final spectral grid is formed.

16 Why a finite spectral grid contains every real metric

Let x_i be the eigenvalues of E, and p_{ki} the diagonal entries of Π_k in an eigenbasis. Define measures $\mu_k = \sum_i p_{ki}\delta_{x_i}$ and $\mu = \sum_k \mu_k$. Each μ_k has mass three; μ has nine unit atoms, counted with multiplicity. Consequently

$$\int (s - x)_+ \, d\mu \ge (s - \lambda_-)_+, \qquad \int (x - s)_+ \, d\mu \ge (\lambda_+ - s)_+.$$

For a parameter box choose 81 rational equally spaced nodes $g_0, \dots, g_{80}$ covering its spectral interval, of spacing h_g. Split every atom barycentrically between its two adjacent nodes, separately for each block weight. The resulting measures $\widehat{\mu}_k$ preserve block mass and first moment exactly. If the current box gives $\lambda_- \le u_-$ and $u_+ \le \lambda_+$, their unit atoms also imply

$$\widehat{\mu}((-\infty, u_- + h_g]) \ge 1, \qquad \widehat{\mu}([u_+ - h_g, \infty)) \ge 1.$$

Indeed every node receiving mass from an atom lies within one spacing of that atom. These two endpoint-mass rows include equality at the cutoffs.
If $f'' \le K_f$ on a cell, its chord minus f is at most $\max(K_f, 0)h_g^2/8$: subtract $K_f x^2/2$ and use concavity. Thus a row $\sum_k q_k \int x^2 d\mu_k \le B$ remains necessary with right side increased by $(3h_g^2/4)\sum_k (q_k)_+$. The inverse-trace allowance is $9h_g^2/[4(1 + g_0)^3]$. Positive logarithm and negative inverse terms are concave and need no allowance. Hinges at grid nodes interpolate exactly. Moreover

$$\int \log(1 + x) \, d\widehat{\mu} \le \log\det S, \qquad \int (1 + x)^{-1} d\widehat{\mu}_k \ge \iota_k.$$

Both grid flag gaps therefore bound their actual counterparts below.
Pointwise matrix constraints need support allowances too. An atom reaching g_j originates between $g_j^- := \max(g_0, g_j - h_g)$ and $g_j^+ := \min(g_{80}, g_j + h_g)$. Let $m_j = \sum_k \widehat{\mu}_{kj}$. The block spectral bounds imply

$$(g_j^- - g_0)\widehat{\mu}_{kj} \le (\beta_k - g_0)m_j, \qquad (g_{80} - g_j^+)\widehat{\mu}_{kj} \le (g_{80} - \alpha_k)m_j.$$

To prove the first, use $E - g_0 I \succeq (x_i - g_0) e_i e_i^{\mathsf{T}}$, compress to the block, then weaken x_i to the lower support endpoint before summing interpolation weights. The second is identical at the upper endpoint. Upper block bounds β_k may be rounded upward and lower bounds α_k downward: the compressed squared norms are nonnegative.
For (20), write $s_k(E)$ for the actual shifts and choose box upper bounds $\bar{s}_k \geq s_k(E)$. Then

$$E + \sum_k \bar{s}_k \Pi_k = E + \sum_k s_k(E)\Pi_k + \sum_k (\bar{s}_k - s_k(E))\Pi_k \succeq 0.$$

Thus $0 \preceq g_{80} I - E \preceq g_{80} I + \sum_k \bar{s}_k \Pi_k$. Retain the row only when all $g_{80} + \bar{s}_k > 0$; no denominator involving a smaller actual shift is used. The rank-one comparison gives the further necessary rows

$$\sum_k \frac{g_j^+ + \bar{s}_k}{g_{80} + \bar{s}_k} \widehat{\mu}_{kj} \geq 0.$$

The coefficients increase with the original eigenvalue, which justifies the upper support endpoint.
Projection variances retain more information. For one projection Π with diagonal p_i, let $r_\Pi = \operatorname{tr}(\Pi E \Pi E)$. The identities

$$\operatorname{tr}(\Pi E^2) - r_\Pi = \sum_{i<j} (x_i - x_j)^2 \Pi_{ij}^2, \qquad \sum_{j \neq i} \Pi_{ij}^2 = p_i(1 - p_i)$$

imply, for $s(x) = \frac{1}{2}\max((x - g_0)^2, (g_{80} - x)^2)$, $r_\Pi \geq \sum_i [p_i x_i^2 - s(x_i) p_i (1 - p_i)]$. Set $s_j = s(g_j)$. After interpolation the necessary grid version is

$$\sum_j \Big[(g_j^2 - s_j)\widehat{\mu}_{kj} + s_j \frac{\widehat{\mu}_{kj}^2}{m_j} \Big] \leq r_k + \frac{3h_g^2}{4}.$$

The allowance follows because subtracting $p_i x^2$ leaves a concave function. Merging contributions decreases the convex perspective term, by Cauchy–Schwarz. Zero total mass contributes zero. For a rational free slope π_j, replace the perspective by $s_j(2\pi_j \widehat{\mu}_{kj} - \pi_j^2 m_j)$; the difference is the nonnegative square $s_j m_j (\widehat{\mu}_{kj}/m_j - \pi_j)^2$.
Finally let $B_F = 5\tau^2/3 + 3k_0^2$. Since $C_c = F_c - A_c/3$, completed squares give, for any $\sigma > 0$ and $0 \leq \theta < 1/9$,

$$r_C \leq \left(\frac{1}{9} + \sigma\right) r_A + \frac{c^2}{3} + \left(1 + \frac{1}{9\sigma}\right) B_F - \left(\frac{1}{9} + \sigma\right) \frac{a^2}{3},$$

$$r_C \geq \theta r_A + \frac{c^2}{3} - \frac{\theta a^2}{3} - \frac{9\theta B_F}{1 - 9\theta}.$$

Both inequalities follow directly from the identities

$$\sigma \Big[\left(\frac{1}{9} + \sigma\right) \|A_c\|_F^2 + \left(1 + \frac{1}{9\sigma}\right) \|F_c\|_F^2 - \|C_c\|_F^2 \Big] = \|\sigma A_c + \frac{F_c}{3}\|_F^2 \geq 0,$$

$$(1 - 9\theta) \Big[\|C_c\|_F^2 - \theta \|A_c\|_F^2 + \frac{9\theta}{1 - 9\theta} \|F_c\|_F^2 \Big] = \|F_c - (1 - 9\theta)\frac{A_c}{3}\|_F^2 \geq 0,$$

using $\|A_c\|_F^2 = r_A - a^2/3$, $\|C_c\|_F^2 = r_C - c^2/3$, and $\|F_c\|_F^2 \leq B_F$. All remaining parameter quadratics are bounded by (26).

17 Exact separation and exhaustion

For $w, w_0 > 0$, convexity gives $-2\log w \geq 2 - 2\log w_0 - 2w/w_0$. Apply finitely many such tangents to the two grid flags. A counterexample has both actual flags negative, so it must satisfy every resulting linear upper bound by zero. There are 246 nonnegative variables: 243 grid masses and three block-square variables $0 \leq r_k \leq 6$. The latter bound follows from

$$0 \leq r_k \leq \operatorname{tr}(\Pi_k E^2) \leq \|E\|_F^2 \leq 6t^2 < 6\left(\frac{11}{15}\right)^2 < 6,$$

using $0 \leq t < 11/15$ on the remaining region. Each mass block sums to three. If the resulting rational rows are $Az \leq b$, then for any rational $y \leq 0$, with $d = q - A^{\mathsf{T}}y$, every feasible point satisfies

$$q^{\mathsf{T}}z \geq y^{\mathsf{T}}b + 3\sum_{k=A,B,C} \min_j d_{kj} + 6\sum_{k=A,B,C} \min(0, d_{r_k}). \tag{62}$$

Multiply the rows by y and bound each block by its fixed mass to prove this formula. A positive right side for $q = 0$ is an infeasibility certificate. Other objectives certify contractions of a, b, τ. To spell out the endpoint contractions, at a grid node s let

$$H_-(s) = \int (s - x)_+ \, d\widehat{\mu}, \qquad H_+(s) = \int (x - s)_+ \, d\widehat{\mu}.$$

A certified lower bound $d \leq -H_-(s)$ gives $s + d \leq \lambda_-$, since $s - \lambda_- \leq (s - \lambda_-)_+ \leq H_-(s) \leq -d$. Similarly $d \leq -H_+(s)$ gives $\lambda_+ \leq s - d$. The lower endpoint is rounded downward, and the upper endpoint upward, before intersection. These implications include repeated eigenvalues and equality at grid nodes.

Every node of the tree is of one of three kinds: a *terminal exclusion*, where a certificate contradicts (62) on the node's region; a *split*, with two children whose regions cover the node's, possibly preceded by a contraction of the region; or a *contraction alone*, with one child, whose region is the contracted one. The completed tree has 400 nodes: 155 terminal exclusions, 154 splits and $400 - 155 - 154 = 91$ contractions alone; the 234 contractions are these 91 and the 143 that precede a split. All 1 091 rational certificates were replayed, independently in FLINT and in standard-library `Fraction` arithmetic, with an empty final frontier. The replay shares the exact row assembler with the constructor, so that assembler's mathematical justification is essential; the interpolation and contact argument above supplies it. Every counterexample in (17) produces feasible grid masses, survives every valid contraction, and reaches a terminal contradiction to (62). With the certified local regions (16), this proves `0008`.

18 What is verified, and where

Proof of Theorem 13. Let $L \supset M \supset N$ be a chain as in Definition 2 whose rank-9 member N has a Gram matrix Q admissibly marked by $P =$ `0008:022143116`, and

suppose $\Delta = \det L < 972$. The chain is admissible for P (Section 2.1), and in the aligned coordinates of Section 3 its section metric $S = I+E$ satisfies $S \succ 0$, $\operatorname{tr} S^{-1} \leq 9$ and $e = 864 \det S < e_{\mathrm{cut}} < 1137$, that is, $\det S < 379/288$; every contact vector of P gives $E[v] \geq 0$ (Definition 10). By Section 4, both flags (14) and (15) are negative and the parameters lie in the root (17). Sections 10 and 11 exclude $t \geq 11/15$, $3t + 4\delta \geq 637/288$, $9\tau + (45/2)\delta \geq 8a + 5b$, $a \geq 15/16$, $r \leq 43/100$ and, for $\eta \geq 1/30$, $\ell \leq 32/125$; Sections 12 to 15 exclude $\eta \leq 1/30$, hence all of (16). Section 16 maps the remaining spectrum of E, with its block weights, into feasible grid masses satisfying every necessary row, and Section 17 and Computational result 14 show that every such point survives every valid contraction and reaches a terminal contradiction to (62). Hence no such chain has $\Delta < 972$. □

The finite checks. Computational result 14 is one statement, but it names the following finite inputs, each a matter of exact rational arithmetic in the formats of Section 2.2:

1. a the aligned basis of the form, the three integral covectors, the twelve integral triangles and their incidence and length identities, and the ray identities that identify the pure, linked and mixed contacts (Sections 3 and 4);
2. b the four cubic identities $P_1, \dots, P_4$ with nonnegative weights $w_{ijk}^{(\nu)}$ and the matrix identity $E + (24/25)\ell I = \sum_i f_i M_i$ with $M_i \succeq 0$, on the complete 45-dimensional contact cone (Section 5);
3. c the 126 sign checks $c_{vpq} \geq 0$ and the two polynomial identities (21) of the linked moment certificate (Section 6);
4. d the five rational comparisons (31), the degree-seven compensation certificate Q_{tr} (Bernstein on $[1, 23/10]$), the bivariate tensor Bernstein certificate for P_a on $[0,1]^2$, the two instances of the cubic R_{m,η_0} and the three radius polynomials $R_{\mathrm{quarter}}, R_{\mathrm{middle}}, R_{\mathrm{right}}$ on $[0, 43/100]$ (Sections 10 and 11);
5. e for each of the eight sign patterns, the ray certificate (44) with its coverage recombinations and the checks $\tau(V_i) = 1$, $\tau(R_j) = 0$, $\operatorname{tr} V_i \leq 18$, $\operatorname{tr} R_j = 2$, $V_i - \frac{7}{5} I \succ 0$, $R_j \succeq 0$, and the four rational inequalities (45) per pair (Section 12);
6. f the coupling-allowance identity with nonnegative c_{ijk} and the coefficient comparison for Ψ (Section 13);
7. g the seven charts of Tables 1 and 2: the main polynomials (60) and (61), the guard (59), and the side checks listed in Section 15, as positive or nonnegative tensor Bernstein expansions on the unit four-cube, or expansions in three variables with univariate Bernstein or Sturm certificates; 18 exceptional rows of $\mathcal{J}_H$ use saved Sturm sequences (Section 15);
8. h the final tree on the root (17): 400 nodes, 155 terminal exclusions, 154 splits, 91 contractions alone (234 contractions in all), 1 091 rational certificates each satisfying (62) for its stated exclusion or outward contraction, with the row assembler of Sections 16 and 17 and an empty final frontier.

Where the records are. The entry point of the proof is the record `0008_complete_2026-10-23/proof_status.json`, which joins the local regions to the final spectral cover; the verification record of the case reads "local reductions, real-to-grid transfer and separation; all seven strip tensors checked against the physical polynomials; final spectral cover replayed in two exact arithmetic implementations". For the

seven strip charts, exact reconstruction checks $1\,415\,718$ main coefficients and $184\,683$ domain coefficients, including coefficients that must vanish, and the power polynomials are compared with independently assembled expressions for the physical quantities of Section 15. A Lean companion checks the printed Gram matrix, its positive definiteness and determinant (saved LDL^{T} decomposition), the eighteen exceptional Sturm polynomials of Section 15 with their scaling identities, and the seven small polynomial certificates of this case, together with the rational trace endpoint. These are `Hermite11.case0008VerifiedRadius`, `Hermite11.case0008VerifiedQuarter`, `Hermite11.case0008VerifiedEll`, the high-a certificate `Hermite11.case0008VerifiedHighA`, and `Hermite11.nineTraceVerifiedCompensation` with `Hermite11.case0008VerifiedTraceEndpoint`. The contact geometry, full η-strip tensors and final tree use the external exact checks. Sections 16 and 17 specify the shared row assembler used by the tree's constructor and checker. The records and their checkers are in the certificate archive `hermite11_local_certificates.zip` of the Zenodo record doi:10.5281/zenodo.23979316. Its `case_map.json` identifies each principal record, its size, the underlying checker sources and recorded result; `certificate_manifest.json` binds the stored files and compressed logical streams. The common archive identifiers and verification procedure are given in [5, Section 9].

What is imported, precisely. Theorem 13 uses the following inputs.

1. The admissible-chain conditions of Definition 2, including the marking and the exact cutoffs.
2. The statements of Section 2.2 (Lemmata 4, 5, 6, 7, 8, 9, 11, 12), whose proofs are given in [6].
3. The reference data of the form: the Gram matrix of Appendix A, its minimum, its 90 contact pairs and the perfection of the form.
4. Computational result 14.

Lemmata 15 to 18 and the derivations of Sections 5 to 17 are proved here, conditional on the finite checks they name.

Lean correspondence. The local analytic statements are `0008`: `Hermite11.case0008_of_computational_facts`. Their finite hypotheses, integral-equivalence wrappers and saved reference records are listed in `case_map.json`. Large cover payloads remain external exact checks.

19 Use of computational tools

Refer to Section 10 of [5].

References

1. Basu, S., Pollack, R., Roy, M.-F.: Algorithms in Real Algebraic Geometry, 2nd edn. Algorithms and Computation in Mathematics, vol. 10. Springer, Berlin (2006). https://doi.org/10.1007/3-540-33099-2
2. Conway, J.H., Sloane, N.J.A.: Sphere Packings, Lattices and Groups, 3rd edn. Springer, New York (1999). https://doi.org/10.1007/978-1-4757-6568-7
3. Coxeter, H.S.M., Todd, J.A.: An extreme duodenary form. Canad. J. Math. **5**, 384–392 (1953). https://doi.org/10.4153/CJM-1953-043-4
4. Dutour Sikirić, M., van Woerden, W.: The lattice packing problem in dimension 9 by Voronoi's algorithm. arXiv:2508.20719 (August 2025; v2, 2026). Data: https://doi.org/10.5281/zenodo.15707640
5. Géraud-Stewart, R., Naccache, D.: The Hermite constants in dimensions 10, 11 and 12. Posted concurrently to this work (2026)
6. Géraud-Stewart, R., Naccache, D.: A focus on Hermite constant 11. Posted concurrently to this work (2026)
7. Géraud-Stewart, R., Naccache, D.: γ_{11}: contact geometry. Posted concurrently to this work (2026)
8. Géraud-Stewart, R., Naccache, D.: γ_{11}: reduction by sections. Posted concurrently to this work (2026)
9. Géraud-Stewart, R., Naccache, D.: γ_{11}: one selected contact, two linked blocks. Posted concurrently to this work (2026)
10. Géraud-Stewart, R., Naccache, D.: γ_{11}: unequal triangles and adapted references. Posted concurrently to this work (2026)
11. Horn, R.A., Johnson, C.R.: Matrix Analysis, 2nd edn. Cambridge University Press, Cambridge (2013). https://doi.org/10.1017/CBO9781139020411
12. Martinet, J.: Perfect Lattices in Euclidean Spaces. Grundlehren der mathematischen Wissenschaften, vol. 327. Springer, Berlin (2003). https://doi.org/10.1007/978-3-662-05167-2
13. Nebe, G., Sloane, N.J.A.: A catalogue of lattices. Entries KAPPA9.2, KAPPA11, K12, http://www.math.rwth-aachen.de/~Gabriele.Nebe/LATTICES/
14. Schrijver, A.: Theory of Linear and Integer Programming. Wiley, Chichester (1986)
15. Voronoï, G.: Nouvelles applications des paramètres continus à la théorie des formes quadratiques. Premier mémoire. Sur quelques propriétés des formes quadratiques positives parfaites. J. Reine Angew. Math. **133**, 97–178 (1908). https://doi.org/10.1515/crll.1908.133.97

A The form `0008:022143116`

The following integral matrix A_P is a representative of the indexed rank-9 form `0008:022143116` of Dutour Sikirić and van Woerden [4], in the basis used by the case certificates, with $P = (4/s_P)A_P$, $s_P = 4$, $\det P = 864$ and 90 contact pairs. This matrix is the record of the data set itself, entry by entry and in the data set's basis: the script `hermite11\_printed\_forms\_check.py` of the Zenodo record [5, Section 9] compares the printed forms of the series with the rows of their keys in the certificate database (which the verification matches to the data set) and reports `PASS` for all eleven, so the chain handed over by [6, Theorem 1], whose rank-9 member is marked by the record, is marked by P. The reference data of the form are that A_P is positive definite, has minimum s_P, that its minimum-vector pairs number 90, that their tensors span the 45-dimensional space of symmetric matrices, and that the normalized determinant is $\det P$; the finite verification is exact elimination (determinant and positive pivots), the bounded LDL^{T} enumeration of all vectors at or below the proposed minimum, and exact linear elimination for the contact-tensor rank.

$$A_P = \begin{pmatrix} 4 & 2 & 2 & -2 & 2 & 2 & -2 & 2 & 2 \\ 2 & 4 & 2 & -2 & 0 & 0 & 0 & 0 & 2 \\ 2 & 2 & 4 & 0 & 2 & 2 & 0 & 0 & 2 \\ -2 & -2 & 0 & 4 & 0 & 0 & 2 & -2 & -1 \\ 2 & 0 & 2 & 0 & 4 & 2 & -2 & 1 & 2 \\ 2 & 0 & 2 & 0 & 2 & 4 & -1 & 2 & 2 \\ -2 & 0 & 0 & 2 & -2 & -1 & 4 & -2 & -1 \\ 2 & 0 & 0 & -2 & 1 & 2 & -2 & 4 & 1 \\ 2 & 2 & 2 & -1 & 2 & 2 & -1 & 1 & 4 \end{pmatrix}.$$

The matrix is symmetric; a direct computation in exact integer arithmetic (Bareiss elimination, then LDL^{T} and Fincke–Pohst enumeration) gives $\det A_P = 864$, positive pivots, $\min A_P = 4$ and exactly 180 vectors of squared length 4, that is, 90 contact pairs, in agreement with the data of [5, Table 4]: the lattice of rank 10 over this form, a section of K_{11} with 276 minimal vectors, has $\det M = 972$ and $u = \min M^* = 8/9$, and the determinant of its rank-9 section by a shortest dual vector is $ud = (8/9)\cdot 972 = 864 = \det P$ [5, Fact 4]. This is the reference determinant of Section 3, where $e = 864 \det S$, that is, $\det S = e/\det P$ in (4).