γ_{11}: Contact Geometry

THE MACADAMIAS 0001, 0014, 0059 AND 0015

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Abstract. The root paper of this series reduces the value $\gamma_{11} = 4/\sqrt[11]{972}$ of the Hermite constant to a statement about chains of lattices $L \supset M \supset N$ of ranks 11, 10 and 9 whose rank-9 member lies, in the sense of Voronoi, over one of 4 487 813 perfect forms; the γ_{11} paper excludes all of them but ten, the *macadamias*. This paper settles four macadamias, the forms `0001:010488327`, `0014:001749549`, `0059:016902802` and `0015:030291998`, by one method: the geometry of the contacts of the form. A section frame of short dual vectors forces the first gluing of M over N to lie near one of finitely many coherent centres; at such a centre the rank-10 metric is compared with an explicit reference lattice, whose Delaunay cells and contact identities give a uniform bound on its covering radius; and a primitive-extension inequality turns that bound into $\det L \geq 972$. The proof is computer-assisted: every finite check it uses is stated as a Computational result with the records of the certificate archive that certify it, and all comparisons are between explicit rational numbers.

Keywords: Hermite constant · Lattice packings · Perfect forms · Covering radius · Delaunay cells · Contact cone · Exact certificates

1 Introduction

This paper is one of a series of seven papers, all made public concurrently, whose aim is to establish the Hermite constants γ_{10}, γ_{11} and γ_{12}. The root paper of the series [3] recalls the problem, proves γ_{10} in full, derives γ_{12} from γ_{11} and reduces γ_{11} to a statement about perfect forms of rank 9: a hypothetical lattice of rank 11 with minimum 4 and determinant below 972 would contain a chain of sections $L \supset M \supset N$ of ranks 11, 10 and 9, and the section N of rank 9 would be marked, in the sense of Voronoi, by one of 4 487 813 explicitly known perfect forms, the *survivors*. The γ_{11} paper [4] excludes every survivor but ten by exact certificates and assembles the value $\gamma_{11} = 4/\sqrt[11]{972}$ from the root paper, the global certificates and ten local arguments. Five companion papers settle nine of these cases; `0052` is settled in [4].
The present paper settles four macadamias, the forms

`0001:010488327`, `0014:001749549`, `0059:016902802`, `0015:030291998`

of the classification of Dutour Sikirić and van Woerden [2] (macadamias 6, 7, 8 and 5 of [3, Table 4]). The four are settled by one idea, the *geometry of the contacts* of the form. Each form carries a *section frame*: twelve (or seventeen) short vectors z_i of the dual of the form, with positive weights q_i of sum 12, whose weighted outer products sum to the form itself. The marking of N by the form forces one of these dual vectors to be short in N^*, and this alone gives $u = \min M^* \le 1$ and the lower bound $\det L \ge 8 \det N/9$; since $8 \det N/9 \ge 972$ once $\det N \ge 2187/2$, only the band $\det N < 2187/2$ remains. On that band the frame does more: the phases of the gluing of M over N at the frame vectors are forced to lie near half-integers, and a coherence lemma turns these twelve near-half-integer phases into a single displacement from one of finitely many *coherent centres* (two for `0001`, one for `0014`, two for `0059`, three for `0015`). At such a centre the rank-10 lattice M is a small deformation of an explicit *reference lattice*, for which a complete list of Delaunay cells and a finite family of contact identities are certified; a covering bound for the deformed metric follows, and a primitive-extension inequality turns the covering bound into $\det L \ge 972$. For `0014` and `0059` the reference lattice is one and the same code lattice $M_{\mathcal{B}}$ of determinant 1024, and one covering proposition serves both; for `0015` the reference lattice is a section of K_{11} [10] of determinant 972, where the bound is attained and the margin is zero. Figure 1 shows the common argument.

Theorem 1. *Assume Computational results 16, 21, 22, 24, 27, 29 and 31. Let P be one of the four forms `0001:010488327`, `0014:001749549`, `0059:016902802`, `0015:030291998`, scaled to minimum 4 (Appendix A). Every chain $L \supset M \supset N$ as in Definition 2 whose rank-9 member N has a Gram matrix admissibly marked by P satisfies $\Delta = \det L \ge 972$.*

Theorem 1 is the statement that [3, Section 6.2] requires from this paper for each of the four forms; it is restated once per form at the end of the section that proves it (Theorems 23, 26, 28 and 30).

The strategy. The shared analytic arguments localise the gluing (Section 4) and prove the covering bounds (Section 3). Their finite inputs are stated as Computational results, whose data are in the certificate archive and can be replayed. The finite comparisons are between explicit rational numbers: no decimal approximation and no floating-point number enters a proof step.

Organisation. Section 2 states the admissible-chain conditions, the common coordinates and the tools used in the four cases. Section 3 is the tool section of this paper: the primitive-extension inequality, the averaging argument on a fixed Delaunay cell, the code lattice $M_{\mathcal{B}}$ with its contact cone and its uniform covering bound, and the lemmata of phase coherence. Section 4 treats the four forms, one subsection each, in the order `0001`, `0014`, `0059`, `0015`. Section 5 states precisely what is verified by computation, which records certify them, and the hypotheses on which Theorem 1 rests. Appendix A prints the Gram matrices of the four forms.

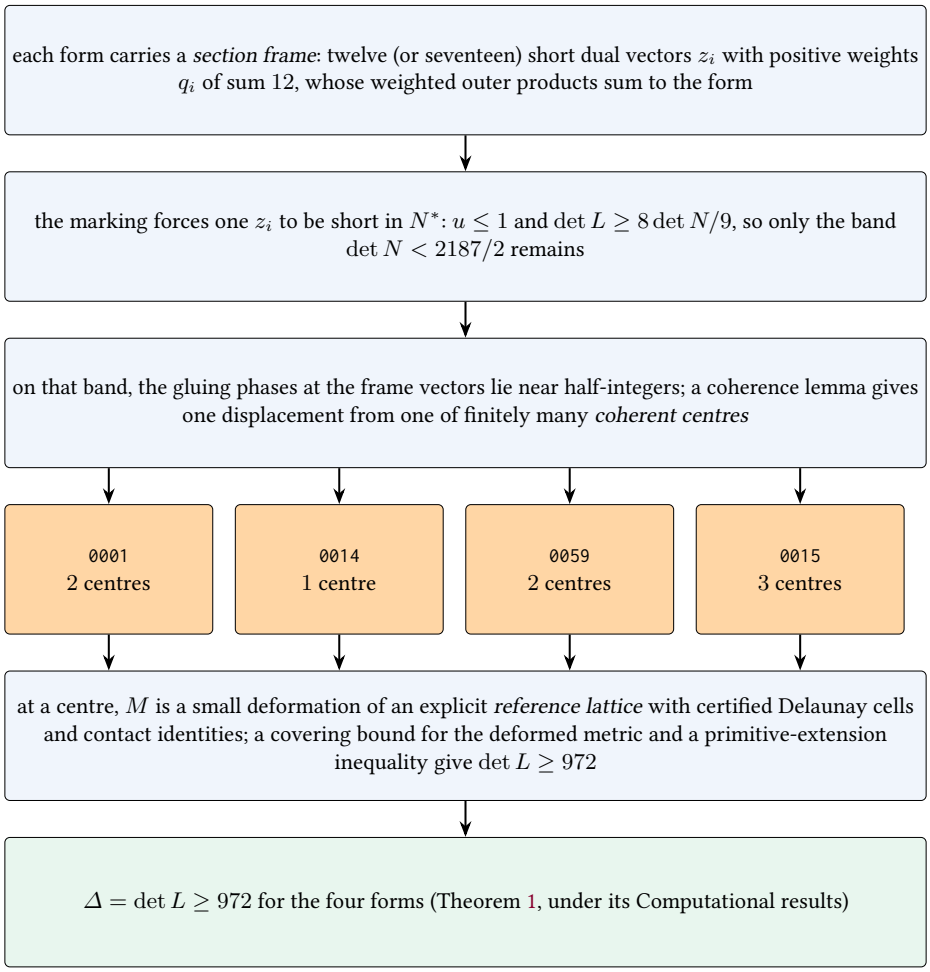


Fig. 1. The common argument of this paper for its four forms. For `0014` and `0059` the reference lattice is one and the same code lattice of determinant 1024; for `0015` it is a section of K_{11} of determinant 972, where the bound is attained and the margin is zero.

2 The local setting and common tools

2.1 Notation and admissible chains

We use the common coordinates of [4, Appendix D]. The determinant of a lattice is the determinant of its Gram matrix, and its minimum is the least nonzero squared length. Its dual is $L^* = \{y \in \operatorname{span} L : y \cdot x \in \mathbb{Z} \text{ for all } x \in L\}$. A sublattice $M \subset L$ is *primitive* if $M = L \cap \operatorname{span} M$; a primitive vector generates a primitive rank-one sublattice. Write $Q[x] = x^{\mathsf{T}} Q x$, with T denoting transpose, and $\operatorname{tr} A$ for the sum of the diagonal entries. For symmetric matrices, $A \succeq 0$ means $A[x] \ge 0$ for every x; $A \succ 0$

means $A[x] > 0$ for $x \neq 0$, and $A \preceq B$ means $B - A \succeq 0$. The nested sections form a flag of ranks 11, 10 and 9; its notation is

$$\begin{gathered} e = \det N, \qquad d = \det M = \frac{e}{u}, \qquad \Delta = \det L = \frac{e}{uv}, \\ u = \min M^*, \qquad v = \min L^*. \end{gathered} \tag{1}$$

The cutoffs are the positive numbers $d_{\mathrm{cut}}^{11} = 4^{11}972^9$ and $e_{\mathrm{cut}}^{55} = 2^{246}3^{197}$.

Definition 2 (admissible residual chain). *A chain $L \supset M \supset N$ is admissible for P if L has rank eleven and minimum at least four, $M = L \cap n_{11}^{\perp}$ and $N = M \cap n_{10}^{\perp}$ for shortest nonzero vectors $n_{11} \in L^*$ and $n_{10} \in M^*$, its rank-nine Gram matrix is marked by P as defined below, and*

$$\begin{gathered} 768 \le d < d_{\mathrm{cut}} = 4\,972^{9/11}, \\ e < e_{\mathrm{cut}} = \Big(\frac{4096}{3}\Big)^{1/10} d_{\mathrm{cut}}^{9/10} < 1137. \end{gathered} \tag{2}$$

Shortest dual vectors are primitive, so the determinant identities above follow by an adapted basis. Rounding a lift of n_{10} across the last cut gives $v \le u + v/4$, hence $v \le 4u/3$ and $\Delta \ge 3d/(4u)$. The global reduction [3, Proposition 15] supplies such a chain from a hypothetical counterexample.

Definition 3 (marking). *Let Q be the Gram matrix of N in an integral basis, and $X = Q^{-1}$. A perfect form P of minimum four marks Q (and Q, or N, is then said to be admissibly marked by P) if $\min Q \ge 4$ and there are weights satisfying*

$$\begin{gathered} X = \sum_{c \in \mathcal{C}(P)} w_c\, cc^{\mathsf{T}}, \qquad w_c \ge 0, \qquad \sum_c w_c \le \frac{9}{4}, \qquad w_c \le \frac{1}{4}, \\ \operatorname{tr}(PX) \le 9, \qquad \det P \le e. \end{gathered} \tag{3}$$

Here $\mathcal{C}(P)$ contains one vector from each opposite pair of minimum vectors of P.

Perfect means that the matrices cc^{T} span the symmetric matrices. A marking is transported by an integral change of basis. Its existence for arbitrary Q is proved in [3, Proposition 7]; here the marker is fixed to the printed form.

Gluing coordinates. Vectors $\beta, \alpha \in \mathbb{R}^9$ and a scalar $\vartheta \in \mathbb{R}$ give the dual squared lengths at $(z, k, k') \in \mathbb{Z}^9 \times \mathbb{Z}^2$ as

$$X[z] + u(\beta^{\mathsf{T}} z + k)^2 + v(\alpha^{\mathsf{T}} z + \vartheta k + k')^2, \tag{4}$$

and the primal squared lengths as

$$Q[x - k\beta - k'(\alpha - \vartheta\beta)] + \frac{(k - \vartheta k')^2}{u} + \frac{k'^2}{v}. \tag{5}$$

Write $\|t\|_{\mathbb{R}/\mathbb{Z}} = \min_{j \in \mathbb{Z}} |t - j|$. The first shortest-normal condition implies

$$X[z] + u\,\|\beta^{\mathsf{T}} z\|_{\mathbb{R}/\mathbb{Z}}^2 \ge u \qquad (0 \neq z \in \mathbb{Z}^9). \tag{6}$$

The second condition applies to every nonzero vector of M^* and every lift to L^*.

Physical coordinates. Choose C_P with $P = C_P^{\mathsf{T}} C_P$ and write

$$Q = C_P^{\mathsf{T}} S C_P, \qquad R = S^{-1}. \tag{7}$$

The columns of C_P realise the reference lattice in Euclidean space. Thus S measures squared lengths in these fixed coordinates, with reference metric I, and R is its inverse: $\det S = e/\det P$, $\operatorname{tr} R = \operatorname{tr}(PX) \le 9$; integral vectors c, z become $C_P c, C_P^{-\mathsf{T}} z$. We write $|x| = \|x\|$ for Euclidean length and $\|E\|_F^2 = \operatorname{tr}(E^{\mathsf{T}} E)$ for the squared Frobenius norm, and $\rho_G^2 = \sup_x \min_{a \in \Lambda} G[x-a]$ for the squared covering radius of the reference lattice Λ under discussion, with $x \in \operatorname{span} \Lambda$. Eigenvalues are listed in decreasing order. The notation $I_n, e_i, \mathbf{1}$ denotes the identity, coordinate vectors and all-one vector; Sym_n denotes the symmetric $n \times n$ matrices. We use $(x)_+ = \max(x, 0)$; subscripts $-$ and $+$ on a box coordinate denote its lower and upper endpoints. Schur complements, cofactors and adjugates have the conventions and block identities of [4, Appendix D.2]. Exact LDL^{T} factorisation and bounded short-vector enumeration are described in [4, Section 5.1]. For the first extension put $h = 1/u$. In the two code-lattice cases we write $\delta_S = \det S$; the displacement of a gluing centre remains the vector δ.

2.2 Tools of the common toolbox

The following statements belong to the common toolbox of the series; their proofs are given in [4], Appendix D, and we only recall the ones that the four cases use.

Lemma 4 (circle triangle; [4, Lemma 22]). *If $t_0 + t_1 + t_2 = 0$ in $\mathbb{R}/\mathbb{Z}$, then $\sum_i \|t_i\|_{\mathbb{R}/\mathbb{Z}} \le 1$. Consequently, if $y_0 + y_1 + y_2 = 0$ are nonzero vectors of M^* of squared lengths b_i, then the squared length v of the last normal satisfies*

$$\sum_{i=0}^{2} \sqrt{\max\left\{0, 1 - \frac{b_i}{v}\right\}} \le 1. \tag{8}$$

In particular $v \le (9/8) \max_i b_i$.

Lemma 5 (one section vector controls both extensions; [4, Lemma 23]). *For every $0 \ne z \in \mathbb{Z}^9$,*

$$\Delta \ge \frac{e}{2X[z]^2}. \tag{9}$$

If three nonzero section covectors sum to zero and all have squared X-length at most x, then

$$\Delta \ge \frac{16e}{27x^2}. \tag{10}$$

Lemma 6 (a D_4 frame; [4, Lemma 25]). *Suppose M^* contains a scaled D_4 root system all of whose roots have squared length at most b. Then $v \le 36b/35$ and $\Delta \ge 35d/(36b)$. In a physical metric G on a reference lattice of determinant d_0, if these roots have reference squared length one, then*

$$\Delta \ge \frac{35}{36}\, d_0 \det G\, \lambda_{\min}(G).$$

Definition 7 (contact cone). *For a reference lattice of minimum four with contact vectors $\mathcal{C}$ in physical coordinates, its* contact cone *consists of the symmetric matrices E satisfying $E[c] \geq 0$ for every $c \in \mathcal{C}$. Every minimum-four metric $I + E$ lies in this cone.*

Positive identities such as

$$K(\operatorname{tr} E)^2 - \operatorname{tr}(E^2) = \sum_{c,c'} a_{cc'} E[c]E[c'], \qquad a_{cc'} \geq 0, \tag{11}$$

or their cubic analogues prove norm bounds for *every* contact-cone matrix. They are checked by comparing polynomial coefficients. An isotropic contact family has nonnegative weights a_c with $\sum_c a_c cc^{\mathsf{T}} = \kappa I$, $\kappa > 0$. Its contact slacks are the values $E[c]$. Their nonnegativity gives $\operatorname{tr} E \geq 0$; for $\varepsilon > 0$ the matrix $E + \varepsilon I$ still has nonnegative contact slacks and has positive trace, and dividing the cubic identity by its positive trace factor and letting $\varepsilon \downarrow 0$ gives the bound for E.

Lemma 8 (marked radial barrier; [4, Lemma 36]). *Let $B = I + E \succ 0$ have size n, E belong to a contact cone, and $\operatorname{tr} B^{-1} \leq n$. Along $B_s = I + sE$, $0 \leq s \leq 1$, both the contact inequalities and marking $\operatorname{tr} B_s^{-1} \leq n$ persist, and $\log\det B_s$ is nondecreasing. If a cone estimate shows $\log\det B_s > \log\rho$ at a prescribed positive trace or norm, while $\det B < \rho$, the actual metric cannot reach that trace or norm. With only the weaker hypothesis $\operatorname{tr} B^{-1} \leq n + \epsilon$, $\epsilon \geq 0$, one still has*

$$\log\det B_s \leq \log\det B + (1 - s)\epsilon. \tag{12}$$

Lemma 9 (a quadratic minorant for the logarithm; [4, Lemma 37]). *Suppose E has size n, $s = \operatorname{tr} E > 0$, $\operatorname{tr} E^2 \leq Ks^2$, and $E \succeq -asI$, with $as < 1$. Let*

$$b = \frac{K + a}{1 + na}, \qquad A = \frac{nb - 1}{a + b}, \qquad B = \frac{na + 1}{a + b}.$$

If $a > 0$ and $K \geq 1/n$, then

$$\log\det(I + E) \geq A\log(1 - as) + B\log(1 + bs). \tag{13}$$

Lemma 10 (a shared trace–inverse logarithm bound; [4, Lemma 38]). *Let $0 < \ell \leq r$, $H > 0$, and let $a_{\log} = 1/(\ell + r)$, $b_{\log} = \ell r/(\ell + r)$. If $q_{\log} \leq \min\{\log\ell - a_{\log}\ell + b_{\log}/\ell,\ \log H - a_{\log}H + b_{\log}/H\}$, then $\log x \geq a_{\log}x - b_{\log}/x + q_{\log}$ for $0 < x \leq H$. In particular,*

$$x - 1 \leq 3\log x + 2\left(\frac{1}{x} - 1\right) \qquad \left(0 < x \leq \frac{14}{5}\right), \tag{14}$$

$$\log(1 + x) \geq \frac{17}{40}x + \frac{23}{40}\,\frac{x}{1 + x} \qquad \left(-1 < x \leq \frac{9}{16}\right). \tag{15}$$

Lemma 11 (automatic marking at small trace; [4, Lemma 39]). *Let $a \geq 0$, $K > 0$, and $B = I + E \succ 0$ have size n. If $s = \operatorname{tr} E \geq 0$, $\operatorname{tr} E^2 \leq Ks^2$, $E \succeq -asI$, and $as < 1$, then*

$$\operatorname{tr} B^{-1} - n \leq -s + \frac{Ks^2}{1 - as}.$$

In particular $s \leq 1/(K + a)$ ensures both $as < 1$ and $\operatorname{tr} B^{-1} \leq n$.

3 Covering bounds from contact geometry

Lemma 12 (a primitive extension; [4, Lemma 41]). *If a lattice L of rank 11 with minimum at least 4 contains a primitive sublattice M of rank 10, then*

$$\Delta \geq d\,\big(4-\rho^2(M)\big). \tag{16}$$

Lemma 13 (averaging in a cell; [4, Lemma 42]). *Let a_i be points, $\pi_i \geq 0$, $\sum_i \pi_i = 1$, and $\sum_i \pi_i a_i = x$. For every positive metric G,*

$$\min_i G[a_i - x] \leq \sum_i \pi_i G[a_i] - G[x].$$

A full-dimensional Euclidean Delaunay cell is the convex hull of lattice points on a sphere whose interior contains no lattice point. Here $\mathrm{conv}\{a_i\}$ means all sums $\sum_i \pi_i a_i$ with $\pi_i \geq 0$ and $\sum_i \pi_i = 1$. These cells tile space [1,9]. We use the *fixed reference tiling* only as a partition, with probability distributions on its vertices. For each deformed metric, Lemma 13 bounds distances using the vertices of this same partition.

Lemma 14 (one mixture bound for deep cells). *Let $K = \mathrm{conv}\{a_i\}$ contain the Euclidean ball of radius R_0, where $R_0 > 2\kappa > 0$. Suppose every $|x| \leq \kappa$ is the mean of a distribution on the a_i with second moment $\sum_i \pi_i a_i a_i^{\mathsf{T}} = \kappa I$. Let $G \succ 0$, $\ell = \lambda_{\min}(G)$, and $G[a_i] \leq \kappa \operatorname{tr} G + B$ for all i, with $B \geq 0$. If $B \leq 4\kappa(R_0 - \kappa)\ell$, then*

$$\min_i G[a_i - x] \leq \kappa \operatorname{tr} G \qquad (x \in K).$$

Proof. For $|x| \leq \kappa$, use the stipulated distribution and Lemma 13. Otherwise let $r = |x|$ and let the ray through x reach the boundary of K at distance $R \geq R_0$. Mix a distribution of mean $\kappa x/r$ and second moment κI with any vertex distribution of mean Rx/r, giving the latter weight $\theta = (r-\kappa)/(R-\kappa)$. The mixture has mean x. Its mean squared G-distance is at most $\kappa \operatorname{tr} G + \theta B - \ell r^2$. Finally

$$r^2 \geq 4\kappa(r-\kappa), \qquad \theta B \leq 4\kappa\ell(r-\kappa),$$

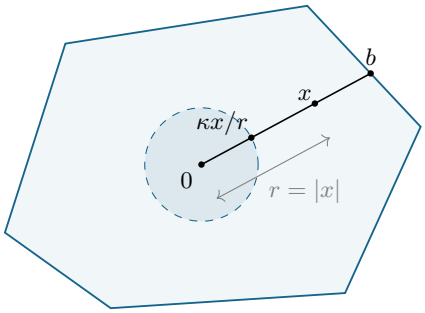
the first inequality being $(r-2\kappa)^2 \geq 0$. □

3.1 The common rank-ten code lattice

A binary linear code is a vector subspace of $\{0,1\}^n$ with addition modulo two. A word's weight is the number of its nonzero coordinates; its support is their index set. The Voronoi cell of a lattice at zero consists of the points at least as close to zero as to any other lattice point: $|x|^2 \leq |x-a|^2$ for every lattice vector a.

Index ten coordinates by the edges of the complete graph on five vertices, in lexicographic order. Let $\mathcal{B} \subset \{0,1\}^{10}$ consist of the edge subsets with even degree at every vertex and an even total number of edges. It is a five-dimensional binary linear code. Define

$$M_{\mathcal{B}} = \{x \in \mathbb{Z}^{10} : x \bmod 2 \in \mathcal{B}\}.$$



Mix two distributions:
central mean $\kappa x/r$,
boundary mean b.
Their mixture has mean x.

Fig. 2. The averaging argument on a fixed reference cell. A planar illustration of the radial mixture used in the ten-dimensional proof.

It has determinant $2^{2(10-5)} = 1024$ and minimum four. Its minimum vectors, up to sign, are the ten vectors $2e_i$ and the 120 sign patterns on the fifteen weight-four codewords. Indeed a nonzero vector of smaller squared length would reduce to a nonzero word of weight at most three, which the defining parity conditions exclude.

Lemma 15 (a finite Voronoi description for a binary code). *Let $\mathcal{C} \subset \{0,1\}^n$ be a binary linear code and $L_{\mathcal{C}} = \{v \in \mathbb{Z}^n : v \bmod 2 \in \mathcal{C}\}$. The part of its Euclidean Voronoi region in the nonnegative orthant is*

$$0 \le x_i \le 1, \qquad \sum_{i:\, c_i=1} x_i \le \frac{1}{2}\sum_i c_i \quad (c \in \mathcal{C}).$$

The full region is obtained by independent coordinate sign changes.

Proof. The Voronoi inequalities for $2e_i$ and for the vectors $c \in \mathcal{C}$ give the stated necessary conditions. Conversely, take $v \in L_{\mathcal{C}}$ and write $|v_i| = 2k_i + c_i$, where k_i is a nonnegative integer and $c \in \mathcal{C}$. Since $0 \le x_i \le 1$,

$$|v_i|x_i - \frac{|v_i|^2}{2} = c_i x_i - \frac{c_i}{2} + 2k_i(x_i - k_i - c_i) \le c_i x_i - \frac{c_i}{2}.$$

Summing gives $v \cdot x \le |v|^2/2$, the Voronoi inequality for the arbitrary vector v. Coordinate sign changes preserve $L_{\mathcal{C}}$ and its Voronoi region. □

Computational result 16 (the code-lattice identities and reference cells). For the explicitly defined lattice $M_{\mathcal{B}}$, the contact polynomial and moment identities in Lemma 17 hold coefficient by coefficient. Exact enumeration of the bounded polytope in Lemma 15 gives 2419 vertices. The archived completeness check verifies all incident edges and agrees with a second exact enumeration. Every vertex has squared norm at most $5/2$, except for 60 vertices of squared norm three. For every listed vertex, the data supply a nonnegative contact identity (20). The corresponding reference Delaunay cells, with all coordinate sign changes and lattice translations, cover space. Every cell is certified as one of two types. A *shallow* cell has $|a|^2 \le 5/2$ at every vertex. A *deep*

cell has $|a|^2 = 3$, an antipodal vertex list, nonnegative weights of mass one and second moment $(3/10)I$, and a finite nonnegative sum-of-squares identity proving

$$\frac{3}{10}|z|^4 - \sum_i w_i (a_i^{\mathsf{T}} z)^4 \geq 0 \qquad \text{as a polynomial in } z.$$

There are six deep-cell classes. Each has 80 errors, namely all eight sign patterns on ten three-element supports. Each coordinate occurs in three supports and each coordinate pair in at most one. The nonnegative part of the polar of each deep cell has 394 vertices, all of squared norm at most two. The archived polar check exhausts 184755 active sets for one class and verifies coordinate permutations carrying it to the other five.

Here is why these records provide the whole geometric input. Lemma 15 accounts for every lattice vector, so no further short-vector search is needed to validate the Voronoi description. A full Voronoi vertex in the nonnegative orthant is also a vertex of the displayed polytope; extra vertices introduced by the coordinate planes cause no difficulty. If c is a Voronoi vertex and v a nearest lattice point, then $c - v$ is another Voronoi vertex. Thus the contact identities on the listed vertices, transported by sign changes, apply to every error of every reference Delaunay cell. The classical Delaunay tiling supplies coverage. Equivalently, its finitely many cells meeting a closed fundamental parallelotope give a rational partition of that parallelotope. Finiteness follows from the bounded circumradii and the discreteness of the lattice. Intersections and a rational triangulation let this partition in the format of the certificates (Section 5). Indeed each new rational vertex has rational convex coordinates in its containing Delaunay cell; interpolating those coordinates on a simplex gives the required affine barycentric functions.

The support counts also make the deep-cell moment check transparent. Give the 80 errors equal weight. Sign cancellation gives mean zero and second moment $(3/10)I$. If $m_{ij} \in \{0, 1\}$ is the number of supports containing both i and j, then

$$\frac{3}{10}|z|^4 - \frac{1}{80}\sum_a (a \cdot z)^4 = \frac{3}{5}\sum_{i<j}(1 - m_{ij}) z_i^2 z_j^2 \geq 0.$$

This supplies the required sum-of-squares identity. For the larger ball, let $K = \operatorname{conv}\{a\}$ for one deep-cell error list. Its polar is $K^\circ = \{y : a \cdot y \leq 1 \text{ for every } a\}$. Independent sign changes reduce its description to $y_i \geq 0$ and $\sum_{i \in S} y_i \leq 1$ for the ten supports S. The checked vertex bound implies $|y|^2 \leq 2$ on this bounded polytope, by convexity, and hence on all of K°. It follows that K contains the ball of radius $1/\sqrt{2}$: otherwise separation from K, followed by normalisation of the separating functional, would give a point of K° whose norm exceeds $\sqrt{2}$. In particular K contains the required closed ball of radius $7/10$, since $49/100 < 1/2$.

The moment identities imply the entire central-mean assertion used below. Write $\kappa = 3/10$. For a unit direction z, the second and fourth moments give $\mathbb{E}(a_i^{\mathsf{T}} z)^2 = \kappa$ and $\mathbb{E}(a_i^{\mathsf{T}} z)^4 \leq \kappa$. Let $X_i = |a_i^{\mathsf{T}} z|$. Two applications of Cauchy–Schwarz give

$$(\mathbb{E}X_i^2)^2 \leq (\mathbb{E}X_i)(\mathbb{E}X_i^3), \qquad (\mathbb{E}X_i^3)^2 \leq (\mathbb{E}X_i^2)(\mathbb{E}X_i^4).$$

Thus $\kappa^3 \le (\mathbb{E}X_i)^2\,\mathbb{E}X_i^4 \le \kappa(\mathbb{E}X_i)^2$, proving $\mathbb{E}|a_i^{\mathsf{T}} z| \ge \kappa$. Choose the sign of each antipodal vertex to make its projection nonnegative. This changes neither its weight nor its outer product, so the resulting mean has projection at least κ and still has second moment κI. The convex hull of these finitely many signed means therefore contains the ball of radius κ: a point outside it would be strictly separated by a linear functional, contradicting this support bound. Convex combinations preserve the prescribed second moment. Thus every point of that ball is the mean of an admissible vertex distribution.

Lemma 17 (code-lattice contact cone). *For a symmetric E nonnegative on these contacts, let $s = \operatorname{tr} E$. Then $s \ge 0$ and*

$$\operatorname{tr} E^2 \le \frac{5}{2}s^2, \qquad E \succeq -\frac{s}{2}I. \tag{17}$$

Proof. The coordinate contacts imply $s \ge 0$. Here is a finite positive-sum certificate for the first inequality. Write $h_a = E[a]$, let $\mathcal{R}$ be the coordinate contacts and $\mathcal{V}$ the other 120 contacts, and form unordered triples as follows:

- $\mathcal{A}$: (r, a, b) with $r \in \mathcal{R}$, $a, b \in \mathcal{V}$ of different supports, all mutually orthogonal;
- $\mathcal{D}$: (r, a, b) with all three pairwise inner products of absolute value two and negative product;
- $\mathcal{E}$: three mutually orthogonal vectors of $\mathcal{V}$ whose support indicators sum to zero over $\mathbb{F}_2$.

There are respectively 5760, 240, and 960 triples. Direct coefficient expansion gives

$$256\,s\Big(\frac{5}{2}s^2 - \operatorname{tr} E^2\Big) = \sum_{\mathcal{A}} h_r h_a h_b + 4\sum_{\mathcal{D}} h_r h_a h_b + 3\sum_{\mathcal{E}} h_a h_b h_c. \tag{18}$$

This is a fully specified integer polynomial identity: the families above determine every term. Independent sign changes imply that only 220 diagonal cubic monomials, 450 monomials $E_{kk}E_{ij}^2$, and 120 monomials $E_{ij}E_{ik}E_{jk}$ can survive. Comparing those 790 coefficients verifies the identity completely. Its right side is nonnegative. At $s = 0$, the identities

$$\sum_{a\in\mathcal{V}} E[a] = 48s, \qquad \sum_{a\in\mathcal{V}} E[a]^2 = 16(s^2 + 2\operatorname{tr} E^2)$$

force $E = 0$. The fourth-moment contact identity also gives

$$\sum_{a\in\mathcal{V}} E[a]\,(a^{\mathsf{T}}y)^2 = 16\big(s\,\|y\|^2 + 2E[y]\big) \ge 0$$

for every y, proving the second bound in (17). All displayed identities follow from the same contact list; no numerical fitting is used to verify the identity. □

Proposition 18 (a uniform code-lattice covering bound). *Let $s = \operatorname{tr} G - 10$ and $\ell = \lambda_{\min}(G)$. If $G \succ 0$, $\min_G M_{\mathcal{B}} \ge 4$, $s \le 5/27$, and $27s/10 \le 12\ell/25$, then*

$$\rho_G^2(M_{\mathcal{B}}) \le \frac{3}{10}\operatorname{tr} G. \tag{19}$$

Proof. Let $E = G - I$; the coordinate contacts give $s = \operatorname{tr} E \geq 0$. The exact reference-cell certificate gives, for each reference Voronoi vertex a,

$$3I - aa^{\mathsf{T}} = \sum_{c \in \mathcal{C}} q_{a,c}\, cc^{\mathsf{T}}, \qquad q_{a,c} \in \mathbb{Q}_{\geq 0}. \tag{20}$$

Thus $E[a] \leq 3s$. Computational result 16 and its following moment argument supply the central distributions at $\kappa = 3/10$ and the inscribed ball of radius $7/10$. Deep-cell errors have squared length three; shallow-cell errors have squared length at most $5/2$. On a deep cell, $G[a] \leq 3 + 3s = (3/10) \operatorname{tr} G + 27s/10$. Apply Lemma 14 with $\kappa = 3/10$, $R_0 = 7/10$, and $B = 27s/10$. Its condition is exactly $27s/10 \leq 12\ell/25$. On a nondeep cell, barycentric averaging gives squared distance at most $5/2 + 3s \leq 3 + (3/10)s$, using $s \leq 5/27$. The cells cover space, so the same bound holds everywhere. □

The criterion holds both when $s \leq 1/8$, $\ell \geq 18/25$, and when $s \leq 4/25$, $\ell \geq 23/25$. It gives respectively $\rho^2 \leq 243/80$ and $\rho^2 \leq 381/125$. Thus one proof supplies the covering estimates for both `0014` and `0059`.

3.2 Phase coherence and metric comparison

Lemma 19 (a coherent half-integer lift). *Let the rows of an integral matrix Z be covectors. Choose nine independent rows forming B, and suppose every row of ZB^{-1} is integral with sum of absolute values at most m. If all phases $Z\beta$ are within ε of specified half-integers, these half-integers respect the row relations modulo $\mathbb{Z}$, and $(m+1)\varepsilon < 1$, then the phase errors equal $Z\delta$ for a single $\delta \in \mathbb{R}^9$. The possible centres modulo $\mathbb{Z}^9$ are obtained from the finite quotient associated with B.*

Proof. Solve $B\delta$ equal to the nine basis errors. For any remaining row, the difference between its error and the corresponding integer combination of basis errors is an integer, because the prescribed centres satisfy the relation modulo one. Its absolute value is at most $(m+1)\varepsilon < 1$, so it is zero. The centre solves an integral linear congruence system; Smith normal form enumerates its cosets. Only the cosets satisfying all rows are retained. □

Lemma 20 (weighted phase energy). *Suppose $\sum_i q_i z_i z_i^{\mathsf{T}} = P$, $q_i > 0$, $\sum_i q_i = 12$, and $\operatorname{tr}(PX) \leq 9$. For a coherent centre with phases $1/2 + r_i$, $|r_i| \leq \varepsilon < 1/2$, let $h = 1/u$ and $\mathcal{E} = P[\delta]$. Then*

$$\mathcal{E} \leq \frac{9\varepsilon}{1-\varepsilon}(h-1). \tag{21}$$

Proof. Equation (6) gives $hX[z_i] \geq 3/4 + |r_i| - r_i^2$. Multiply by q_i and sum to obtain $\sum_i q_i(|r_i| - r_i^2) \leq 9(h-1)$. Since $r_i^2 \leq \varepsilon(|r_i| - r_i^2)/(1-\varepsilon)$, the claim follows from $\mathcal{E} = \sum_i q_i r_i^2$. □

In section and height coordinates centred at such a gluing, the actual rank-ten metric is

$$G = \begin{pmatrix} S & -Sb \\ -b^{\mathsf{T}}S & h + b^{\mathsf{T}}Sb \end{pmatrix}, \qquad \|b\|^2 = P[\delta], \qquad \det G = h \det S. \tag{22}$$

The formulas follow by substituting the displaced gluing into its quadratic form. In particular

$$\operatorname{tr} G = \operatorname{tr} S + h + b^{\mathsf{T}} S b, \qquad \operatorname{tr} G^{-1} = \operatorname{tr} S^{-1} + \frac{1 + \|b\|^2}{h}. \tag{23}$$

These two formulas explain why some cases use primal trace and others inverse trace. If $S \preceq qI$, $h \leq p$, and $\|b\|^2 \leq \epsilon$, then $G \prec cI$ follows from $c > q$, $c > p$, and $(c-q)(c-p) > cq\epsilon$, by the Schur complement. Indeed the worst coupling direction reduces to a two-dimensional block; its determinant is $(c-q)(c-p) - cq\|b\|^2$.

4 Four cases controlled by contact geometry

Computational result 21 (the four section frames). For each of `0001`, `0014`, `0059` and `0015`, an integral dual frame with positive rational weights satisfies $\sum_i q_i z_i z_i^{\mathsf{T}} = P$ and $\sum_i q_i = 12$. Each z_i is nonzero. The more specific frame lengths, weights, ranks and congruences used in each subsection are checked in that subsection's finite data, relative to the printed P. All outer-product identities are checked entry by entry.

All four cases start with the same reduction. Their finite dual frames satisfy $\sum_i q_i z_i z_i^{\mathsf{T}} = P$, $q_i > 0$, $\sum_i q_i = 12$. Marking gives $\min_i X[z_i] \leq 3/4$; the first-normal inequality (6) and Lemma 5 then give

$$u \leq 1, \qquad d \geq e \geq \det P, \qquad \Delta \geq \frac{8e}{9}. \tag{24}$$

Thus $e \geq 2187/2$ closes all four cases at once. Throughout this section assume $e < 2187/2$. The remaining task is to locate the first gluing, bound its rank-ten metric, and apply a covering or root-frame bound.

4.1 Case `0001`: a small marked section and two centres

Computational result 22 (the finite data for `0001`). The frame, 72-contact second moment, nonnegative quadratic cone identity and phase-row relations specified below hold for the printed form. The congruence system has precisely the two centres $\pm\beta_0$. The reference rank-ten extension at these centres has a complete rational cell cover with squared covering radius at most $22/9$. The reported cover has 243 phase files, 3762 cells and 1 462 650 vertices. For the quadratic identity, the checker verifies the orbit partition and all 17 coefficients.

Let $e_0 = 262144/243$ and

$$\beta_0 = \frac{1}{4}(0,0,1,1,0,3,3,0,2)^{\mathsf{T}}.$$

The frame consists of the 17 shortest reference dual pairs, all of squared length $3/4$: weight $z = e_9$ by $4/3$ and the other sixteen by $2/3$. Their outer products sum to P. These identities are reconstructed from the printed Gram matrix.

Write $S = I + E$ and $r = \|E\|_F$, so $r^2 = \operatorname{tr} E^2$. The 72 contact pairs satisfy $\sum cc^{\mathsf{T}} = 32P^{-1}$ in integral coordinates. A nonnegative quadratic identity of the form (11) gives

$$\operatorname{tr} E \ge 0, \qquad r \le 3 \operatorname{tr} E.$$

Its seventeen contact-pair orbit coefficients, in the certificate's lexicographic orbit order, are

$$\left(0,\ \frac{1}{64},\ \frac{1}{96},\ \frac{3}{64},\ 0,\ 0,\ \frac{3}{64},\ 0,\ 0,\ 0,\ \frac{1}{48},\ 0,\ \frac{1}{96},\ 0,\ \frac{5}{192},\ \frac{5}{96},\ 0\right).$$

The orbit partition and every matrix coefficient are checked, so the inequality applies to the full 45-dimensional section metric.
At $r = 1/22$, the logarithm series gives

$$\log\det(I + E) \ge \frac{r}{3} - \sum_{k\ge 2} \frac{r^k}{k} > \frac{7153}{524288} \ge \log\left(\frac{531441}{524288}\right).$$

Indeed $|\operatorname{tr} E^k| \le r^k$ for $k \ge 2$. Since $e/e_0 < 531441/524288$ and the section is marked, Lemma 8 gives

$$r < \frac{1}{22}, \qquad \frac{21}{22} I \prec S \prec \left(\frac{23}{22}\right) I.$$

Let $p = 5159/5000$ and $h = 1/u$. The exact cutoff comparison $(e_0 p)^{11} > 4^{11} 972^9$ gives $h < p$. Every one of the 17 covectors has $X[z] < 11/14$, and

$$1 - \frac{11}{14} p - \left(\frac{87}{200}\right)^2 = \frac{3}{40000} > 0.$$

Thus each first phase is within $\varepsilon = 13/200$ of a half-integer. The nine rows numbered $0, 1, 2, 3, 4, 5, 7, 8, 11$ in sorted order form a matrix B of determinant -2. Every remaining row is an integer combination of these rows, with odd coefficient sum and coefficient absolute sum at most seven. The equations $B\beta \equiv \frac{1}{2}\mathbf{1} \pmod{\mathbb{Z}^9}$ have exactly two solutions modulo $\mathbb{Z}^9$, because $B^{-1}\mathbb{Z}^9/\mathbb{Z}^9$ has order two. Both β_0 and $-\beta_0$ satisfy all the phase equations, and they are distinct modulo $\mathbb{Z}^9$ since $2\beta_0 \notin \mathbb{Z}^9$. They therefore exhaust the possible centres. Since $8\varepsilon < 1$, Lemma 19 supplies one displacement δ.
By Lemma 20,

$$P[\delta] \le \frac{9\varepsilon}{1-\varepsilon}(h-1) < \frac{18603}{935000} < \frac{1}{50}.$$

Using (22), the Schur comparison gives $G \prec (5/4)I$ because

$$\left(\frac{5}{4} - \frac{23}{22}\right)\left(\frac{5}{4} - p\right) - \frac{5}{4}\cdot\frac{23}{22}\cdot\frac{1}{50} = \frac{4069}{220000} > 0.$$

The reference rank-ten lattice at $(P, \beta_0, u = 1)$ has squared covering radius $22/9$. Its exact certificate consists of 243 phase files, 3762 cells and 1 462 650 vertices. Hence $\rho_G^2 \le (5/4)(22/9) = 55/18$, while $d = eh \ge e_0$. The extension lemma (Lemma 12) gives

$$\Delta \ge e_0\left(4 - \frac{55}{18}\right) = \frac{2228224}{2187} > 972.$$

This closes `0001`.

Theorem 23. *Assume Computational results 21, 22 and 31. Every chain $L \supset M \supset N$ as in Definition 2 whose rank-9 member is admissibly marked by* `0001:010488327` *has* $\Delta \geq 972$.

Proof. If $e \geq 2187/2$, (24) gives $\Delta \geq 8e/9 \geq 972$. If $e < 2187/2$, the argument above gives $\Delta \geq 2228224/2187 > 972$. □

4.2 Case `0014`: layer contacts bound the complete trace

Computational result 24 (the finite data for `0014`). The required data are fixed relative to the printed form P, with $\det P = 1024$. They consist of a physical basis C_P with $C_P^{\mathsf{T}} C_P = P$; a nonnegative quadratic contact identity (11) with $K = 5$; the two frames and integer row identities specified below; and a reference-cell certificate of Computational result 16 for the extension at β_0. Each frame has twelve nonzero integral rows z_i, with $\sum_i z_i z_i^{\mathsf{T}} = P$ and $P^{-1}[z_i] = 3/4$. The 48 layer errors and their integral lifts are the vectors specified below. The only scalar certificate inputs are positive Bernstein expansions for the three explicitly defined polynomials f_1, f_2, f_3 in (29). All these requirements are finite identities or coefficient signs.

We argue by contradiction, assuming $\Delta < 972$ and $1024 \leq e < 2187/2$. Let

$$S = C_P^{-\mathsf{T}} Q C_P^{-1}, \qquad \delta_S = \det S = \frac{e}{1024}, \qquad r = \|S - I\|_F,$$
$$s = \operatorname{tr} S - 9, \qquad J = \operatorname{tr} S^{-1}, \qquad k = \frac{447}{1000}.$$

Marking gives $J \leq 9$, hence $s \geq 0$ by Cauchy–Schwarz on the nine eigenvalues. The contact identity gives $r^2 \leq 5s^2$, and therefore $kr \leq s$ since $5k^2 < 1$. The rank-ten cutoff gives

$$d < 1024p, \qquad p := \frac{87}{80},$$

by the rational comparison $4^{11} 972^9 < (1024p)^{11}$. Since $d = eh$, this implies $h\delta_S < p$ and $1 \leq h < p$.

Lemma 25 (a common radius estimate). *Let $S \succ 0$ have size nine, with the notation r, s, δ_S, J above. For $0 \leq r \leq R < 1$, set*

$$C_R = \frac{1}{2} + \frac{R}{3(1-R)}.$$

Then

$$\log \delta_S \geq s - C_R r^2, \qquad J \leq 9 - s + \frac{r^2}{1-R}. \tag{25}$$

If $kr \leq s$, it follows that

$$\log \delta_S - 2 \log\left(\frac{J}{9}\right) \geq K_R r, \qquad K_R = k - C_R R + \frac{2}{9}\left(k - \frac{R}{1-R}\right).$$

For $\delta_0 \geq 1$ we also have $\log \delta_0 \leq L(\delta_0) := (\delta_0 - \delta_0^{-1})/2$.

Proof. Write the eigenvalues as $1+x_i$, so $\sum x_i = s$, $\sum x_i^2 = r^2$ and $x_i \ge -R$. The remainder $q(x) = \log(1+x) - x + x^2/2$ has derivative $x^2/(1+x)$. For $x \ge 0$ it is nonnegative. For $-R \le x \le 0$, integration gives $q(x) \ge x^3/[3(1-R)] \ge -Rx^2/[3(1-R)]$. Summation proves the first estimate. The exact identity $(1+x)^{-1} = 1 - x + x^2/(1+x)$ proves the second. Use $\log(J/9) \le J/9 - 1$, $s \ge kr$ and $r^2 \le Rr$ for the joint estimate. Finally, $L'(\delta_0) - (\log \delta_0)' = (\delta_0 - 1)^2/(2\delta_0^2) \ge 0$, and both functions vanish at one. □

The actual section has radius less than $4/25$. The twelve-vector frame and the one-covector flag give $72e \le J^2\Delta$: choose a frame entry at most $J/12$ and apply Lemma 5. Consequently

$$\log \delta_S - 2\log\left(\frac{J}{9}\right) < \log\left(\frac{2187}{2048}\right) \le L\left(\frac{2187}{2048}\right) = \frac{588665}{8957952} =: L_*.$$

First, Lemmata 8 and 25 give $r < 1/5$, since $L_* < k/5 - C_{1/5}/25$. For each of the four pairs

$$(R, D) = \left(\frac{1}{5}, \frac{9}{50}\right),\ \left(\frac{9}{50}, \frac{1}{6}\right),\ \left(\frac{1}{6}, \frac{13}{80}\right),\ \left(\frac{13}{80}, \frac{4}{25}\right),$$

the rational comparisons $K_R > 0$ and $L_* \le K_R D$ hold. If $r \le R$, the joint estimate therefore gives $r < D$. These four successive applications prove $r < 4/25$. All comparisons just stated involve the displayed rational numbers; clearing their positive denominators proves them. No logarithm series or sampled radius is required.

The actual gluing has one coherent reference centre. The reference centre is

$$\beta_0 = \frac{1}{2}(0,0,1,1,0,1,1,0,0)^{\mathsf{T}}.$$

For a row z in either frame, write $x = Q^{-1}[z]$. The determinant estimate

$$\delta_S \ge A(r) := 1 + kr - \frac{r^2}{2(1-r)}$$

follows by summing $\log(1+x_i) \ge x_i - x_i^2/[2(1-r)]$ and using $\log \delta_S \le \delta_S - 1$. We claim $hx < 2211/2500$ for every frame row. If $r \le 29/250$, then $(1-r)x \le 3/4$, and

$$\delta_S(1-r) \ge 1 - \frac{553}{1000}r - \frac{947}{1000}r^2 \ge \frac{57694323}{62500000}.$$

Together with $h\delta_S < p$, these inequalities give

$$hx < \frac{p\left(\frac{3}{4}\right)}{\frac{57694323}{62500000}} < \frac{2211}{2500}.$$

If $29/250 \le r < 4/25$, the same determinant estimate gives $\delta_S \ge 5219/5000$. One way to check this is to replace $r^2/[2(1-r)]$ by $25r^2/42$; the resulting quadratic is increasing on this interval and its left endpoint exceeds $5219/5000$. The failed flag

$\Delta < 972$ then forces every nonzero integral covector to have Q^{-1}-length greater than $5931/8000$, because $1944(5931/8000)^2 < 1024(5219/5000)$. The other eleven rows in the same frame give $x \le 9 - 11(5931/8000)$, and hence

$$hx < \frac{p\left[9 - 11\left(\frac{5931}{8000}\right)\right]}{\frac{5219}{5000}} < \frac{2211}{2500}.$$

The first-normal inequality is $hx \ge 3/4 + |\varepsilon_z| - \varepsilon_z^2$, where $|\varepsilon_z| \le 1/2$ is the error from a half-integer phase. Its right side is increasing in $|\varepsilon_z|$ and equals $2211/2500$ at $4/25$. Thus every frame error has absolute value less than $4/25$.

Here are the precise integer identities needed to turn these individual errors into a common displacement. Nine rows of the first frame form an invertible integral matrix B_0. Every row of that frame is an integral combination of these rows, with odd coefficient sum and sum of absolute coefficients at most three. There is an integral matrix D_0 with $D_0 B_0 = 2I$. Both frames have half-integral phases at β_0, and their union contains a unimodular row basis. These are part of Computational result 24. Lemma 19, using $4(4/25) < 1$, gives $\beta = c + \delta$ with all first-frame phases at c half-integral. Lemma 20 gives

$$P[\delta] \le \frac{12}{7}(h - 1) < \frac{3}{20}.$$

The identity $D_0 B_0 = 2I$ makes $2(c - \beta_0)$ integral. For every row of the other frame, Cauchy–Schwarz gives $|z^{\mathsf{T}}\delta| < \sqrt{(3/4)(3/20)} < 17/50$. Its observed phase error is less than $4/25$, and $17/50 + 4/25 = 1/2$. Thus $z^{\mathsf{T}}c$ must have the same half-integral class as $z^{\mathsf{T}}\beta_0$: the other class is a distance $1/2$ away. The unimodular row basis now gives $c - \beta_0 \in \mathbb{Z}^9$. An integral change of the first lift removes this integer vector. The reference lattice and all its cell and layer data are therefore fixed, while the actual metric is exactly (22) with $b = C_P\delta$. We have proved

$$r < \frac{4}{25}, \quad s \ge kr, \quad |b|^2 < \frac{3}{20}, \quad h \det S < p, \quad 1 \le \det S < \frac{2187}{2048}. \tag{26}$$

Layer contacts and the three finite scalar checks. The 48 first-layer errors are all sign patterns on the six row and column triples of

$$\begin{pmatrix} 5 & 6 & 3 \\ 7 & 1 & 9 \\ 4 & 8 & 10 \end{pmatrix}.$$

Coordinates are numbered from one to ten; the normal coordinate, number two, is omitted. Their mean is zero and covariance (the average outer product about the mean) is $I/3$ on the section. Their average absolute projection in any direction z is at least $|z|/3$: sign averaging on a triple is at least its largest absolute entry; if the row and column maxima sum to R_0, C_0, the average is at least $(R_0 + C_0)/6$, whereas $|z|^2 \le R_0 C_0$.

Let $y^2 = S[b]$ and $z_0 = h - 1 + s/3$. For each opposite pair, the minimum-four inequalities give $S[a] + y^2 + h - 4 \ge 2|a^{\mathsf{T}}Sb|$. Average and use $|Sb| \ge \sqrt{1 - r}\, y$ to obtain

$$y^2 - \frac{2}{3}\sqrt{1 - r}\, y + z_0 \ge 0. \tag{27}$$

Also $y^2 \le 3(1+r)/20$. Define $Z(r) = p/A(r) - 1 + kr/3$. We require precisely

$$\begin{aligned} Z(r) &< \frac{40}{441}(1-r), \\ [\frac{3(1+r)}{20} + Z(r)]^2 &< \frac{1-r^2}{15}, \\ \frac{17}{15}kr + \frac{7}{5}\Big(\frac{p}{A(r)} - 1\Big) &< \frac{1}{8} \qquad \Big(0 \le r \le \frac{4}{25}\Big). \end{aligned} \tag{28}$$

Set

$$N_0 = 2(1-r)(1+kr) - r^2, \qquad N_1 = 2p(1-r) - N_0 + \Big(\frac{kr}{3}\Big)N_0.$$

Then $N_0 \ge 1034/625 > 0$, $A = N_0/[2(1-r)]$ and $Z = N_1/N_0$. The required three polynomials are

$$\begin{aligned} f_1(r) &= \frac{40}{441}(1-r)N_0 - N_1, \\ f_2(r) &= \frac{1-r^2}{15}N_0^2 - \big[\frac{3}{20}(1+r)N_0 + N_1\big]^2, \\ f_3(r) &= \big[\frac{1}{8} - \frac{17}{15}kr + \frac{7}{5}\big]N_0 - \frac{7}{5}p\,2(1-r). \end{aligned} \tag{29}$$

For each f_i, the input is a degree n and positive rational coefficients c_j satisfying the polynomial identity $f_i(4t/25) = \sum_{j=0}^{n} c_j t^j (1-t)^{n-j}$. At every $t \in [0,1]$ at least one basis term is positive, so the identity proves $f_i > 0$. Dividing by N_0, N_0^2, N_0, respectively, proves (28) including both endpoints.

We now derive the scalar bounds used in the covering argument. Write $q = r^2/[2(1-r)]$. The bounds on δ_S give $s < 1/10$ and $0 < A(r) \le 1 + s - q \le 11/10$. For $(\alpha, \beta) = (1/3, 1)$ and $(17/15, 7/5)$, the function

$$\alpha s + \beta\Big(\frac{p}{1+s-q} - 1\Big)$$

is decreasing on this interval: its derivative is nonpositive because $\alpha(11/10)^2 < \beta p$. Its value at $s = kr$ therefore bounds its value at the actual s. In particular $z_0 \le Z(r)$. Let $a_* = 2\sqrt{1-r}/3$ and $M = 3(1+r)/20$. The first two comparisons make $x^2 - a_* x + z_0$ negative at both $x = 2a_*/7$ and $x = \sqrt{M}$. Convexity and (27), with $0 \le y \le \sqrt{M}$, force $y < 2a_*/7$. Thus $y^2 \le (2/5)z_0$. The first and third comparisons now yield

$$|b|^2 < \frac{16}{441}, \qquad \operatorname{tr} G - 10 = s + h - 1 + y^2 < \frac{1}{8}.$$

The lower Schur comparison with $S \succ (21/25)I$ and $h \ge 1$ gives $G \succ (18/25)I$; the scalar margin is $51/4375$. Proposition 18 gives $\rho_G^2 \le 243/80$, and

$$\Delta \ge 1024\Big(4 - \frac{243}{80}\Big) = \frac{4928}{5} > 972,$$

contradicting the assumed counterexample.

Theorem 26. *Assume Computational results 16, 21, 24 and 31. Every chain $L \supset M \supset N$ as in Definition 2 whose rank-9 member is admissibly marked by* `0014:001749549` *has $\Delta \geq 972$.*

Proof. By (24), $e \geq \det P = 1024$ and the range $e \geq 2187/2$ gives $\Delta \geq 972$; on $1024 \leq e < 2187/2$ the argument above shows that $\Delta < 972$ is contradictory. □

4.3 Case `0059`: weak marking becomes exact marking

Computational result 27 (the finite data for `0059`). The printed form has determinant 1024 and a quadratic contact identity with $K = 9$. A twelve-vector integral frame has unit weights, covariance P, and reference dual lengths $3/4$. The two phase-row systems below have integer relations of odd coefficient sum and absolute coefficient sum at most three, and the specified finite cyclic-quotient identities. The two good centres have fixed reference-cell certificates of Computational result 16, and the code contact identities hold in their fixed physical bases. The layer data are finite first, second and third moment tensors, integral lifts, and nonnegative contact majorants of coefficient $25/7$. Four further integral covectors satisfy the projector and centre-incidence identities stated below. These are all fixed finite inputs. Neither a covering bound for a varying metric nor a localisation assertion on a determinant band is assumed.

Use S, r, s, δ_S as above and $p = 87/80$; now the cone gives $r \leq 3s$. Again $1 \leq h < p/\delta_S$ and $1 \leq \delta_S < 2187/2048$. The two reference centres are $\pm\beta_0$ modulo $\mathbb{Z}^9$, where

$$\beta_0 = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)^{\mathsf{T}}.$$

Their rank-ten reference lattices are integrally isometric to $M_{\mathcal{B}}$ (Appendix A prints the isometry).

Radius, phase and centre bounds on four adjoining bands. For a closed band $a \leq \delta_S \leq \delta_0$, let

$$H = \frac{p}{a}, \qquad \bar{s} = L(\delta_0) + C_R R^2.$$

Use the following four rows:

a	δ_0	R	ε
1	$51/50$	$27/400$	$3/20$
$51/50$	$41/40$	$7/80$	$3/20$
$41/40$	$103/100$	$27/250$	$7/40$
$103/100$	$31/30$	$129/1000$	$1/5$.

For each row, direct rational arithmetic gives

$$L(\delta_0) < \frac{R}{3} - C_R R^2, \qquad \frac{3H}{4(1-R)} < \frac{3}{4} + \varepsilon - \varepsilon^2. \tag{30}$$

The first inequality and the marked radial barrier give $r < R$; Lemma 25 then gives $s \le \bar{s}$. Every phase-system row z has $P^{-1}[z] \le 3/4$, so $hQ^{-1}[z] \le 3H/[4(1-R)]$. The second inequality and the first-normal bound give phase error less than ε at every such row. Since $4\varepsilon < 1$, Lemma 19 gives a common centre.
The following integer identities enumerate the possible gluing centres. For the chosen integral row basis B_0, the data supply integral matrices and vectors satisfying

$$D_0 B_0 = nI, \qquad D_0 = a_{\mathrm{res}} b_{\mathrm{res}}^{\mathsf{T}} + nM, \qquad n\gamma = a_{\mathrm{res}}.$$

Fix a listed base centre c_0. If $B_0(c - c_0) = \mathbf{k} \in \mathbb{Z}^9$, multiplication gives $c - c_0 = M\mathbf{k} + (b_{\mathrm{res}}^{\mathsf{T}}\mathbf{k})\gamma$. Reducing the integer $b_{\mathrm{res}}^{\mathsf{T}}\mathbf{k}$ modulo n leaves exactly the listed n candidate classes. The low-band row system uses $n = 2$ and lists $\pm\beta_0$. Consequently, throughout the four bands,

$$\beta = \pm\beta_0 + m + \delta, \qquad m \in \mathbb{Z}^9, \qquad P[\delta] \le \frac{9\varepsilon}{1-\varepsilon}(h-1).$$

Changing the integral lift removes m and places the actual metric in (22) in the corresponding fixed reference basis. This change preserves every minimum and determinant.

One layer proof for the first three bands. Write the fixed layer errors as a_i with weights $w_i \ge 0$. The finite identities required are

$$\sum_i w_i = 1, \quad \sum_i w_i a_i = 0, \quad \sum_i w_i a_i a_i^{\mathsf{T}} = \frac{I}{3}, \quad \sum_i w_i (a_i)_j (a_i)_k (a_i)_\ell = 0$$

for all coordinate triples, together with $|a_i|^2 = 3$ and $(S - I)[a_i] \le (25/7)s$ from the contact majorants. For any direction z, let $x_i = a_i^{\mathsf{T}} z$ and $M_z = \max_i x_i$. The moment identities give $M_z \ge 0$, $\sum_i w_i x_i^2 = |z|^2/3$ and $\sum_i w_i x_i^3 = 0$. If $M_z = 0$ all nonzero-weight x_i vanish. Otherwise, averaging $(M_z - x_i)(M_z + x_i)^2 \ge 0$ gives $M_z(M_z^2 - |z|^2/3) \ge 0$. Thus $M_z \ge |z|/\sqrt{3}$ in all cases.
Let $E_b = S[b]$, and let

$$\mathcal{E}_0 = \frac{9\varepsilon}{1-\varepsilon}(H-1), \qquad I_0 = (1+R)\mathcal{E}_0, \qquad Z_0 = H - 1 + \frac{25}{7}\bar{s}.$$

The actual displacement satisfies $0 \le E_b \le I_0$. Expanding the minimum-four inequality at $(a_i, 1)$ and using the support bound with $z = Sb$ gives

$$E_b - 2\sqrt{\frac{1-R}{3}}\,\sqrt{E_b} + Z_0 \ge 0.$$

For the first three rows above, use respectively (η is the allowance for the squared-distance excess of a cell, used below; it is not the band end δ_0)

row	Y	ℓ	T	η
1	4/125	4/5	32/225	0
2	7/200	39/50	33/250	0
3	47/1000	3/4	29/200	16443/640000.

All the following comparisons are between the displayed rational numbers, with $\bar{s}, \mathcal{E}_0, I_0, Z_0$ defined above:

$$(Y+Z_0)^2 < \frac{4}{3}(1-R)Y, \qquad (I_0+Z_0)^2 < \frac{4}{3}(1-R)I_0,$$
$$\bar{s}+H-1+Y \le T, \qquad \ell Y \le (1-R-\ell)(1-\ell),$$
$$\left(\frac{27}{4}T\right)^2 \le 4\ell\left(\eta+\frac{81}{40}T\right), \qquad \frac{27}{10}T-\frac{1}{2} \le \eta.$$

The upward quadratic in $\sqrt{E_b}$ is negative at $\sqrt{Y}$ and $\sqrt{I_0}$, so $E_b < Y$. The next two comparisons give $\operatorname{tr} G - 10 \le T$ and $G \succeq \ell I$ by the lower Schur bound.
Here is the small extension of the mixture argument needed in row three. On a deep cell, the vertex excess is at most $27T/10$. For a point at distance $x \ge 3/10$ from the centre, its mixture weight is at most $(x-3/10)/(7/10-3/10)$. With $A_0 = 27T/4$, the squared-distance excess is therefore at most

$$A_0\left(x-\frac{3}{10}\right) - \ell x^2 \le \frac{A_0^2}{4\ell} - \frac{3A_0}{10} \le \eta.$$

On a shallow cell it is at most $(27/10)T - 1/2 \le \eta$. Thus the whole reference partition gives $\rho_G^2 \le 3 + (3/10)T + \eta$. The resulting bounds are

δ_S band	ρ_G^2 upper bound	Δ lower bound
$[1, 51/50]$	$1141/375$	$367616/375$
$[51/50, 41/40]$	$7599/2500$	$15673728/15625$
$[41/40, 103/100]$	$1964283/640000$	$24424397/25000.$

Indeed $d = eh \ge 1024a$, so the last column is $1024a(4-\rho_G^2)$. Every entry exceeds 972. The closed bands share endpoints, leaving no boundary gap.

The final band and the two unwanted centre classes. On $31/30 \le \delta_S < 2187/2048$, the failed flag forces every nonzero integral covector to have Q^{-1}-length greater than $q_* = 737771/10^6$, since $1944q_*^2 < 1024(31/30)$. The twelve unit-weight frame rows consequently satisfy $Q^{-1}[z_i] \le 9 - 11q_*$. Since $h < 261/248$,

$$hQ^{-1}[z_i] < \frac{261}{248}(9-11q_*) < \frac{3}{4} + \frac{6}{25} - \left(\frac{6}{25}\right)^2.$$

Their phase errors are less than $6/25$. The high-band row system uses the same integer construction with $n = 4$, giving four candidate centres and

$$P[\delta] \le \frac{54}{19}(h-1) \le \frac{351}{2356}.$$

Two classes are $\pm\beta_0$ modulo $\mathbb{Z}^9$. At each other class, four nonzero integral covectors w_i have integral phase. Their finite identities are

$$K = \sum_{i=1}^{4} w_i w_i^{\mathsf{T}}, \qquad KP^{-1}K = K, \qquad \operatorname{tr}(P^{-1}K) = 3.$$

Whitening by P makes K an orthogonal projection of rank three; hence $0 \preceq K \preceq P$. The available integral lifts give $hQ^{-1}[w_i] \geq 1 - (w_i^{\mathsf{T}}\delta)^2$. Summing, and using $K[\delta] \leq P[\delta]$, proves

$$\operatorname{tr}(Q^{-1}K) \geq \frac{4 - \frac{351}{2356}}{\frac{261}{248}} = \frac{18146}{4959} =: t_*.$$

Let $\alpha_* = 14877/18146 = 3/t_*$ and $\beta_* = 29754/26485 = 6/(9 - t_*)$. Then $0 < \alpha_* < \beta_*$ and

$$\operatorname{tr}\big(Q^{-1}(\alpha_* K + \beta_*(P - K))\big) \leq 9\beta_* + (\alpha_* - \beta_*)t_* = 9.$$

The marked determinant inequality and the projection identity yield

$$e \geq \det[\alpha_* K + \beta_*(P - K)] = 1024\,\alpha_*^3\beta_*^6 > \frac{2187}{2},$$

a contradiction. This explicitly eliminates both unwanted classes.

The same covering proof closes the two upper bands. The fourth band has $\varepsilon = 1/5$, $h \leq 435/412$ and $P[\delta] \leq (9/4)(h - 1)$. The final band has $\varepsilon = 6/25$, $h \leq 261/248$ and $P[\delta] \leq (54/19)(h - 1)$, at one of the two good centres. Equation (23) therefore gives

$$\operatorname{tr} G^{-1} \leq 10 + \frac{455}{4959}, \qquad \det G < \frac{87}{80}. \tag{31}$$

The two inverse-trace excess bounds are respectively $23/348$ and $455/4959$.
Let $s_G = \operatorname{tr} G - 10 \geq 0$. The code cone has $K = 5/2$ and $a = 1/2$ in Lemma 9, giving masses four and six at $-s_G/2$, $s_G/2$. If $s_G \geq 1/3$, apply that lemma to $G_0 = I + (G - I)/(3s_G)$:

$$\det G_0 \geq \left(\frac{5}{6}\right)^4 \left(\frac{7}{6}\right)^6 > \frac{29}{24}.$$

But (12) and $455/4959 < 1/10$ imply

$$\log\det G_0 < \log\left(\frac{87}{80}\right) + \frac{1}{10} < \log\left(\frac{29}{24}\right).$$

Here $\log(10/9) > 1/10$ follows by integrating $1/t$ on $[1, 10/9]$, and $(87/80)(10/9) = 29/24$. Thus $s_G < 1/3$, and Lemma 11 gives exact marking $\operatorname{tr} G^{-1} \leq 10$. The same minorant at $s_G = 4/25$ gives $\det G_0 \geq (23/25)^4(27/25)^6 > 87/80$. The marked radial barrier yields $s_G < 4/25$; the code cone then gives $G \succ (23/25)I$. Proposition 18 gives $\rho_G^2 \leq 381/125$, so

$$\Delta \geq 1024\left(4 - \frac{381}{125}\right) = \frac{121856}{125} > 972.$$

Together with the three lower bands and the already closed range $e \geq 2187/2$, this proves `0059`.

Theorem 28. *Assume Computational results 16, 21, 27 and 31. Every chain $L \supset M \supset N$ as in Definition 2 whose rank-9 member is admissibly marked by* `0059:016902802` *has $\Delta \geq 972$.*

Proof. By (24), $e \geq 1024$, that is $\delta_S = \det S \geq 1$, and $e \geq 2187/2$ gives $\Delta \geq 972$. The range $1 \leq \delta_S < 2187/2048$ is covered by the four closed bands above and the final band $31/30 \leq \delta_S < 2187/2048$: on the first three, Δ is bounded below by the last column of the table, every entry of which exceeds 972; on the fourth and the final band the gluing is at one of the two good centres and $\Delta \geq 121856/125$, while the two unwanted centre classes are contradictory. □

4.4 Case `0015`: three centres, one common trace theorem

Computational result 29 (the finite geometry for `0015`). For the printed section and reference extension M_2, the two contact lists, positive cubic cone identities, second and fourth moments, and the lower and upper nonnegative contact expansions for each of the 54 layer vectors stated below hold. Their first and third moments vanish and their second moment has the displayed value. The complete reference-cell certificate has the three deep-cell classes and nondeep radius bound stated below; each deep class has the stated quartic sum-of-squares identity and convex-hull radius. The dual D_4 frame is integral with the normalisation required by Lemma 6. The 13-vector section frame, its four triples, index-three congruence system, 27 integral-phase covectors and 27 additive triangles have the identities and incidences printed below. The exact scalar comparisons in the finite tables are verified with the displayed formulas.

The proof below derives the mean-body assertion, all phase ranges, and the exclusion of all three centres from these finite identities.
Set $e_0 = 972$ and

$$\beta_0 = \left(\frac{2}{3}, \frac{1}{6}, \frac{2}{3}, \frac{5}{6}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, 0, 0\right)^{\mathsf{T}}.$$

Let M_2 be the rank-ten reference extension with integral Gram matrix

$$P_{10} = \begin{pmatrix} P & -P\beta_0 \\ -\beta_0^{\mathsf{T}} P & 1 + P[\beta_0] \end{pmatrix}.$$

Its determinant is $\det P = 972$.

The two contact cones. The section has 81 contact pairs and M_2 has 135. Their positive cubic identities give, respectively,

$$\operatorname{tr}((S-I)^2) \leq 5(\operatorname{tr} S - 9)^2, \qquad \operatorname{tr}((G-I)^2) \leq \frac{47}{14}(\operatorname{tr} G - 10)^2.$$

Their second moments are $36I$ and $54I$ in physical coordinates. Thus zero trace excess forces zero deformation by the cubic identity argument above. The rank-ten fourth moment is $18(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, so contact positivity gives $G - I \succeq -(\operatorname{tr} G - 10)I/2$. For each of the 54 first-layer section errors a, the exact identity

$$P^{-1} + aa^{\mathsf{T}} = \sum_{c \in \mathcal{C}(P)} q_c \, cc^{\mathsf{T}}, \qquad q_c \geq 0,$$

gives $E[C_P a] \ge -\operatorname{tr} E$. Lifting $E = S - I$ to $\operatorname{diag}(E, \operatorname{tr} E)$ makes every reference rank-ten contact nonnegative. The rank-ten fourth-moment bound therefore implies $E \succeq -(\operatorname{tr} E) I$. Hence Lemma 9 gives the two functions

$$b_9(s) = \frac{11}{4}\log(1-s) + \frac{25}{4}\log\Big(1 + \frac{3s}{5}\Big), \tag{32}$$

$$b_{10}(s) = \frac{19}{4}\log\Big(1 - \frac{s}{2}\Big) + \frac{21}{4}\log\Big(1 + \frac{9s}{14}\Big), \tag{33}$$

which bound the corresponding log determinants below.

The common rank-ten theorem. Every minimum-four metric G on M_2 with trace excess $s \le 203/1000$ excludes an admissible $\Delta < 972$. The reference cell certificate has three deep-cell classes. Each has 80 errors a of squared length three, with second moment $(3/10)I$, and their convex hull contains the ball of radius $3/4$. Nondeep cells have squared circumradius at most $8/3$. For the uniform distribution on each deep error set, the exact quartic certificate proves just the inequality needed here:

$$\mathbb{E}(a^{\mathsf{T}} z)^4 \le \frac{3}{10}|z|^4.$$

Its residual is a sum of positive multiples of squares of quadratic forms; expansion checks all 715 quartic coefficients. The absolute-moment estimate follows from one polynomial identity:

$$t^4 - 3t^2 + 2t = t(t-1)^2(t+2) \ge 0 \qquad (t \ge 0).$$

For $z \ne 0$, substitute $t = |a^{\mathsf{T}} z|/|z|$ and average. The second moment is $\mathbb{E}t^2 = 3/10$, while $\mathbb{E}t^4 \le 3/10$, so $2\mathbb{E}t \ge 3\mathbb{E}t^2 - \mathbb{E}t^4 \ge 3/5$. Thus $\mathbb{E}|a^{\mathsf{T}} z| \ge (3/10)|z|$; the same conclusion is immediate for $z = 0$. Redistribute mass within opposite pairs, keeping each pair's total mass fixed. The second moment stays $(3/10)I$, while the largest possible projection of the mean on z is $\mathbb{E}|a^{\mathsf{T}} z|$. These means form a compact convex set: it is the convex hull of the finitely many choices sending each antipodal pair entirely to one side. If a point x with $|x| \le 3/10$ lay outside, choose a closest mean y and let $z = x - y \ne 0$. Minimality along every segment from y to another mean v, after expanding squared distance, gives $\langle v - y, z\rangle \le 0$. Thus $\langle x, z\rangle$ exceeds every mean projection by at least $|z|^2$, contradicting $\langle x, z\rangle \le (3/10)|z| \le \mathbb{E}|a^{\mathsf{T}} z|$. Thus the whole ball of radius $3/10$ is contained in the mean set.
Lemma 14, with $\kappa = 3/10$, $R_0 = 3/4$, and $B = 3\lambda_{\max}(G) - (3/10)\operatorname{tr} G$, gives

$$\rho_G^2 \le \max\Big\{\frac{3}{10}\operatorname{tr} G,\ \Big(\frac{8}{3}\Big)\lambda_{\max}(G)\Big\}$$
$$\text{if} \quad 3\lambda_{\max}(G) - \frac{3}{10}\operatorname{tr} G \le \frac{27}{50}\lambda_{\min}(G).$$

For $0 \le s \le 2/25$, the cone bounds give $\lambda_{\min} \ge 1 - s/2$ and $\lambda_{\max} \le 1 + 11s/6$, and the displayed condition holds. Hence

$$\rho_G^2 \le \max\Big\{3 + \frac{3s}{10},\ \frac{8}{3} + \frac{44s}{9}\Big\}.$$

The first term dominates for $s \le 9/125$. To control both terms where needed, let

$$\Phi_{10}(s) = \Big(1 - \frac{s}{2}\Big)^{19}\Big(1 + \frac{9s}{14}\Big)^{21} = \exp(4b_{10}(s)).$$

The following three ranges use one argument:

range of s	$f(s)$	corresponding bound
$[0, 2/25]$	$1 - 3s/10$	deep-cell term
$[9/125, 2/25]$	$4/3 - 44s/9$	nondeep-cell term
$[2/25, 203/1000]$	$(35/36)(1 - s/2)$	D_4 frame.

For each row, f is positive and affine, so $b_{10}(s) + \log f(s)$ is concave. At both endpoints the exact rational comparison is

$$\Phi_{10}(s) f(s)^4 \ge 1;$$

equality occurs only at $s = 0$ in the first row. Concavity gives $\det G\, f(s) \ge 1$ throughout each range. The first row holds on the entire small-trace interval; the second controls the additional covering term when it can be larger. Thus the covering lemma gives $\Delta \ge 972$ for $0 \le s \le 2/25$. For the third row, the integral D_4 frame and Lemma 6 give $\Delta \ge (35/36)\, 972 \det G\, (1 - s/2)$, so again $\Delta \ge 972$. At $s = 0$ the cone forces $G = I$, and the same conclusion follows from the reference cover. If G is marked, its trace is automatically below $203/1000$: the exact inequality

$$\Big[\Big(1 - \frac{s}{2}\Big)^{19}\Big(1 + \frac{9s}{14}\Big)^{21}\Big]^{11} 972^8 > 4^{44} \tag{34}$$

at $s = 203/1000$ is the radial barrier against $\det G < 4 \cdot 972^{-2/11}$. The same inequality holds at $s = 83/200$. Concavity of b_{10} therefore forbids the entire trace interval $[203/1000, 83/200]$, even for unmarked metrics.

All first gluings have three coherent centres. Twelve reference dual covectors z_i satisfy $\sum_i z_i z_i^{\mathsf{T}} = P$. They split into four triples, with indices $(0, 3, 10)$, $(1, 2, 11)$, $(4, 7, 9)$, $(5, 6, 8)$, each summing to $z_\Sigma = (1, \dots, 1)^{\mathsf{T}}$. All thirteen have squared reference dual length $3/4$; the frame has rank nine and an integral basis of determinant three. We have $h \ge 1$. The global cutoff gives $d^{11} < 4^{11} 972^9 < (11131/10)^{11}$, where the last comparison is exact rational arithmetic. Since $d = eh$, this proves $h < 11131/(10e)$. The signed error at a covector z is $\beta^{\mathsf{T}} z - \lfloor \beta^{\mathsf{T}} z \rfloor - 1/2$; denote the twelve frame errors by r_i.

For $e \le 1055$, the marked section barriers (32) at the following endpoints give $S \succ (1 - s_0) I$ and let all thirteen phase errors within $1/4$:

e lower	e upper	s_0
972	1010	$1/20$
1010	1040	$1/10$
1040	1055	$13/100$.

The exact tests are

$$(1-s_0)^{11}\left(1+\frac{3s_0}{5}\right)^{25} > \left(\frac{e_+}{972}\right)^4 \qquad \text{and} \qquad \frac{3}{4}\,\frac{11131}{10e_-}\,\frac{1}{1-s_0} < \frac{15}{16}.$$

For each triple, the sum of its errors minus the error at z_Σ is an integer of absolute value less than one, so it is zero. For $e \geq 1055$, summing the first-normal inequalities with these raw errors gives

$$\sum_i (|r_i| - r_i^2) \leq 9(h-1) < \frac{1}{2}.$$

If two triple sums differed by a nonzero integer, their six errors would have absolute sum at least one; since $|r| - r^2 \geq |r|/2$, this is impossible. In both ranges, the four triple error sums are equal. Select the first two covectors from each triple and the third covector from the first triple. These nine covectors are independent: a vector annihilated by them is annihilated by the other three, using the common triple-sum relation, and hence is zero because $\sum_i z_i z_i^{\mathsf{T}} = P \succ 0$. Solve the nine selected equations $z_i^{\mathsf{T}}\delta = r_i$. The equal triple sums then give the remaining three equations. Consequently $\beta - \delta$ has half-integral phase at every frame covector. The integral congruences place this centre in exactly one of the classes β_0, $-\beta_0$, $3\beta_0$ modulo $\mathbb{Z}^9$.
The same barrier argument gives the sharper error bound $1/6$ through $e = 1028$, using the following adjoining bands:

e lower	e upper	s_0
972	1000	$31/1000$
1000	1020	$57/1000$
1020	1028	$69/1000$.

In each row the exact tests are

$$(1-s_0)^{11}\left(1+\frac{3s_0}{5}\right)^{25} > \left(\frac{e_+}{972}\right)^4, \qquad \frac{3}{4}\,\frac{11131}{10e_-}\,\frac{1}{1-s_0} < \frac{8}{9}.$$

These also supply $S \succeq (1-s_0)I$. The displacement already constructed realises the raw phase errors, so it satisfies all applicable bounds simultaneously. Writing $\mathcal{E} = P[\delta]$, Lemma 20 gives

$$\mathcal{E} \leq c(h-1), \qquad c = \begin{cases} \dfrac{9}{5}, & e \leq 1028, \\ 3, & e \leq 1055, \\ 9, & \text{all } e. \end{cases}$$

The last bound uses the universal raw-error bound $|r_i| \leq 1/2$.

The third centre is excluded globally. At $3\beta_0$, 27 covectors of integral phase satisfy $K := \sum_w ww^{\mathsf{T}} = 3P - 4z_\Sigma z_\Sigma^{\mathsf{T}}$. An integral translation of the centre preserves

these integral phases. For each w, its integral centre phase is an available lift, so $hX[w] \geq 1 - (\delta^{\mathsf{T}} w)^2$. Also

$$\sum_w (\delta^{\mathsf{T}} w)^2 = 3\mathcal{E} - 4(\delta^{\mathsf{T}} z_\Sigma)^2 \leq 3\mathcal{E}.$$

Summing gives $h \operatorname{tr}(KX) \geq 27 - 3\mathcal{E}$. The normal inequality at z_Σ and the universal phase-distance bound $1/2$ give $hX[z_\Sigma] \geq 3/4$. Marking therefore implies

$$27 - 3\mathcal{E} \leq h \operatorname{tr}(KX) = 3h \operatorname{tr}(PX) - 4hX[z_\Sigma] \leq 27h - 3, \qquad 9h + \mathcal{E} \geq 10.$$

If $\mathcal{E} \leq c(h-1)$, this forces $h \geq (10+c)/(9+c)$. The coherent phase bounds give $c = 9/5$ through $e = 1028$, $c = 3$ through 1055, and $c = 9$ above 1055. For $e \geq 1020$, these contradict, respectively,

$$h < \frac{11131}{10200} < \frac{59}{54}, \qquad h < \frac{11131}{10280} < \frac{13}{12}, \qquad h < \frac{11131}{10550} < \frac{19}{18}.$$

For the lower range use the 27 additive triangles on the integral-phase covectors, each covector having incidence three. Averaging their three squared displacements gives a triangle with sum at most $\mathcal{E}/3$. If $S \succeq \ell I$, each of its three integral lifts has squared dual length at most $8/(9\ell) + u\mathcal{E}/3$, where $u = 1/h$. The lifted vectors still sum to zero: their integral centre phases do. Thus Lemma 4 gives $v \leq 1/\ell + 3u\mathcal{E}/8$. Throughout $e \leq 1020$, we have $\mathcal{E} \leq (9/5)(h-1)$ and $h \geq 59/54$. Consequently

$$0 < u \leq r := \frac{54}{59}, \qquad uv \leq g_\ell(u) := \left(\frac{1}{\ell} + \frac{27}{40}\right)u - \frac{27}{40}u^2.$$

On the two bands with lower endpoint and spectral bound

$$(a, \ell) = \left(972, \frac{969}{1000}\right), \qquad \left(1000, \frac{943}{1000}\right),$$

the exact rational comparisons are

$$2\left(\frac{27}{40}\right)r \leq \frac{1}{\ell} + \frac{27}{40}, \qquad g_\ell(r) < \frac{a}{972}.$$

The first makes g_ℓ increasing on $[0, r]$. The second gives $uv < a/972 \leq e/972$, and therefore $\Delta = e/(uv) > 972$. This excludes the third centre in every range.

Aligning the two reference centres. For a centre $\beta_0 + n$, the integral shear $(x, k) \mapsto (x + kn, k)$ replaces the gluing vector by $\beta_0 + \delta$. For a centre $-\beta_0 + n$, follow this shear by $x \mapsto -x$, giving $\beta_0 - \delta$. These transformations are unimodular, preserve both minima and determinants, and preserve the reference displacement energy $P[\delta]$. It is therefore enough to treat the centre β_0 in the following arguments.

The two reference centres below $e = 1000$. Write the 54 integer layer points as z_j, their section errors as $a_j = z_j - \beta_0$, and their physical errors as $\alpha_j = C_P a_j$. The finite identities required here are

$$P[a_j] = 3, \qquad 5P^{-1} - a_j a_j^{\mathsf{T}} = \sum_{c \in \mathcal{C}(P)} q_{jc}\, cc^{\mathsf{T}}, \quad q_{jc} \geq 0,$$

and, with uniform expectation over the 54 points,

$$\mathbb{E}\alpha_j = 0, \qquad \mathbb{E}\alpha_j \alpha_j^{\mathsf{T}} = \frac{I}{3}, \qquad \mathbb{E}(\alpha_j)_p(\alpha_j)_q(\alpha_j)_r = 0 \quad \text{for every } p, q, r.$$

These are coefficient and matrix identities, with no geometric support bound assumed. Here is their support consequence. For any direction z, set $x_j = \alpha_j^{\mathsf{T}} z$ and $M = \max_j x_j$. The first moment gives $M \geq 0$; the second gives $\mathbb{E}x_j^2 = |z|^2/3$; and expansion of the third moment gives $\mathbb{E}x_j^3 = 0$. If $M = 0$, all x_j vanish because they are nonpositive with mean zero, so $z = 0$. If $M > 0$, average the nonnegative polynomial

$$(M - x_j)(M + x_j)^2 = M^3 + M^2 x_j - M x_j^2 - x_j^3$$

to obtain $M(M^2 - |z|^2/3) \geq 0$. Thus in all cases

$$\max_j \alpha_j^{\mathsf{T}} z \geq \frac{|z|}{\sqrt{3}}.$$

Let $b = C_P \delta$, $y^2 = S[b]$, $s = \operatorname{tr} S - 9$, and $t = h - 1$. Contact positivity applied to the upper expansions gives $(S - I)[\alpha_j] \leq 5s$, hence $S[\alpha_j] \leq 3 + 5s$. The nonzero integer layer vector $(z_j, 1)$ has squared length $S[\alpha_j - b] + h \geq 4$. Expanding and using the support consequence with $z = Sb$ yields

$$y^2 + t + 5s \geq 2 \max_j \alpha_j^{\mathsf{T}} Sb \geq \frac{2|Sb|}{\sqrt{3}} \geq 2\sqrt{\frac{\ell}{3}}\, y,$$

where $S \succeq \ell I$ implies $|Sb|^2 \geq \ell S[b]$ by Cauchy–Schwarz and $\ell |b|^2 \leq S[b]$ (the zero-energy case is immediate).
One interval suffices for the whole range $972 \leq e \leq 1000$. The section barrier (32), the variance bound, and the height cutoff give

$$s < s_0 := \frac{31}{1000}, \qquad \ell := \frac{969}{1000}, \qquad m := \frac{4279}{4000},$$
$$\ell I \preceq S \preceq mI, \qquad t < T := \frac{1411}{9720}.$$

The one-sixth phase bound gives $y^2 \leq (9/5)mt$. Set $s_* := 83/200$, $Y := s_* - s_0 - T$, and $Z := T + 5s_0$. The only remaining checks are the rational inequalities

$$\left(\frac{9}{5}\right) mT < \frac{\ell}{3}, \qquad 0 < Y < \frac{\ell}{3}, \qquad (Y + Z)^2 < \left(\frac{4\ell}{3}\right) Y. \tag{35}$$

The layer inequality says $p(y) \geq 0$ for $p(x) = x^2 - 2\sqrt{\ell/3}\,x + Z$, while the last check gives $p(\sqrt{Y}) < 0$. Both y and $\sqrt{Y}$ lie below the vertex $\sqrt{\ell/3}$. If $y \geq \sqrt{Y}$, then

$$p(y) - p(\sqrt{Y}) = (y - \sqrt{Y})\left(y + \sqrt{Y} - 2\sqrt{\frac{\ell}{3}}\right) \leq 0,$$

a contradiction. Hence $y^2 < Y$, and the exact relative trace identity (23) gives $\operatorname{tr} G - 10 = s + t + y^2 < s_*$. The common unmarked trace theorem therefore excludes $\Delta < 972$.

The reference centres on $1000 \leq e \leq 1080$. At section trace $1/4$, the cone gives largest eigenvalue displacement at most $\sqrt{5}/4 \leq 9/16$. Summing (15) and using the intermediate marking from Lemma 8 would give $\log\det S \geq 17/160 > \log(10/9)$, a contradiction. Therefore $s < 1/4$, and the same inequality gives

$$s \leq \frac{40}{17}\log\!\Big(\frac{e}{972}\Big).$$

The logarithm comparisons need no numerical approximation. For $x \geq 0$,

$$\log(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3},$$

because the derivative of the right side minus the left side is $x^3/(1+x) \geq 0$ and the difference vanishes at zero. At $x = 1/9$, this gives

$$\log\!\left(\frac{10}{9}\right) \leq \frac{461}{4374} < \frac{17}{160}, \qquad \frac{40}{17}\left(\frac{461}{4374}\right) < \frac{24799}{100000}.$$

On each row below, set $h_* = 11131/(10a)$, $m = 1 + 9s_*/4$, and $c = 9\varepsilon/(1-\varepsilon)$:

a	b	s_*	ε
1000	1025	3161/50000	1701/10000
1025	1047	4989/50000	598/3125
1047	1061	6689/50000	5451/25000
1061	1069	4427/25000	21091/100000
1069	1080	24799/100000	15033/100000.

The section bound follows from (32) in the first four rows and the logarithm inequality in the last. The phase bound uses either $X[z] < (3/4)/(1-s_*)$ or $X[z] < 9 - 11r$ with $r^2 < a/1944$; the latter follows because by Lemma 5, a counterexample ($\Delta < 972$) has $X[z] > \sqrt{e/1944}$ for every frame covector, and their twelve lengths sum to $\operatorname{tr}(PX) \leq 9$. In the last two rows one may take, respectively, $r = 73877/100000$ and $r = 14831/20000$. In every row $h_* X[z] < 1 - (1/2 - \varepsilon)^2$, and

$$\operatorname{tr} G - 10 \leq s_* + (1 + mc)(h_* - 1) < \frac{83}{200}.$$

The forbidden trace interval (34) forces $\operatorname{tr} G - 10 < 203/1000$, where the common theorem applies.

The upper range. For every $e \geq 1080$, the frame argument gives $\varepsilon = 1/10$ using $\sqrt{5/9} > 559/750$ and $(207/200)(9 - 11 \cdot 559/750) < 21/25$. Then Lemma 20 gives $P[\delta] \leq h - 1$, and (23) gives $\operatorname{tr} G^{-1} \leq 10$. The common marked theorem applies. The ranges meet at their endpoints, so every first and second gluing in `0015` has now been covered.

Theorem 30. *Assume Computational results 21, 29 and 31. Every chain $L \supset M \supset N$ as in Definition 2 whose rank-9 member is admissibly marked by* `0015:030291998` *has $\Delta \geq 972$.*

Proof. By (24), $e \geq 972$ and the range $e \geq 2187/2$ gives $\Delta \geq 972$. On $972 \leq e < 2187/2$ the first gluing is near one of the three coherent centres; the centre $3\beta_0$ is excluded in every range, the centres $\pm\beta_0$ are aligned, and the ranges $972 \leq e \leq 1000$, $1000 \leq e \leq 1080$ and $e \geq 1080$ each bring the rank-ten metric G into the common rank-ten theorem, which gives $\Delta \geq 972$. □

5 What is verified, and where

Three obligations. A finite computer-assisted argument has three logically separate parts. First, every actual counterexample must lie in the stated root and satisfy every necessary inequality. Second, a terminal witness must imply the claimed contradiction throughout its closed box. Third, the complete collection of boxes must cover the root. Sections 3 and 4 prove the geometric and analytic implications; the certificate archive of the series (its location is given at the end of this section) contains the rational data and the programs that check their coefficient identities, terminal witnesses and coverage. The certificates of this paper have three elementary formats.

1. *Matrix and polynomial identities.* All entries or power coefficients are compared exactly. For a positive contact identity, every coefficient must be nonnegative and every factor must be one of the stated contact slacks or a square. A matrix identity is tested entry by entry, including zero entries. For a tensor Bernstein certificate, the supplied coefficients are expanded back to the power basis and every coefficient is compared.
2. *Finite vector and cell lists.* Integrality, contact lengths, basis determinants, and isometries are exact rational calculations. A claimed complete short-vector list is checked by exhaustive bounded enumeration using a rational LDL^{T} decomposition. A cell list must additionally have a completeness certificate: either a verified partition covering a fundamental domain, or a full rational polytope enumeration with the stated bounding inequalities.
3. *Linear or scalar terminal witnesses.* The checker reconstructs the necessary inequality system. Nonnegative rational multipliers and a residual formula prove separation on the entire variable box. For a scalar inequality the checker uses rational interval arithmetic, polynomial signs, or positive Bernstein coefficients. Every square-root and logarithm enclosure includes its error bound.

The precise checking rules are those of [4]: the rational linear certificates, the Bernstein rule for univariate polynomials, the logarithm and square-root enclosures and the three obligations of a finite cover of its Appendix D.7, applied by the verifier of its Section 7.

The finite checks of this paper. Table 1 lists, for each of the four forms, the Computational results its argument uses, the written argument, and the verification record; every row also uses the reference data of Computational result 31. The catalogue scan and the preliminary reduction (the certificate database of [4]) are separate from these four local exclusions.

Computational result 31 (the printed reference forms). Each printed A_P of Appendix A is positive definite, has minimum s_P, and its minimum-vector pairs have the number stated there. Their tensors span the 45-dimensional space of symmetric matrices. The normalised determinant is the displayed $\det P$.

The finite verification is fully specified by the matrices: exact elimination checks determinants and positive pivots; the bounded LDL^{T} enumeration checks all vectors at or below the proposed minimum; and exact linear elimination checks the contact-tensor ranks. The exact check establishes these statements for each of the eleven printed reference forms of the series recorded in the certificate archive (the ten forms of the ten local cases and `0060:012815567`, the rank-10 survivor of [3, Section 5]); the four forms of this paper are among them, their input matrices and contact lists match the data used here, and the contact lists are in `paper_2026-10-25/data/printed_forms.json`. The finite reference facts have two roles: they identify the actual lattice and they supply the identities used for all its deformations. For `0001`, the two-centre argument and the normalisation of the cone identity are checked alongside the reference covering record. The cases `0014` and `0059` share the code-lattice geometry of Computational result 16: Lemma 15, the moment identity and the polar argument explain why its saved finite lists imply full coverage and the required ball inclusions; the integral embeddings, phase relations and layer identities are checked separately. The `0015` checks cover both contact cones, all three centre classes, the layer moments and the reference-cell records.

Table 1. Coverage of the four local arguments and their verification records.

form	finite checks	written argument and verification	final record
`0001`	21, 22	radial bound, two centres and covering transfer. Frame, cone identity and two-centre classification checked; reference covering supported by its exact replay.	`0001_structural_closure_verified.json`
`0014`	16, 21, 24	localisation, layer quadratic and common cover. Frames, phases, embeddings and contact identities checked; complete reference-cell records archived.	`0014_structural_closure_verified.json`
`0059`	16, 21, 27	five bands, automatic marking and common cover. Phase, layer, projector and embedding checks connect the band and reference-cell records.	`0059_structural_closure_verified.json`
`0015`	21, 29	two cones, three centres, mean-body proof and all trace ranges. Structural, cone, mean-body and gluing records archived.	`0015_structural_closure_verified.json`

The four *structural* records link the positive identities, phase enumerations and extension bounds to their underlying verification records. The accompanying Lean development verifies the printed Gram matrices, their positive definiteness and determinants

from saved LDL^{T} decompositions, and the three Bernstein certificates of the scalar checks for `0014` (`Hermite11/Certificates/Case0014Scalars.lean`), three of the thirteen polynomial certificates of the series [3, Section 8]. The contact geometry and cell covers use the external exact checks listed above.

Where the records are. The finite records on which the Computational results of this paper rely are the four structural records of Table 1, the reference-cell, contact and moment records of Computational results 16, 22, 24, 27 and 29 that they bind, and the reference data `paper_2026-10-25/data/printed_forms.json` of Computational result 31. The records and their checkers are in the certificate archive `hermite11_local_certificates.zip` of the Zenodo record doi:10.5281/zenodo.23979316. Its `case_map.json` identifies each principal record, its size, the underlying checker sources and recorded result; `certificate_manifest.json` binds the stored files and compressed logical streams. The common archive identifiers and verification procedure are given in [3, Section 9].

The hypotheses, precisely. Theorem 1 rests on:

- The admissible-chain conditions of Definition 2, including the marking and the exact cutoffs.
- the γ_{11} paper [4]: the statements of Section 2.2 (Lemmata 4–11) and of Section 3 imported from it (Lemmata 12 and 13), whose proofs are given there, and the coordinate system of Section 2;
- the Computational results 16, 21, 22, 24, 27, 29 and 31 of this paper, each a finite check whose data are in the certificate archive.

Proof of Theorem 1. Let P be one of the four forms and let $L \supset M \supset N$ be a chain as in Definition 2 whose rank-9 member has a Gram matrix admissibly marked by P. Each of Sections 4.1–4.4 starts from the common admissibility conditions of Section 2 and states its coordinate changes. The four structural cases exhaust their height and phase ranges before applying the covering or flag inequalities; every necessary contraction retains the counterexample, and the cited Computational results assert exactly the finite identities and complete covers used. Theorems 23, 26, 28 and 30 give $\Delta \geq 972$ in the four cases. □

Together with the root paper, the certificate database and case `0052` of [4], and the companion papers [5,6,7,8] for the other five cases, Theorem 1 is one of the inputs of [3, Theorem 14]: every lattice of rank 11 with minimum 4 has determinant at least 972, that is, $\gamma_{11} = 4/\sqrt[11]{972}$, attained by K_{11}. The γ_{11} paper assembles that theorem from these statements exactly as the root paper assembles γ_{10}.

Lean correspondence. The local analytic statements are `0001`: `Hermite11.case0001_of_finite_computational_facts`; `0014`: `Hermite11.case0014_from_reference_data`; `0015`: `Hermite11.case0015_of_computational_facts`; `0059`: `Hermite11.case0059_from_reference_data`. Their finite hypotheses, integral-equivalence wrappers and saved reference records are listed in `case_map.json`. Large cover payloads remain external exact checks.

6 Use of computational tools

Refer to Section 10 of [3].

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A The four reference forms

The following integer matrices are A_P, with $P = (4/s_P)A_P$. They are representatives of the indexed rank-nine forms of Dutour Sikirić and van Woerden [2], in the bases used by the case certificates. For each form we print $s_P = \min A_P$, the determinant $\det P = (4/s_P)^9 \det A_P$ of the form at minimum 4, and the number of its contact pairs (Computational result 31). Each of the four matrices is the record of the data set itself, entry by entry and in the data set's basis: the script `hermite11\_printed\_forms\_check.py` of the Zenodo record [3, Section 9] compares the printed forms of the series with the rows of their keys in the certificate database (which the verification matches to the data set) and reports `PASS` for all eleven, so the chain handed over by [4, Theorem 1], whose rank-9 member is marked by the record, is marked by P. The determinants agree with the root paper's data [3, Table 4]: the lattice M of rank 10 that lies over each form has $\min M^* = 1$, so that $\det M = \det N = \det P$ there.

`0001:010488327`. $s_P = 6$, $\det P = \frac{262144}{243} = \frac{2^{18}}{3^5}$, 72 contact pairs (macadamia 6: the rescaled dual of the 270-vector section of K_{11} lies over it).

$$A_P = \begin{pmatrix} 6 & 2 & 2 & 0 & 2 & 0 & 2 & -5 & 2 \\ 2 & 6 & 0 & 2 & 0 & 2 & 0 & -3 & -2 \\ 2 & 0 & 6 & 2 & 0 & -2 & -2 & -1 & 0 \\ 0 & 2 & 2 & 6 & 2 & -2 & -2 & -1 & 0 \\ 2 & 0 & 0 & 2 & 6 & 0 & 2 & -5 & 2 \\ 0 & 2 & -2 & -2 & 0 & 6 & 2 & -3 & 0 \\ 2 & 0 & -2 & -2 & 2 & 2 & 6 & -5 & 0 \\ -5 & -3 & -1 & -1 & -5 & -3 & -5 & 10 & -1 \\ 2 & -2 & 0 & 0 & 2 & 0 & 0 & -1 & 6 \end{pmatrix}.$$

`0014:001749549`. $s_P = 4$, $\det P = 1024$, 81 contact pairs (macadamia 7: the lattice M_{260} of [3, Table 4] lies over it; M_{260} is the code lattice $M_{\mathcal{B}}$ of Section 3.1, see the end of this appendix).

$$A_P = \begin{pmatrix} 4 & 2 & 2 & 0 & -1 & -1 & -1 & 2 & 2 \\ 2 & 4 & 0 & 2 & -1 & -1 & -1 & 0 & 2 \\ 2 & 0 & 4 & -2 & -1 & -1 & -1 & 2 & 0 \\ 0 & 2 & -2 & 4 & 1 & 1 & 1 & -2 & 0 \\ -1 & -1 & -1 & 1 & 4 & 2 & 2 & -2 & -2 \\ -1 & -1 & -1 & 1 & 2 & 4 & 0 & 0 & -2 \\ -1 & -1 & -1 & 1 & 2 & 0 & 4 & -2 & 0 \\ 2 & 0 & 2 & -2 & -2 & 0 & -2 & 4 & 1 \\ 2 & 2 & 0 & 0 & -2 & -2 & 0 & 1 & 4 \end{pmatrix}.$$

`0059:016902802`. $s_P = 4$, $\det P = 1024$, 78 contact pairs (macadamia 8: a second section of the same lattice M_{260}).

$$A_P = \begin{pmatrix} 4 & 0 & 0 & 0 & -2 & -2 & 0 & 0 & 2 \\ 0 & 4 & 0 & 0 & -2 & -2 & 0 & 2 & 0 \\ 0 & 0 & 4 & 0 & 0 & -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & -2 & 0 & 2 & 0 & 0 \\ -2 & -2 & 0 & -2 & 4 & 2 & -1 & -1 & 0 \\ -2 & -2 & -2 & 0 & 2 & 4 & -1 & 0 & -1 \\ 0 & 0 & 2 & 2 & -1 & -1 & 4 & 1 & 1 \\ 0 & 2 & 0 & 0 & -1 & 0 & 1 & 4 & 1 \\ 2 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 4 \end{pmatrix}.$$

`0015:030291998`. $s_P = 4$, $\det P = 972$, 81 contact pairs (macadamia 5: the 270-vector section of K_{11} lies over it; that section is the reference extension M_2 of Section 4.4).

$$A_P = \begin{pmatrix} 4 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 2 \\ 2 & 4 & 2 & 0 & 0 & 1 & 1 & 0 & 2 \\ 2 & 2 & 4 & 1 & 1 & 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 4 & 0 & 2 & 2 & 2 & 0 \\ 1 & 0 & 1 & 0 & 4 & 1 & 1 & 2 & 2 \\ 1 & 1 & 0 & 2 & 1 & 4 & 2 & 1 & 2 \\ 0 & 1 & 1 & 2 & 1 & 2 & 4 & 2 & 0 \\ 0 & 0 & 2 & 2 & 2 & 1 & 2 & 4 & 0 \\ 2 & 2 & 1 & 0 & 2 & 2 & 0 & 0 & 4 \end{pmatrix}.$$

A direct computation in exact rational arithmetic (bounded LDL^{T} enumeration for the minimum and the minimal vectors, Gaussian elimination for the determinants and for the rank of the contact tensors) confirms, for each of the four matrices, symmetry, the printed minimum s_P, the printed number of contact pairs, the printed determinant $\det P$, and perfection (the cc^{T} span Sym_9). It also confirms the frame identity of Section 4.1 (17 dual pairs of P^{-1}-length $3/4$ with $\sum_i q_i z_i z_i^{\mathsf{T}} = P$, $\sum_i q_i = 12$, and $\sum cc^{\mathsf{T}} = 32P^{-1}$ over the 72 contact pairs), the integrality and the invariants of P_{10} for `0015` ($\det P_{10} = 972$, minimum 4, 135 contact pairs), and that the reference extension of `0059` at $(P, \beta_0, u = 1)$ has determinant 1024, minimum 4 and 130 contact pairs, the invariants of $M_{\mathcal{B}}$. The reference checks are specified by these printed matrices. The certificate archive contains `paper_2026-10-25/verify_paper_data.py`, the exact printed-form records and their contact data.

The lattice over `0014` and `0059`. The root paper describes the lattice M_{260} of macadamias 7 and 8 by its invariants: rank 10, minimum 4, determinant 1024, 260 minimal vectors, $\min M_{260}^* = 1$, with `0014` and `0059` as two of its sections by shortest dual vectors. The code lattice $M_{\mathcal{B}}$ of Section 3.1 has exactly these invariants: an exact enumeration gives

minimum 4 with 130 minimal pairs (the ten $2e_i$ and the 120 sign patterns), determinant 1024, and dual minimum 1 with 130 shortest pairs. Both forms are sections of it. For each of the two forms let

$$P_{10} = \begin{pmatrix} P & -P\beta_0 \\ -\beta_0^{\mathsf{T}} P & 1 + P[\beta_0] \end{pmatrix}$$

be the Gram matrix of the extension of P by its reference centre β_0 (Section 4.2, resp. 4.3) at $u = 1$, in the basis $(b_1, \ldots, b_9, -\beta_0 + f)$, where $b_1, \ldots, b_9$ is the basis of N with Gram matrix P and $f \perp N$ with $|f|^2 = 1/u = 1$, so that the squared lengths are those of (5) with $k' = 0$; it is integral, of determinant 1024, minimum 4, 130 minimal pairs and dual minimum 1. Let $G_B = BB^{\mathsf{T}}$ be the Gram matrix of $M_{\mathcal{B}}$ in the basis whose rows are those of B below. The unimodular matrices U_{0014} and U_{0059},

$$B = \begin{pmatrix}
2&0&0&0&0&0&0&0&0&0\\
0&2&0&0&0&0&0&0&0&0\\
0&0&2&0&0&0&0&0&0&0\\
0&0&0&2&0&0&0&0&0&0\\
0&0&0&0&2&0&0&0&0&0\\
0&1&1&0&1&1&0&0&0&0\\
0&1&0&1&1&0&1&0&0&0\\
1&0&1&0&1&0&0&1&0&0\\
1&0&0&1&1&0&0&0&1&0\\
1&1&1&1&1&0&0&0&0&1
\end{pmatrix}, \qquad U_{0014} = \begin{pmatrix}
1&1&0&1&0&0&-1&0&0&0\\
1&1&1&0&-1&0&-1&1&0&0\\
1&1&0&1&0&0&0&0&0&-1\\
1&1&1&0&0&0&-1&0&0&0\\
1&2&0&1&0&0&-1&0&0&0\\
0&0&0&0&0&0&0&0&1&0\\
-1&-1&-1&1&1&1&1&-1&-1&-1\\
-1&-1&1&-1&0&0&0&0&-1&0\\
0&0&0&0&-1&-1&1&0&1&0\\
-1&-1&-1&-1&0&0&0&0&0&1
\end{pmatrix},$$

$$U_{0059} = \begin{pmatrix}
1&1&0&0&-1&-1&1&1&1&-1\\
0&1&-1&1&-1&0&0&0&0&0\\
0&1&0&1&-1&-1&1&0&0&0\\
1&1&0&0&-1&-1&0&0&0&0\\
1&1&0&1&-2&-1&1&0&0&0\\
0&0&0&-2&1&0&-1&0&0&1\\
0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&-1&0&0&0\\
-2&0&0&0&1&1&0&0&-1&0\\
0&-2&0&0&1&1&0&-1&0&0
\end{pmatrix},$$

satisfy $U^{\mathsf{T}} G_B U = P_{10}$ exactly, with $\det U_{0014} = -1$ and $\det U_{0059} = 1$: the two reference extensions are integrally isometric to $M_{\mathcal{B}}$, which therefore lies over both forms with the invariants of M_{260}. The local proof uses these explicit reference extensions. It needs no identification of a separately named virtual row in the global database.