# The Hermite Constants in Dimensions 10, 11 and 12

**PROOF'S ROOT PAPER**

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**Abstract.** This paper introduces a series of seven papers establishing $\gamma_{10}^{10} = 4^{10}/768$, $\gamma_{11}^{11} = 4^{11}/972$ and $\gamma_{12} = 4/\sqrt{3}$, under the attributed classification of perfect forms of rank nine and the explicitly stated computational inputs. The argument treats the dimensions in order. In dimension ten, a section orthogonal to a shortest dual vector is compared with a perfect form. A completed exact census leaves three possible reference forms, and matrix inequalities exclude a determinant below 768 in each case. In dimension eleven, two successive sections give a chain of ranks eleven, ten and nine. The dimension-ten bound controls the determinant of the last section. The verified global census and preliminary certificates reduce the problem to ten local cases: nine are treated in five companion papers, and one in *A Focus on Hermite Constant 11*. The local arguments combine contact geometry, section bounds and finite rational covers. Mordell's inequality then gives the dimension-twelve upper bound, matched by the Coxeter–Todd lattice. We prove the common reduction, explain the dependency between the papers, and state exactly which finite assertions must be checked. The accompanying Lean development verifies the complete analytic deduction from those inputs; external exact checkers verify the large certificate data.

**Keywords:** Hermite constant · Lattice packings · Perfect forms · Laminated lattices · Coxeter–Todd lattice · Mordell's inequality

## 1 Introduction

How densely can equal balls be packed in $\mathbb{R}^n$ if their centres must form a lattice? Hermite asked the question in this form in 1850 and gave the first general answer, a bound that grows geometrically with $n$. The exact answer, the *Hermite constant* $\gamma_n$, was known in nine dimensions only: $n \leq 8$, by a sequence of proofs spread over two centuries, and $n = 24$, by the work of Cohn and Kumar on the Leech lattice. The classification of the perfect forms of rank 9, recently completed by Dutour Sikirić and van Woerden, adds $n = 9$.

This paper opens a series of seven papers (all made public concurrently) whose aim is to settle the cases $n = 10$, 11 and 12. The three dimensions are treated in order, and each step uses the one before it. $\gamma_{10}$ is obtained from the classification of the

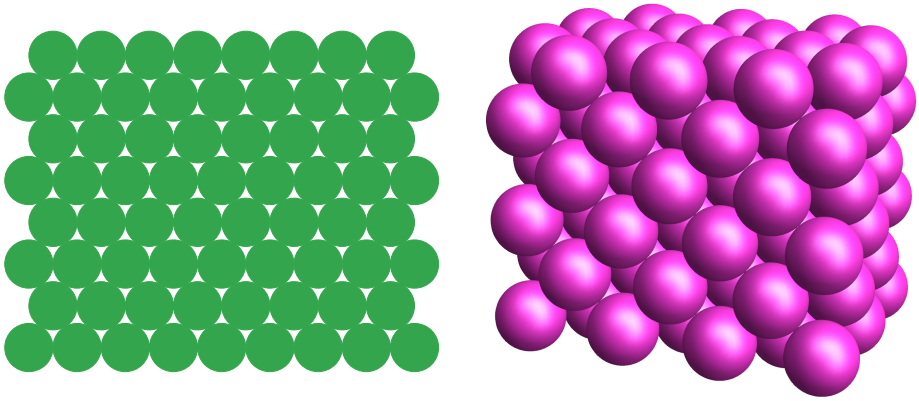

**Fig. 1.** The packing problem illustrated in dimension 2 (in green) and in dimension 3 (in magenta).

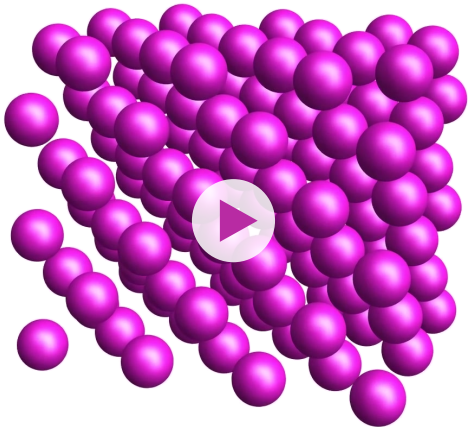

**Fig. 2.** The packing problem in dimension 3, animated.

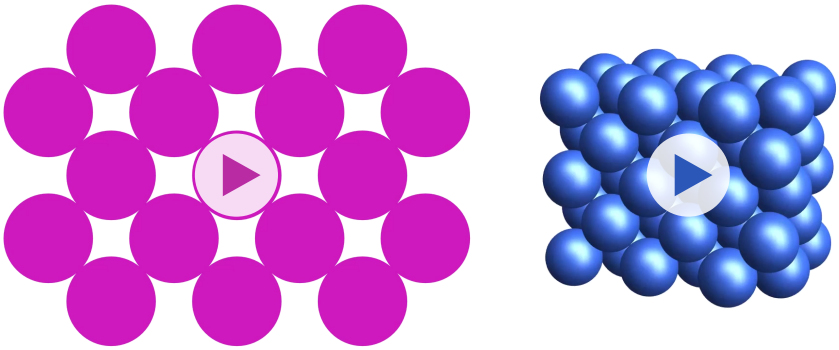

**Fig. 3.** The packing problem in 3D (magenta) and 4D (blue) where one dimension is replaced by time (traversal).

perfect forms of rank 9 by cutting a hypothetical counterexample orthogonally to a shortest dual vector. $\gamma_{11}$ is obtained by the same method applied twice: a first cut takes a hypothetical counterexample of rank 11 to a section of rank 10, a second cut takes that section to rank 9, and $\gamma_{10}$ is needed between the two cuts: it bounds the dual minimum of the rank-10 section, hence the determinant of the rank-9 section, which is then compared with the classification. $\gamma_{12}$ then follows from $\gamma_{11}$ by Mordell's inequality, with no numerical computation at all, because the inequality happens to be an equality between the two record lattices $K_{11}$ and $K_{12}$. All three values are the ones predicted by [25], which guided the whole enterprise: $\gamma_n^n = 4^n/\alpha_{4,n}$, where $\alpha_{4,n}$ is the least determinant of an integral lattice of rank $n$ and minimum 4.

The paper is organised as follows. Section 2 fixes the definitions and Section 3 recalls the problem's history. Section 4 states the plan of the paper series, the tree of papers that carries it out, and what each paper depends on. Section 5 contains the complete proof for $n = 10$. Section 6 states the result for $n = 11$, proves the reduction that connects it to $n = 10$, names the perfect forms that exact certificates settle and the ten that do not, and describes the next paper *A Focus on Hermite Constant 11* [14], where the proof of $\gamma_{11}$ is given. Section 7 derives $n = 12$ from $n = 11$. Section 8 states what the accompanying Lean development checks, and Section 9 what is published with the paper. Appendix A explains the jump from 11 to 12 and to which other pairs of dimensions the same jump applies; Appendices B and C contain the reference forms and a table used in Section 5; Appendix D explains the name *macadamia*, and Appendix E lists where each finite claim of the series is checked. Section 10 records the computational tools used.

The proof combines analytic reductions with finite certificates. The reductions specify the full domain to be checked and prove what each certificate implies. Exact checkers verify the saved witnesses and their coverage.

## 2 Definitions

A *lattice* of rank $k$ in $\mathbb{R}^n$ is $L = B\mathbb{Z}^k$, where $B$ is a real $n \times k$ matrix of rank $k$. Its *Gram matrix* is $Q = B^T B$ ($Q$ depends on the basis $B$), its *determinant* is $\det L = \det Q$, and its *minimum* $\min L$ is the least squared length of a nonzero vector of $L$. Thus $\det L$ is the square of the volume of a fundamental cell. The minimum always denotes a squared length. The *dual* of $L$ is $L^* = \{y \in \operatorname{span} L : y \cdot x \in \mathbb{Z} \text{ for all } x \in L\}$; the dual basis $B(B^T B)^{-1}$ has Gram matrix $Q^{-1}$, so $\det L^* = 1/\det L$.
For a positive definite matrix $Q$ we write $Q[x] = x^T Q x$, $\min Q = \min\{Q[x] : x \in \mathbb{Z}^k \setminus \{0\}\}$ and $\operatorname{Min} Q = \{x \in \mathbb{Z}^k \setminus \{0\} : Q[x] = \min Q\}$; thus $\min L = \min Q$ and $\min L^* = \min Q^{-1}$. Matrices $Q$ and $U^T Q U$ with $U \in GL_k(\mathbb{Z})$ (an integer matrix of determinant $\pm 1$) are *equivalent*; they are the Gram matrices of one lattice in two bases. A positive definite $P$ is *perfect* if the matrices $xx^T$, $x \in \operatorname{Min} P$, span the space $\operatorname{Sym}_k$ of symmetric $k \times k$ matrices; perfection is a property of the equivalence class. We write $I_k$ for the identity matrix, $e_i$ for coordinate vectors, and $\mathbf{1}$ for the all-one vector. The superscript $\mathsf{T}$ denotes transpose; $\operatorname{tr} A$ denotes the sum of the diagonal entries of $A$. For symmetric matrices, $A \succeq 0$ means $x^T A x \geq 0$ for every $x$, and $A \succ 0$ means strict positivity for $x \neq 0$; $A \preceq B$ means $B - A \succeq 0$. The notation $\operatorname{cone}\{A_i\}$ means all finite nonnegative linear combinations of the $A_i$. The Frobenius norm is $\|A\|_F^2 = \operatorname{tr}(A^T A)$; the operator norm is $\|A\|_{\mathrm{op}} = \sup_{|x|=1} |Ax|$.
The *Hermite invariant* of a lattice $L$ of rank $n$ is

$$\gamma(L) = \frac{\min L}{\sqrt[n]{\det L}},$$

unchanged when $L$ is replaced by a similar lattice $cAL$ ($c > 0$, $A$ orthogonal), and the *Hermite constant* is $\gamma_n = \sup \gamma(L)$ over the lattices of rank $n$. Hermite's bound $\gamma_n \leq (4/3)^{(n-1)/2}$ shows that the *supremum* is finite, and Mahler's compactness theorem that it is attained [26]. If $\min L = 4$, the balls of radius 1 centred at the points of $L$ have disjoint interiors, with one centre per cell of volume $\sqrt{\det L}$; so $\gamma_n$ describes the densest packing of equal balls whose centres form a lattice, and

$$\gamma_n \leq c \iff \text{every lattice of rank } n \text{ with } \min L \geq 4 \text{ has } \det L \geq \left(\frac{4}{c}\right)^n. \qquad (1)$$

For $\Leftarrow$, rescale any lattice to minimum exactly 4. For $\Rightarrow$, if $\min L = m \geq 4$ then $\det L \geq (m/c)^n \geq (4/c)^n$. Because of (1), every statement of this series is written in the form "*every lattice of rank $n$ with minimum 4 has determinant at least $D_n$*", and

$$\gamma_n = \frac{4}{\sqrt[n]{D_n}}.$$

## 3 History

*Small dimensions.* Lagrange's reduction of binary quadratic forms (1773) gives $\gamma_2 = 2/\sqrt{3}$ [24]; the hexagonal lattice attains it. Gauss, in his 1831 review of Seeber's book on ternary forms, obtained $\gamma_3 = \sqrt[3]{2}$ [13], attained by the face-centred cubic lattice

$A_3$. Hermite's letters to Jacobi of 1850 introduced the constants $\gamma_n$ for all $n$ and the bound $\gamma_n \le (4/3)^{(n-1)/2}$ [20]. Korkine and Zolotareff determined $\gamma_4 = \sqrt{2}$ (1872) and $\gamma_5 = \sqrt[5]{8}$ (1877) [22,23], attained by $D_4$ and $D_5$, and introduced the reduction theory and the notion of *extreme form* that carry their names. Blichfeldt determined $\gamma_6, \gamma_7$ and $\gamma_8$ in 1935 [1], attained by $E_6$, $E_7$ and $E_8$; Vetčinkin [29] later completed the proofs and established uniqueness. Mordell [27] observed in 1944 that the inequality $\gamma_n^{n-2} \le \gamma_{n-1}^{n-1}$, which he proved by cutting a lattice orthogonally to a shortest dual vector, gives $\gamma_8$ from $\gamma_7$ in one line; the same inequality is our tool in Section 7. No value of $\gamma_n$ was added for more than seventy years. Chaundy published values for $n = 9$ and $n = 10$ in 1946 [2], but the proof was considered incomplete (see [25] for the review); his values are the ones established in [12] and in Section 5 below.



DES SCIENCES ET BELLES-LETTRES. 265

RECHERCHES D'ARITHMÉTIQUE.

PAR M. DE LA GRANGE.

Ces recherches ont pour objet les nombres qui peuvent être représentés par la formule $Bt^2 + Ctu + Du^2$, où $B$, $C$, $D$ sont supposés des nombres entiers donnés, & $t$, $u$ des nombres aussi entiers mais indéterminés. Je donnerai d'abord la maniere de trouver toutes les différentes formes dont les diviseurs de ces sortes de nombres sont susceptibles; je donnerai ensuite une méthode pour réduire ces formes au plus petit nombre possible; je montrerai comment on en peut dresser des Tables pour la pratique; & je ferai voir l'usage de ces Tables dans la recherche des diviseurs des nombres. Je donnerai enfin la démonstration de plusieurs théorèmes sur les nombres premiers de la même forme $Bt^2 + Ctu + Du^2$, dont quelques-uns sont déjà connus, mais n'ont pas encore été démontrés, & dont les autres sont entierement nouveaux.

AVERTISSEMENT.

1. On suppose toujours dans la suite que toutes les lettres désignent des nombres entiers positifs ou négatifs; & on représentera ordinairement par les premieres lettres de l'alphabet les nombres donnés, & par les dernieres les nombres indéterminés.

OBSERVATION.

2. La formule du premier degré $Bt + Cu$, où $B$ & $C$ sont des nombres quelconques donnés & premiers entr'eux, peut représenter un nombre quelconque; mais il n'en est pas de même de la formule du second degré $Bt^2 + Ctu + Du^2$; car nous avons prouvé ailleurs (voyez les Mémoires de l'Académie pour les années 1767 & 1768) que l'équation $A = Bt + Cu$ est toujours résoluble en nombres entiers, quels que soient les

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270 NOUVEAUX MÉMOIRES DE L'ACADÉMIE ROYALE

$y$ & $z$ *étant aussi premiers entr'eux, &* P, Q, R *étant tels que l'on ait* $4PR - Q^2 = 4BD - C^2$ *& de plus* Q *n'étant ni* $\succ$ P *ni* $\succ$ R, *abstraction faite des signes de* P, Q & R.

La démonstration de ce Théoreme suit naturellement des deux Théorèmes précédens, & de leurs Corollaires.

COROLLAIRE 1.

9. Si $4BD - C^2$ est un nombre positif, il faudra que $4PR$ soit aussi positif; donc, à cause que $P = \text{ou} \succ Q$, & $R = \text{ou} \succ Q$, il est clair que $4PR$ sera aussi $= \text{ou} \succ 4Q^2$, & par conséquent $4PR - Q^2 = \text{ou} \succ 3Q^2$; donc on aura aussi $4BD - C^2 = \text{ou} \succ 3Q^2$, & de là

$$Q = \text{ou} \prec V\left(\frac{4BD - C^2}{3}\right).$$

COROLLAIRE 2.

10. Soit maintenant $4BD - C^2$ un nombre négatif, en sorte que $C^2 - 4BD$ soit positif; on aura donc dans ce cas $Q^2 - 4PR \succ 0$, ce qui, à cause que Q n'est jamais $\succ P$ ni $\succ R$, ne peut avoir lieu à moins que $4PR$ ne soit un nombre négatif; ainsi $-4PR$ sera un nombre positif $= \text{ou} \succ 4Q^2$, à cause de $P = \text{ou} \succ Q$ & $R = \text{ou} \succ Q$; de sorte que $Q^2 - 4PR$ sera $= \text{ou} \succ 5Q^2$, & par conséquent $C^2 - 4BD$ sera aussi $= \text{ou} \succ 5Q^2$; donc il faudra que

$$Q = \text{ou} \prec V\left(\frac{C^2 - 4BD}{5}\right).$$

COROLLAIRE 3.

11. Donc, puisque Q doit être un nombre entier, on ne pourra prendre pour Q que les nombres entiers positifs ou négatifs qui ne surpasseront pas les limites trouvées, en comprenant aussi le zéro parmi les nombres entiers; d'où l'on voit que Q ne pourra jamais avoir qu'un certain nombre de valeurs différentes.

**Fig. 4.** Where it starts. Lagrange's *Recherches d'arithmétique* in the *Nouveaux Mémoires* of the Berlin Academy for 1773 (left, p. 265), and its Corollary 1 (right, p. 270, art. 9): a reduced form $Py^2 + Qyz + Rz^2$ satisfies $Q \le \sqrt{(4BD - C^2)/3}$, which is $\gamma_2 = 2/\sqrt{3}$. Scan: Princeton University copy, HathiTrust.

*Perfect forms and Voronoi.* Voronoi proved in 1908 that a form whose Hermite invariant is a local maximum is perfect, that there are finitely many perfect forms of each rank up to equivalence and scaling, and gave an algorithm that enumerates them by walking from one perfect form to its neighbours [31]. The numbers of perfect forms of ranks 1 to 8 are $1, 1, 1, 2, 3, 7, 33, 10916$ (the last by Dutour Sikirić, Schürmann and Vallentin [11]); in rank 9 there are $2\,237\,251\,040$ perfect forms, enumerated by Dutour Sikirić and van Woerden [12], who infer $\gamma_9 = 2$, attained by the laminated lattice $\Lambda_9$. The data set of that classification is the computational foundation of the present series. Martinet's book [26] is the reference for the theory.

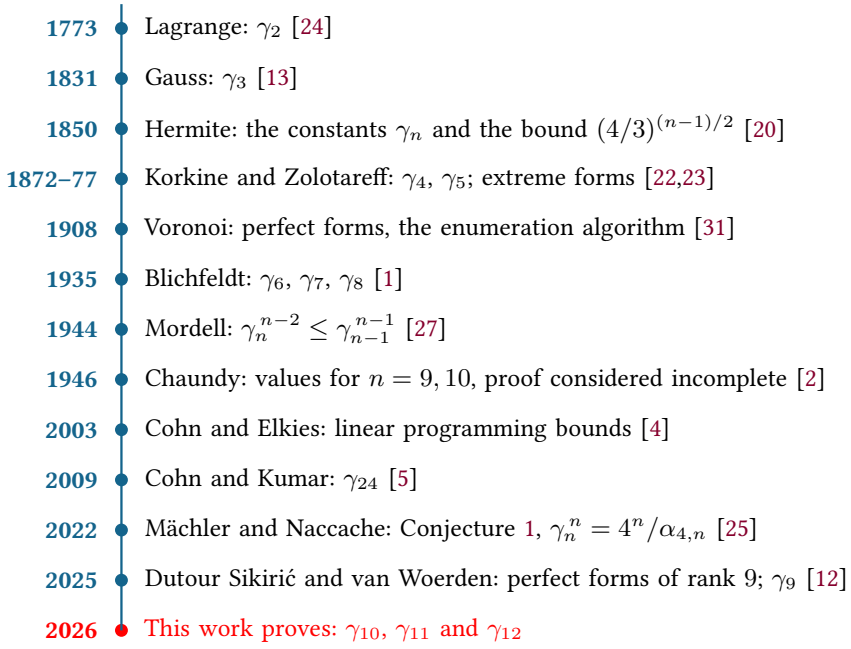



**Fig. 5.** Progress on the Hermite constants: the values established, the tools, the conjecture of [25] that this series pursues, and (in red) the values proved here.

*Dimension 24 and sphere packings.* Cohn and Kumar proved $\gamma_{24} = 4$ in 2009, with equality only for the Leech lattice [5], using the linear programming bounds of Cohn and Elkies [4]. Viazovska [30] and Cohn, Kumar, Miller, Radchenko and Viazovska [6] proved the optimality of $E_8$ and of the Leech lattice among all sphere packings in 2017. In dimensions 10, 11 and 13, the nonlattice packings $P_{10c}$, $P_{11a}$ and $P_{13a}$ of [9, Ch. 5] are denser than $\Lambda_{10}$, $K_{11}$ and $K_{13}$, respectively. The three-point bounds of Cohn, de Laat and Salmon [3] give upper bounds for general packings.

*The record lattices.* Conway and Sloane's laminated lattices $\Lambda_n$ [7] and the lattices $K_n$ built around the Coxeter–Todd lattice $K_{12}$ [10] are the densest known lattices for $n \le 24$ [9, Ch. 1, Table 1.2]. Scaled to minimum 4, the determinants of the records are the numbers $D_n$ of Table 1: $D_{10} = 768$ ($\Lambda_{10}$), $D_{11} = 972$ ($K_{11}$), $D_{12} = 729$ ($K_{12}$).

*The conjecture.* Mächler and Naccache [25] noticed that all nine known values of $(4/\gamma_n)^n$ coincide with $\alpha_{4,n}$, the least determinant of an integral lattice of rank $n$ with minimum 4, a quantity tabulated by Conway and Sloane [9], and conjectured the equality for all $n \le 24$.

*Conjecture 1 (Mächler–Naccache [25]).* For $1 \le n \le 24$, $\gamma_n^n = 4^n/\alpha_{4,n}$.

The values of $\alpha_{4,n}$ for $9 \le n \le 23$ are those of the record lattices, so the conjecture says that the records are optimal. Table 1 lists them, and Figure 6 plots the corresponding constants. The present series hunts the conjectured values for $n = 10$, 11 and 12.

**Table 1.** The conjectured values $\gamma_n^n = 4^n/\alpha_{4,n}$ of [25] and their status; for $n = 10, 11, 12$ they are $2^{12}/3$, $2^{20}/3^5$ and $2^{24}/3^6$. The proofs for $n = 10$, 11, 12 assume Hypothesis 2. Previously known results are in blue, the three values reported in this paper are in red.

| $n$ | $\alpha_{4,n} = D_n$ | record lattice | status of $\gamma_n^n = 4^n/\alpha_{4,n}$ |
|---|---|---|---|
| 1–4 | 4, 12, 32, 64 | $\mathbb{Z}$, $A_2$, $A_3$, $D_4$ | proved, classical |
| 5–8 | 128, 192, 256, 256 | $D_5$, $E_6$, $E_7$, $E_8$ | proved, classical |
| 9 | 512 | $\Lambda_9$ | proved [12] |
| 10 | 768 | $\Lambda_{10}$ | This work: proved (Thm. 3) |
| 11 | 972 | $K_{11}$ | This work: proved (Thm. 14, cf. [14]) |
| 12 | 729 | $K_{12}$ | This work: proved from $n = 11$ (Thm. 17) |
| 13 | 972 | $K_{13}$ | still conjectured [25] |
| 14 | 768 | $\Lambda_{14}$ | still conjectured [25] |
| 15 | 512 | $\Lambda_{15}$ | still conjectured [25] |
| 16 | 256 | $\Lambda_{16} = BW_{16}$ | still conjectured [25] |
| 17–20 | 256, 192, 128, 64 | $\Lambda_{17}, \ldots, \Lambda_{20}$ | still conjectured [25] |
| 21–23 | 32, 12, 4 | $\Lambda_{21}$, $\Lambda_{22}$, $\Lambda_{23}$ | still conjectured [25] |
| 24 | 1 | Leech | proved [5] |

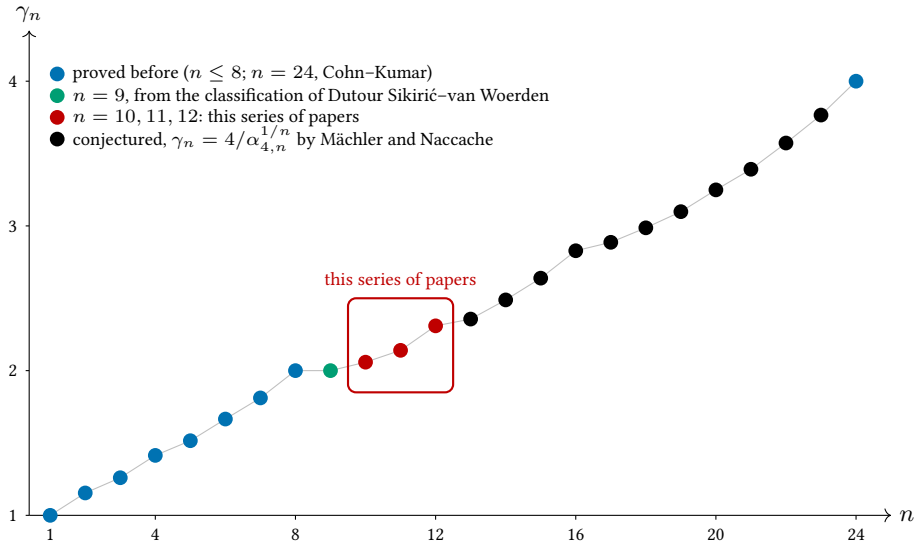



**Fig. 6.** The Hermite constants $\gamma_n = 4/D_n^{1/n}$ for $n \leq 24$, with $D_n = \alpha_{4,n}$ the determinant of the record lattice at minimum 4 (Table 1): $\gamma_{10} = 4/\sqrt[10]{768} \simeq 2.0584$, $\gamma_{11} = 4/\sqrt[11]{972} \simeq 2.1402$, $\gamma_{12} = 4/\sqrt{3} \simeq 2.3094$. For the open dimensions the points are the values conjectured in [25].

## 4 The plan

The series proves the three values in the order 10, 11, 12. The inputs are the classification below, the stated computational results, and the cited classical lattice results. Each computational result specifies a finite assertion that can be checked separately; it is an explicit premise of the analytic proof.

**Hypothesis 2.** The data set of [12] contains, for every perfect positive definite $9 \times 9$ matrix, an integral Gram matrix equivalent to a positive multiple of it.

Hypothesis 2 says that the data set of [12] records a *complete* classification [12, Theorem 1]. The arguments use it only for the perfect matrices whose determinant at minimum 4 is below an explicit bound: $c^* = 4 \cdot 768^{4/5} \simeq 813.5$ for $n = 10$ and $e_{\mathrm{cut}} = 4\,(4 \cdot 972^{9/11})^{9/10}/\sqrt[10]{768} \simeq 1136.0$ for $n = 11$.

*Step 1: dimension 10 (Section 5).* Suppose a lattice of rank 10 has minimum 4 and determinant below 768. Replacing it by a multiple of its dual if necessary, we cut it orthogonally to a shortest dual vector; the section $M$ has rank 9, minimum at least 4 and determinant $d < c^*$, and the determinant of the lattice is at least $3d/(4u)$ with $u = \min M^*$. Voronoi's theory provides a perfect form $P$ of rank 9 with $\mathrm{tr}(Q^{-1}P) \leq 9$ for a Gram matrix $Q$ of $M$ and $\det P \leq d$. An exact search of the data of [12] finds only three perfect forms of determinant below $c^*$ at minimum 4, and for each of them a matrix inequality turns $\mathrm{tr}(Q^{-1}P) \leq 9$ and $\min Q \geq 4$ into $d \geq 1024u$, so the determinant of the lattice is at least 768. The step uses neither $\gamma_8$ nor $\gamma_9$; its only computational input is the search.

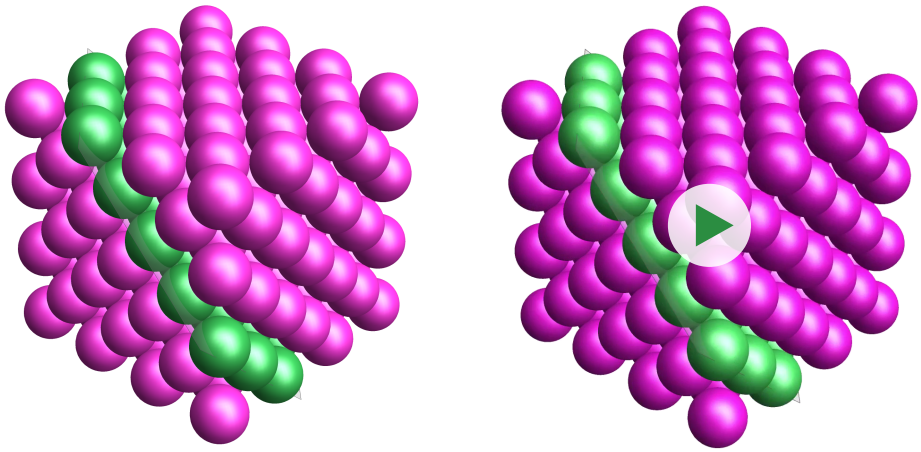


**Fig. 7.** The cut, in dimension three. Left: the densest packing of dimension 3 sliced by a plane orthogonal to one of its shortest dual vectors; the balls centred on the plane (in green) form the densest packing of dimension 2. Right, animated (click to play): the plane sweeps through the packing and lights up one layer after another. Every step of the series is this picture, one or two dimensions up: the section of a lattice by the hyperplane orthogonal to a shortest dual vector is a lattice of one dimension less.

*Step 2: dimension 11 (Section 6 and [14]).* Suppose a lattice of rank 11 has minimum 4 and determinant below 972. The same cut gives a lattice $M$ of rank 10 with minimum at least 4 and determinant below $d_{\mathrm{cut}} = 4 \cdot 972^{9/11} \simeq 1113.0$; this is where Step 1 enters: $\gamma_{10}$ bounds the minimum of $M^*$, so a second cut, orthogonal to a shortest vector of $M^*$, gives a lattice $N$ of rank 9 with minimum at least 4 and determinant below $e_{\mathrm{cut}}$. Voronoi's theory and the data of [12] then provide a perfect form of rank 9 of determinant below $e_{\mathrm{cut}}$ for $N$; there are 4 487 813 of them instead of three. For each of them we must show that every chain $L \supset M \supset N$ of this kind has $\det L \geq 972$. For all but ten of the forms an exact certificate shows more: no lattice $M$ of rank 10 with minimum 4 and determinant below $d_{\mathrm{cut}}$ has a section $N$ over the form at all. The ten exceptions, called here the *macadamias*,[1] are the forms that lie under an actual lattice of rank 10 of determinant below $d_{\mathrm{cut}}$ (such as $\Lambda_9$ under $\Lambda_{10}$, or the sections of $K_{11}$). Their rank-eleven extensions must therefore be controlled directly. Several cases share one contact or section argument; the others use their own finite reductions. Section 6 names them.

[1] After the macadamia nut (Appendix D); these ten forms require the local arguments of the series.

*The chain, and where $\gamma_{10}$ enters.* The two cuts of Step 2 produce a chain of three lattices, and the role of $\gamma_{10}$ is to bound the dual minimum of the middle one:

$$\begin{array}{ccccc} L & \supset & M & \supset & N \\ \text{rank } 11 & & \text{rank } 10 & & \text{rank } 9 \\ \Delta < 972 & & 768 \le d < d_{\text{cut}} & & \det N = ud < e_{\text{cut}} \end{array}$$

Here $\Delta = \det L$, $d = \det M$ and $u = \min M^*$; each cut is orthogonal to a shortest dual vector, so that $\Delta \ge 3d/(4u)$ (first cut) and $\det N = ud$ (second cut). The dimension-ten theorem bounds $u$ by $4/\sqrt[10]{768\,d}$, giving the cutoff $e_{\text{cut}}$ for the rank-nine search. Step 1 applies the same construction in ranks 10 and 9, with the cutoff $c^*$ in place of $d_{\text{cut}}$. Section 6 starts with a table of these quantities.

*Step 3: dimension 12 (Section 7).* Mordell's inequality $\gamma_{12}^{10} \le \gamma_{11}^{11}$ and the identity $4^{11}/972 = 4^{10}/3^5$ give $\gamma_{12} \le 4/\sqrt{3}$, which $K_{12}$ attains. No computation is involved, and no slack is lost: the inequality is an equality between $K_{11}$ and $K_{12}$. Appendix A shows that this is an instance of a general mechanism (a record lattice similar to its dual, whose section by a minimal dual vector is the record one dimension lower) and that the same jump is exact for the pairs $(3,4)$, $(7,8)$, $(11,12)$, $(15,16)$ and $(23,24)$, and for no other pair up to 24.

*The tree of papers.* Figure 8 shows how the thread of our seven papers[2] is organised. This paper, the root paper of the series, proves Step 1 and Step 3 and states Step 2. The *$\gamma_{11}$ paper*, entitled *A Focus on Hermite Constant 11* [14], proves Step 2: it recalls the root paper, states the plan for 11, proves the exclusion of every form but the ten macadamias, explains the kinds of certificates used and why they work, and points to a data pack that contains the certificates together with the code that tests them; it then lists the ten macadamias and names, for each, the paper that cracks it. Five *companion papers* settle nine macadamias; `0052` is settled in [14]. The methods are contacts [15] (four forms), reduction by sections [16] (two forms), a finite spectral relaxation [17], one selected contact with two linked blocks [18], and unequal triangles with adapted references [19] (one form each). Every paper of the tree closes one clearly delimited problem and states the hypotheses it inherits from the others.

## 5 Dimension 10

**Theorem 3.** *Assume Hypothesis 2 and Computational result 9. Every lattice of rank 10 with minimum 4 has determinant at least 768. Hence $\gamma_{10} = \sqrt[10]{4096/3} = 2.0583720\ldots$, attained by $\Lambda_{10}$.*

*Outline.* Let $c^* = 4 \cdot 768^{4/5}$, so $c^{*5} = 2^{42}3^4$ and $c^* = 813.48\ldots$ Suppose a lattice of rank 10 has minimum 4 and determinant below 768. The proof has four steps.

[2] All made public concurrently.

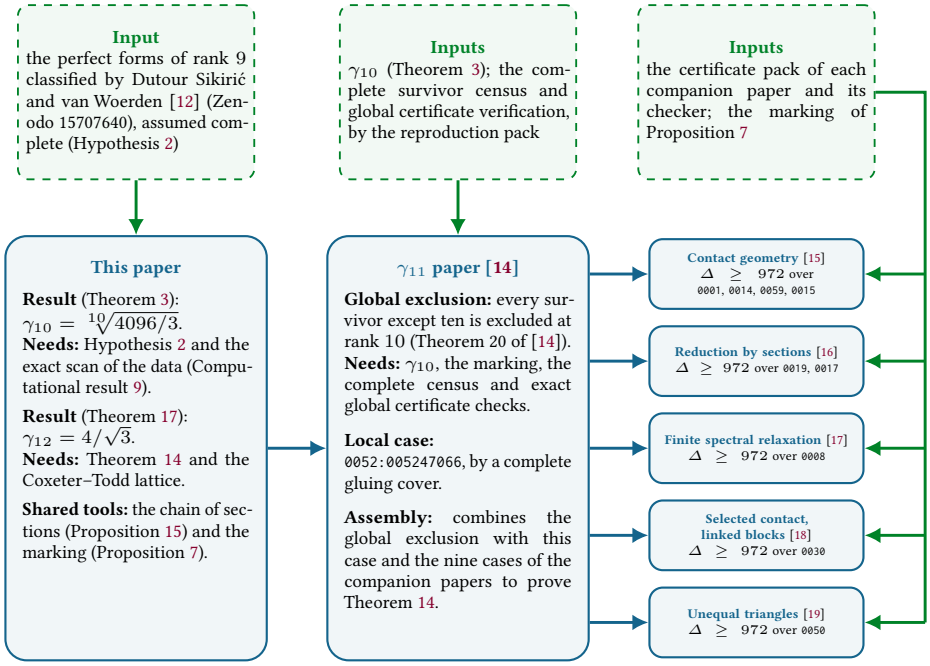



**Fig. 8.** The tree of papers, with one result per box and the input it requires; the dashed boxes above name the mathematical and finite inputs that enter at each level. The root paper (this paper) proves 10, announces 11 and derives 12; the $\gamma_{11}$ paper [14] performs the global assembly and settles 0052; five companions settle the other nine cases.

1. (Section 5.2.) Replacing the lattice by a multiple of its dual if necessary (this keeps minimum $4$ and determinant below $768$), we cut it orthogonally to a shortest dual vector. The resulting sublattice $M$ of rank $9$ has determinant $d < c^*$, and the determinant of the lattice is at least $3d/(4u)$, where $u = \min M^*$.
2. (Section 5.3.) For a Gram matrix $Q$ of $M$ there is a perfect $P$ with $\min P = 4$, $\operatorname{tr}(Q^{-1}P) \le 9$ and $\det P \le d < c^*$.
3. (Section 5.4.) A computation over the data of [12] shows that, up to equivalence, only three perfect matrices have minimum $4$ and determinant below $c^*$: $P_\Lambda$ (the lattice $\Lambda_9$), $P_S$ (an extension of $E_8$ by one vector; Appendix B), and $P_K$, which [28] lists as KAPPA9.2.
4. (Section 5.5.) For each of the three, $\operatorname{tr}(Q^{-1}P) \le 9$ and $\min Q \ge 4$ imply $d \ge 1024u$, so the determinant of the lattice is at least $768$, a contradiction.

Computational result 9 is the finite input in Step 3; Hypothesis 2 supplies the classification. The remaining steps are proved below. Section 8 describes their Lean formalisation. Figure 9 shows the four steps.

### 5.1 Sections of a lattice

**Fact 4.** *Let $L$ be a lattice of rank $n$ in $\mathbb{R}^n$, let $v \in L^*$ be primitive (not $k$ times a vector of $L^*$ for an integer $k \ge 2$), let $M = L \cap v^\perp$, and let $\pi$ be the orthogonal projection onto $v^\perp$. Then* a *$M$ has rank $n-1$ and $\det M = |v|^2 \det L$;* b *$M^* = \pi(L^*)$;* c *every $w \in M^*$ has a preimage $y \in L^*$ under $\pi$ with $|y|^2 \le |w|^2 + |v|^2/4$.*

*Proof.* a The set $\{v \cdot x : x \in L\}$ is a subgroup $g\mathbb{Z}$ of $\mathbb{Z}$. Since $v/g \in L^*$ and $v$ is primitive, $g = 1$. So some $b \in L$ has $v \cdot b = 1$, and $L = M \oplus \mathbb{Z}b$ because $x - (v \cdot x)b \in M$; hence $M$ has rank $n-1$. Appending $b$ to a basis of $M$ gives a basis of $L$, and Gram–Schmidt gives $\det L = \det M \cdot \operatorname{dist}(b, v^\perp)^2 = \det M/|v|^2$.
b For $y \in L^*$ and $x \in M$, $\pi(y) \cdot x = y \cdot x \in \mathbb{Z}$. Conversely, for $w \in M^*$ the vector $y = w - (w \cdot b)v$ satisfies $y \cdot x = w \cdot x$ for $x \in M$ and $y \cdot b = 0$, so $y \in L^*$ and $\pi(y) = w$.
c The preimages of $w$ are $y + kv$, $k \in \mathbb{Z}$, since $L^* \cap \mathbb{R}v = \mathbb{Z}v$. Writing $y = w + \theta v$, we get $|y + kv|^2 = |w|^2 + (\theta + k)^2|v|^2$; choose $k$ with $|\theta + k| \le 1/2$. □

**Fact 5.** *Every positive definite $n \times n$ matrix $X$ satisfies*

$$\det X \ge \left(\frac{3}{4}\right)^{\frac{n(n-1)}{2}} (\min X)^n.$$

*Proof.* Induction on $n$; the case $n = 1$ is clear. Let $L$ be a lattice with Gram matrix $X$, let $x \in L$ be a shortest nonzero vector, $\mu = |x|^2 = \min X$, and $\pi$ the orthogonal projection onto $x^\perp$. As $x$ is primitive in $L$, it extends to a basis $x, y_2, \dots, y_n$ of $L$.
Then $\pi(L)$ has basis $\pi(y_2), \dots, \pi(y_n)$, and Gram–Schmidt gives $\det L = \mu \det \pi(L)$.
For $y \in L$ with $\pi(y) \ne 0$, choose $k \in \mathbb{Z}$ so that the component of $y + kx$ along $x$ has length at most $|x|/2$; since $y + kx \ne 0$, $|\pi(y)|^2 \ge |y + kx|^2 - \mu/4 \ge 3\mu/4$.
So $\min \pi(L) \ge 3\mu/4$, and by induction:

$$\det L \ge \mu \left(\frac{3}{4}\right)^{\frac{(n-1)(n-2)}{2}} \left(\frac{3\mu}{4}\right)^{n-1} = \left(\frac{3}{4}\right)^{\frac{n(n-1)}{2}} \mu^n.$$

□

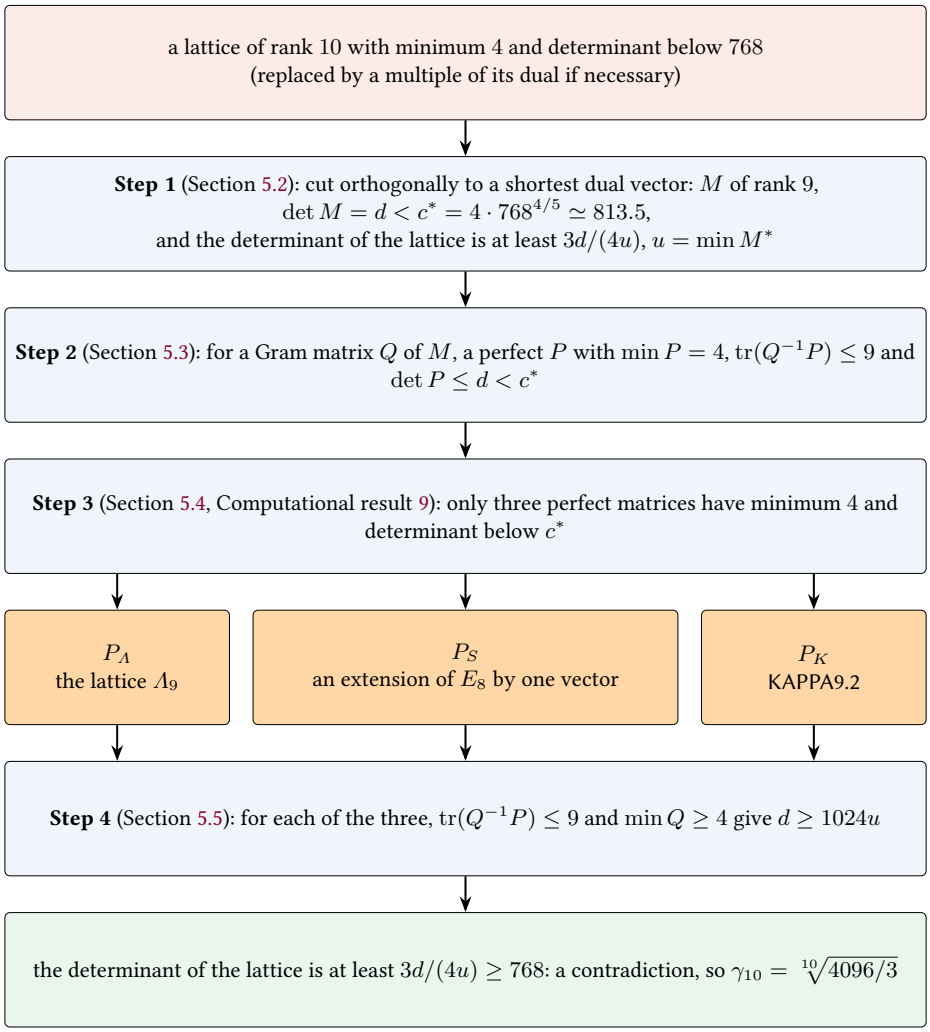



**Fig. 9.** The proof of Theorem 3 in four steps. It assumes Hypothesis 2 (the classification is complete) and Computational result 9 (the three forms).

### 5.2 Step 1: a sublattice of rank 9

**Proposition 6.** *Let $L$ have rank 10, minimum 4 and determinant $D < 768$. There is a lattice $L'$ of rank 10 with minimum 4 and determinant $D' < 768$, and a sublattice $M \subset L'$ of rank 9, with*

$$\min M \geq 4, \qquad d := \det M < c^*, \qquad D' \geq \frac{3d}{4u} \text{ where } u = \min M^*.$$

*Proof.* Write $\gamma(K) = \min K/(\det K)^{1/10}$. Let $L'$ be the multiple with minimum 4 of whichever of $L$, $L^*$ has the larger $\gamma$. Then $\gamma(L'^*) \le \gamma(L')$ and $\gamma(L') \ge \gamma(L)$, so $D' = (4/\gamma(L'))^{10} \le (4/\gamma(L))^{10} = D < 768$.
Let $v$ be a shortest nonzero vector of $L'^*$; it is primitive. Let $a = |v|^2$ and $M = L' \cap v^\perp$. Then $\min M \ge 4$, and $d = aD'$ by Fact 4 a. From $aD'^{1/10} = \gamma(L'^*) \le \gamma(L') = 4D'^{-1/10}$ we get $a \le 4D'^{-1/5}$, so $d \le 4D'^{4/5} < c^*$.
Let $w \in M^*$ with $|w|^2 = u$. By Fact 4 b c there is $y \in L'^*$ with $\pi(y) = w \ne 0$ and $|y|^2 \le u + a/4$. Since $v$ is a shortest nonzero vector, $a \le u + a/4$. So $a \le 4u/3$ and $D' = d/a \ge 3d/(4u)$. □

### 5.3 Step 2: a perfect form $P$ with $\mathrm{tr}(Q^{-1}P) \le 9$

The set of positive definite matrices with minimum at least 4 is convex, and Voronoi showed that its corners are perfect matrices [31]. Minimizing the linear function $X \mapsto \mathrm{tr}(Q^{-1}X)$ over this set reaches a corner, where the value is at most the value 9 at $Q$.

**Proposition 7.** *Let $Q$ be a positive definite $9 \times 9$ matrix with $\min Q \ge 4$. There is a perfect $P$ with $\min P = 4$, $\mathrm{tr}(Q^{-1}P) \le 9$ and $\det P \le \det Q$.*
(Marking.) *More precisely, one can choose a perfect minimum-four form $P$ and nonnegative numbers $w_c$, indexed by one representative $c$ of each opposite contact pair $\pm c$ of $P$ (the contacts of $P$ are the vectors of $\mathrm{Min}\, P$), such that*

$$Q^{-1} = \sum_c w_c\, cc^T, \quad \sum_c w_c \le \frac{9}{4}, \quad w_c \le \frac{1}{4}, \quad \mathrm{tr}(Q^{-1}P) \le 9, \quad \det P \le \det Q. \tag{2}$$

The first statement is what dimension 10 uses. The second, which we call a *marking* of $Q$ by $P$, is what the domain certificates of dimension 11 use (Section 6.2): it says that $Q^{-1}$ lies in the cone spanned by the $cc^T$, $c \in \mathrm{Min}\, P$ (the Voronoi domain of $P$), with explicit bounds on the weights. The marking contains more information than the trace bound. For example, in dimension two, $P = \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}$ and $Q = \begin{pmatrix} 8 & 4 \\ 4 & 8 \end{pmatrix}$ have $\min P = 4$, $\min Q = 8$, $\mathrm{tr}(Q^{-1}P) = 5/3 \le 2$, and the contacts of $P$ up to sign are $e_1$, $e_2$, $e_1 + e_2$, so every matrix in their cone has a nonnegative off-diagonal entry, whereas $(Q^{-1})_{12} = -1/12$.

*Proof.* Let $\mathcal{R}$ be the convex set of positive definite $X \in \mathrm{Sym}_9$ with $\min X \ge 4$, and let $f(X) = \mathrm{tr}(Q^{-1}X)$. Then $Q \in \mathcal{R}$ and $f(Q) = 9$.
*$f$ attains its minimum on $\mathcal{R}$.* Let $X \in \mathcal{R}$ with $f(X) \le 9$. Then $\mathrm{tr}\, X \le \lambda_{\max}(Q) f(X)$, so the eigenvalues of $X$ are at most $\Lambda_Q = 9\lambda_{\max}(Q)$.
Fact 5 gives $\det X \ge \delta_0 := 4^9(3/4)^{36}$, so the eigenvalues of $X$ are at least $\delta_0/\Lambda_Q^8$. Hence the set $\{X \in \mathcal{R} : f(X) \le 9\}$ is bounded and closed, so compact, and $f$ attains a minimum $f_0 \le 9$ on it. The set of minimizers is compact; let $P$ be a minimizer with the largest Frobenius norm $\|P\|_F$. Since $\|\cdot\|_F^2$ is strictly convex, $P$ is not the midpoint of two distinct minimizers.

*$P$ is perfect.* Let $T = \{x \in \mathbb{Z}^9 : P[x] = 4\}$. Suppose the $xx^T$, $x \in T$, do not span $\mathrm{Sym}_9$, and choose $N \neq 0$ in $\mathrm{Sym}_9$ with $N[x] = 0$ for $x \in T$; let $\|N\|$ be its operator norm. Only finitely many $x \in \mathbb{Z}^9$ have $P[x] \leq 5$, so $\eta = \inf\{P[x] - 4 : x \notin T \cup \{0\}\} > 0$. If $|s|\,\|N\| \leq \lambda_{\min}(P)\,\eta/(4+\eta)$, then $P + sN$ is positive definite, equals $4$ on $T$, and for $x \notin T \cup \{0\}$ satisfies $(P+sN)[x] \geq P[x](1-|s|\,\|N\|/\lambda_{\min}(P)) \geq 4$. So $P \pm sN \in \mathcal{R}$. Minimality of $f_0$ forces $f(N) = 0$, so $P \pm sN$ are minimizers with midpoint $P$, a contradiction. Hence the $xx^T$, $x \in T$, span $\mathrm{Sym}_9$; in particular $\min P = 4$ and $P$ is perfect.

*Bounds.* $\mathrm{tr}(Q^{-1}P) = f_0 \leq 9$. The eigenvalues of $Q^{-1/2}PQ^{-1/2}$ are positive with sum at most $9$, so by AM–GM their product $\det P/\det Q$ is at most $1$.

*Marking.* Let $X = Q^{-1}$; recall that

$$\mathcal{R} = \{Y \in \mathrm{Sym}_9 : Y \text{ positive definite},\ Y[z] \geq 4 \text{ for } z \in \mathbb{Z}^9 \setminus \{0\}\}.$$

Use precisely the minimizer $P$ of $Y \mapsto \mathrm{tr}(XY)$ constructed above. The argument so far establishes existence, perfection, $\mathrm{tr}(XP) \leq 9$, and $\det P \leq \det Q$. It remains to prove the contact expansion.

Let $\mathcal{C}$ be the finite set of opposite-pair representatives of the contacts of $P$. There is a positive inactive-contact gap

$$\eta = \inf\{P[z] - 4 : z \in \mathbb{Z}^9 \setminus \{0\},\ P[z] > 4\} > 0.$$

Indeed, positive definiteness makes every bounded sublevel finite. Consider a symmetric matrix $H$ with $H[c] \geq 0$ for all $c \in \mathcal{C}$. For sufficiently small $t > 0$, choose

$$\delta = \frac{t\|H\|_{\mathrm{op}}}{\lambda_{\min}(P)} \leq \frac{\eta}{4+\eta} < 1.$$

Then $P + tH$ is positive definite. Its contact values are at least four, and every inactive integer vector satisfies

$$(P + tH)[z] \geq (1-\delta)P[z] \geq (1-\delta)(4+\eta) \geq 4.$$

Thus $P + tH \in \mathcal{R}$, and minimality gives $\mathrm{tr}(XH) \geq 0$.

The finite-dimensional dual-cone theorem, equivalently Farkas' lemma, now gives $X = \sum_{c \in \mathcal{C}} w_c\, cc^T$ with $w_c \geq 0$. Explicitly, if $X$ were outside this finitely generated closed cone, separation would produce exactly an $H$ with all $H[c] \geq 0$ but $\mathrm{tr}(XH) < 0$, contradicting the preceding paragraph.

Since $P[c] = 4$, taking the trace against $P$ yields $4\sum_c w_c = \mathrm{tr}(XP) \leq 9$. Also $w_c\, cc^T \preceq X$. Conjugating by $Q^{1/2}$ gives $w_c Q[c] \leq 1$, and $Q[c] \geq 4$ implies $w_c \leq 1/4$. □

A marking is carried along by a change of basis: if $Q' = U^{-T}QU^{-1}$ and $P' = U^{-T}PU^{-1}$ with $U \in GL_9(\mathbb{Z})$, then $Q'^{-1} = \sum_c w_c\,(Uc)(Uc)^T$ and the $Uc$ are the contacts of $P'$, with the same weights. So the marking of a Gram matrix of a lattice by a perfect form $P$ may be transported to any reference matrix equivalent to $P$, as in the proof of Theorem 3 below, where $P = U^T P_{\mathrm{ref}} U$.

### 5.4 Step 3: the perfect matrices of small determinant

Appendix B gives the matrices $P_\Lambda$, $P_S$, $P_K$. Proposition 8 is a proof; it uses one computational result, stated separately as Computational result 9.

**Proposition 8.** *Assume Hypothesis 2 and Computational result 9. Every perfect* $9 \times 9$ *matrix* $P$ *with* $\min P = 4$ *and* $\det P < c^*$ *is equivalent to exactly one of* $P_\Lambda$, $P_S$, $P_K$.

*Proof.* Let $P$ be perfect with $\min P = 4$ and $\det P < c^*$. By Hypothesis 2 there are a matrix $G$ of the data and $\lambda > 0$ with $P$ equivalent to $\lambda G$. Equivalent matrices have the same minimum and determinant, so $\lambda = 4/\min G$. Let $m$ be the least diagonal entry of $G$. Since $\min G \le m$,

$$c^* > \det P = \frac{4^9 \det G}{(\min G)^9} \ge \frac{4^9 \det G}{m^9},$$

and $c^* < 814$. So $4^9 \det G < 814\, m^9$. By Computational result 9, $G$ is one of three matrices, $\min G = m$, and $P$, being equivalent to $(4/m)G$, is equivalent to the corresponding reference matrix. The three reference matrices have different determinants (Appendix B), so they are pairwise inequivalent. □

**Computational result 9.** For a matrix $G$ of the data of [12], let $m$ be its least diagonal entry. Exactly three of the 2 237 251 040 matrices satisfy $4^9 \det G < 814\, m^9$. For each of the three, $\min G = m$, and $(4/m)G$ is equivalent to one of $P_\Lambda$, $P_S$, $P_K$; each of $P_\Lambda$, $P_S$, $P_K$ occurs once.

Computational result 9 is an exact census. Its verification is a streaming calculation: for each matrix compute $\det G$ by integer elimination and compare $4^9 \det G$ with $814 m^9$ as integers. A fixed-width implementation must use arbitrary-precision fallback whenever its intermediate bounds would be exceeded. For each retained matrix, check $\min G = m$ and an explicit $U \in GL_9(\mathbb{Z})$ satisfying $U^T P_{\mathrm{ref}} U = (4/m)G$. The inequality above proves that this test cannot discard a form needed by Proposition 8.

The program `result9.py` of the reproduction pack (Section 9) performs exactly this calculation: every one of the 2 237 251 040 determinants is computed by Bareiss elimination in integers of unbounded size, and no floating-point value takes part in a decision. It finds the three forms `0017:001746552`, `0019:030884313`, `0030:030301719`, prints the three matrices $U$, determines the runner-up $22528/27$ (at `0060:012815567`) exactly over all matrices, and writes a hashed transcript, one SHA-256 per block of 10 000 matrices, with root `52e9bdf703a98eaa92308082f73ae3abe6871bca82155f8d1ae136a93679dd93`. Any block of the transcript can be recomputed in under a second; an independent implementation gave the same root.

### 5.5 Step 4: the three forms

**Theorem 10.** *Let* $P_{\mathrm{ref}} \in \{P_\Lambda, P_S, P_K\}$ *and let* $Q$ *be a positive definite* $9 \times 9$ *matrix with* $\min Q \ge 4$ *and* $\mathrm{tr}(Q^{-1} P_{\mathrm{ref}}) \le 9$. *Let* $d = \det Q$ *and* $u = \min Q^{-1}$. *Then* $d \ge 1024\, u$.

We treat $P_K$ first, because its proof is the shortest. Throughout, $u \le Q^{-1}[z]$ for every $z \in \mathbb{Z}^9 \setminus \{0\}$; $e_j$ and $f_j$ denote the $j$-th unit vectors of $\mathbb{R}^9$ and $\mathbb{R}^8$.

### The form $P_K$.

**Lemma 11.** *Let $P, Q$ be positive definite with $\operatorname{tr}(Q^{-1}P) \le 9$, and $u = \min Q^{-1}$. Let $z_i \in \mathbb{Z}^9 \setminus \{0\}$ and $\omega_i > 0$ ($i = 1, \dots, k$), let $W = \sum \omega_i$, and suppose $R = P - \sum \omega_i z_i z_i^T$ is positive definite. Then $\det Q \ge (10/9)^{10}\, W \det R \cdot u$.*

*Proof.* $9 \ge \sum \omega_i Q^{-1}[z_i] + \operatorname{tr}(Q^{-1}R) \ge uW + \operatorname{tr}(Q^{-1}R)$. The ten positive numbers $uW$ and the eigenvalues of $Q^{-1/2}RQ^{-1/2}$ have sum at most 9, so by AM–GM their product $uW \det R / \det Q$ is at most $(9/10)^{10}$. □

Take $P = P_K$, the vectors $z_1 = e_7$, $z_2 = e_8 - e_9$, $z_3 = e_7 + e_8 - e_9$ with weight $2/9$, and $z_4 = e_3 - e_4 - e_5$, $z_5 = e_2 - e_3$, $z_6 = -e_2 + e_4 + e_5 - e_9$, $z_7 = e_9$ with weight $1/8$. Let $Z_1 = (z_1, z_2, z_3)$ and $Z_2 = (z_4, \dots, z_7)$. A direct computation gives

$$Z_1^T P_K^{-1} Z_1 = \frac{1}{3}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \qquad Z_2^T P_K^{-1} Z_2 = I_4 - \frac{1}{4} J_4, \qquad Z_1^T P_K^{-1} Z_2 = 0,$$

where $J_4$ is the all-ones matrix. The first two matrices equal their own squares and have ranks 2 and 3, and $\Pi_i = P_K^{-1/2} Z_i Z_i^T P_K^{-1/2}$ has the same nonzero eigenvalues as $Z_i^T P_K^{-1} Z_i$; so $\Pi_1, \Pi_2$ are orthogonal projections of ranks 2 and 3, with orthogonal ranges by the third identity.
Hence $P_K^{-1/2} R P_K^{-1/2} = I - \frac{2}{9}\Pi_1 - \frac{1}{8}\Pi_2$ has eigenvalues $7/9$ (twice), $7/8$ (three times) and 1. So $R$ is positive definite, $\det R = 768\,(7/9)^2(7/8)^3 = 16807/54$, and $W = 7/6$. Lemma 11 gives

$$\frac{d}{u} \ge \left(\frac{10}{9}\right)^{10} \cdot \frac{7}{6} \cdot \frac{16807}{54} = \frac{2^8 5^{10} 7^6}{3^{24}} \simeq 1041.4.$$

### The form $P_\Lambda$.

*The change of coordinates.* For $P_\Lambda$ and $P_S$ we use coordinates adapted to the reference form. Each of $P_\Lambda$, $P_S$ is $2C^TC$ for a matrix $C$ whose first eight columns are the simple roots of $E_8$ (Appendix B). Replacing $Q$ by $Y = \frac{1}{2} C^{-T} Q C^{-1}$ sends the reference form to the identity ($Y = I$ exactly when $Q = P_{\mathrm{ref}}$) and turns the known lattice vectors into simple directions: the hypothesis $\min Q \ge 4$ becomes $Y[x] \ge 2$ for every nonzero $x \in C\mathbb{Z}^9$, and $C\mathbb{Z}^9$ contains, for $P_\Lambda$, the thirty-two vectors $\pm e_i \pm e_9$, and for $P_S$ nine vectors forming a simplex. The trace hypothesis becomes $\operatorname{tr}(Y^{-1}) \le 9$, the determinant $d$ becomes a multiple of $\det Y$, and $u$ is bounded by a multiple of $(Y^{-1})_{99}$. Lemmata 12 and 13 are stated for $Y$; the paragraphs "Application" after each lemma carry out the translation.

*The idea of Lemma 12.* We write $Y$ as an eight-dimensional block $A$ and a ninth direction, with Schur complement $q$. The desired inequality is $q^2 \det A \ge 1$. Suppose it fails. The inverse-trace bound then restricts $q$ and the eigenvalues of $A$. We use these restrictions to show that $\operatorname{tr} A + 8t - 2\|b\|_1 < 16$, whereas the sixteen contact

inequalities force the same expression to be at least 16. The estimates below establish this contradiction.

Each intermediate bound has one job: the bound b on the largest eigenvalue of $A$ allows the logarithmic trace estimate c; the estimate a for $r$ (the slack in the trace hypothesis, which controls $|\xi|^2$) controls the contribution of the off-diagonal block in d; and the Bernstein calculation in f closes the final scalar interval. The central quantities are $q, \xi$ and $r$; the other letters ($\lambda, \alpha, S, \kappa, \zeta, \rho, T(\varepsilon)$) are temporary variables of individual estimates. We write $(s)_+ = \max(s, 0)$ for the positive part.

**Lemma 12.** *Let $Y$ be a positive definite $9 \times 9$ matrix with $\operatorname{tr}(Y^{-1}) \le 9$ and $(e_i + \sigma e_9)^T Y (e_i + \sigma e_9) \ge 2$ for $i = 1, \dots, 8$ and $\sigma = \pm 1$. Then $\det Y \ge (Y^{-1})_{99}$.*

*Proof.* Write $Y = \begin{pmatrix} A & b \\ b^T & t \end{pmatrix}$ with $A$ of size 8, $\xi = A^{-1}b$ and $q = t - b^T\xi > 0$.

Block inversion (Schur complement) gives

$$\det Y = q \det A, \qquad (Y^{-1})_{99} = \frac{1}{q}, \qquad \operatorname{tr}(Y^{-1}) = \operatorname{tr}(A^{-1}) + \frac{1 + |\xi|^2}{q}. \tag{3}$$

We must show $q^2 \det A \ge 1$. Suppose $q^2 \det A < 1$. Let $U = \operatorname{tr}(A^{-1})$ and $r = q(9 - U) - 1$. By (3) the trace hypothesis says $|\xi|^2 \le r$, so $r \ge 0$. Steps a–c bound $q$, the eigenvalues of $A$ and $\operatorname{tr} A$; step d turns the sixteen hypotheses on $e_i \pm e_9$ into (5); steps e–f show that (5) fails.

a $q < 1$. AM–GM and $\det A^{-1} > q^2$ give $U > 8q^{1/4}$, so $r < 9q - 8q^{5/4} - 1$. With $\zeta = q^{1/4}$, $(1 - q) - (9q - 8q^{5/4} - 1) = 2(\zeta - 1)^2(4\zeta^3 + 3\zeta^2 + 2\zeta + 1) \ge 0$. Hence $0 \le r < 1 - q$. Let $\varepsilon = 1 - q \in (0, 1)$.

b *The largest eigenvalue $\lambda$ of $A$.* Let $\alpha = 1/\lambda$ be the least eigenvalue of $A^{-1}$, $S = U - \alpha$ and $\kappa = (1 + r)/q = 9 - U$, so $\kappa + S = 9 - \alpha$. AM–GM on the other seven eigenvalues of $A^{-1}$, then weighted AM–GM with weights 2 and 7, give $q^2 < \alpha(S/7)^7$ and $(\kappa/2)^2(S/7)^7 \le ((9 - \alpha)/9)^9$. Hence

$$(1 + r)^2 = q^2\kappa^2 < \varphi(\alpha) := 4\alpha\Big(1 - \frac{\alpha}{9}\Big)^9.$$

$\varphi$ increases on $(0, 9/10)$, and $\varphi(5/14) < 1$, $\varphi(4/9) < (17/16)^2$. Since $(1 + r)^2 \ge 1$ and $5/14 < 4/9 < 9/10$, we get $\alpha > 5/14$, that is $\lambda < 14/5$; and if $r > 1/16$, then $(1 + r)^2 > (17/16)^2$, so $\alpha > 4/9$ and $\lambda < 9/4$.

c $\operatorname{tr} A$. For $0 < x \le 14/5$, $x \le 3 \ln x + 2/x - 1$: the difference $\psi(x)$ satisfies $\psi'(x) = -(x - 1)(x - 2)/x^2$ and $\psi(1) = 0$, so $\psi \ge \min(0, \psi(14/5))$ on $(0, 14/5]$; and $\psi(14/5) = 3\ln(14/5) - 108/35 > 0$, because

$$\ln\Big(\frac{14}{5}\Big) = \ln \frac{1 + \frac{9}{19}}{1 - \frac{9}{19}} = 2 \sum_{j \ge 0} \frac{\left(\frac{9}{19}\right)^{2j+1}}{2j + 1}$$

and the first four terms exceed $\frac{36}{35}$. Summing over the eigenvalues of $A$ (all in $(0, 14/5)$ by b) and using $\ln \det A < -2 \ln q$:

$$\operatorname{tr} A \le 3 \ln \det A + 2U - 8 < 10 - 6 \ln q - \frac{2(1 + r)}{q}. \tag{4}$$

d *Using the vectors* $e_i \pm e_9$. The hypothesis reads $A_{ii} + 2\sigma b_i + t \geq 2$. Choosing $\sigma$ with $\sigma b_i = -|b_i|$ and summing over $i$ gives $\operatorname{tr} A + 8t - 2\|b\|_1 \geq 16$. Let $s = |\xi| \leq \sqrt{r}$. Since $b = A\xi$ and $t = q + \xi^T A\xi$,

$$\begin{aligned} 16 &\leq \operatorname{tr} A + 8q + 8\xi^T A\xi - 2\|A\xi\|_1 \leq \operatorname{tr} A + 8q + (8s-2)\|A\xi\|_2 \\ &\leq \operatorname{tr} A + 8q + \lambda(8r - 2\sqrt{r})_+. \end{aligned}$$

The second inequality uses Cauchy–Schwarz, $\xi^T A\xi \leq s\|A\xi\|_2$, and $\|A\xi\|_1 \geq \|A\xi\|_2$. For the third: the middle term is at most 0 if $s \leq 1/4$; if $s > 1/4$, it is at most $\lambda(8s^2 - 2s) \leq \lambda(8r - 2\sqrt{r})$, because $\|A\xi\|_2 \leq \lambda s$ and $8x^2 - 2x$ increases for $x \geq 1/8$. With (4), and $F(q) := 10 - 2/q - 6\ln q + 8q$,

$$16 < F(q) - \frac{2r}{q} + \lambda(8r - 2\sqrt{r})_+. \tag{5}$$

e $16 - F(q) \geq \varepsilon(4-\varepsilon)$. The function $\Delta(q) = 16 - F(q) - (1-q)(3+q)$ satisfies $\Delta(1) = 0$ and $\Delta'(q) = 2(q-1)^3/q^2 \leq 0$ on $(0, 1]$.
f (5) *fails.* If $r \leq 1/16$, the last term of (5) is 0, and e gives $F(q) - 2r/q \leq 16 - \varepsilon(4-\varepsilon) < 16$. If $r > 1/16$, then $\lambda < 9/4$, and the right side of (5) is at most $F(q) + g(r)$ with

$$g(r) = \left(18 - \frac{2}{q}\right) r - \frac{9}{2}\sqrt{r}.$$

By e it suffices to show $g(r) < \varepsilon(4-\varepsilon)$.

- $r < \rho := \varepsilon(4 - 5\varepsilon)/4$: by a, $r < 9q - 8q^{5/4} - 1$, and with $q = \zeta^4$, $\zeta < 1$, $\rho - (9q - 8q^{5/4} - 1) = \frac{1}{4}(1-\zeta)^3(5\zeta^5 + 15\zeta^4 + 30\zeta^3 + 18\zeta^2 + 9\zeta + 3) \geq 0$.
- Since $1/16 < r < \rho$, we have $\rho > 1/16$, that is $5\varepsilon^2 - 4\varepsilon + 1/4 < 0$, so $\varepsilon < (4 + \sqrt{11})/10 < 3/4$.
- $g$ is convex and $g(1/16) = -1/(8q) < 0$, so on $(1/16, \rho]$ it suffices that $g(\rho) < \varepsilon(4-\varepsilon)$, that is, $T(\varepsilon) < \frac{9}{2}\sqrt{\rho}$ with the scalar $T(\varepsilon) = (18 - 2/q)\rho - \varepsilon(4-\varepsilon) = \varepsilon(43\varepsilon^2 - 66\varepsilon + 24)/(2(1-\varepsilon))$.
- For $3/5 \leq \varepsilon < 3/4$, $T(\varepsilon) < 0$: the quadratic factor equals $-3/25$ at $3/5$ and $-21/16$ at $3/4$, and it is convex; hence $T(\varepsilon) < \frac{9}{2}\sqrt{\rho}$, as $\rho > 0$. For $0 < \varepsilon < 3/5$, $\frac{81}{4}\rho - T(\varepsilon)^2 = \varepsilon\, p(\varepsilon)/(16(1-\varepsilon)^2)$ with

  $$p(x) = 324 - 3357x + 13806x^2 - 26085x^3 + 22704x^4 - 7396x^5,$$

  and $p > 0$ on $[0, 3/5]$ by the Bernstein coefficients in Appendix C; so $|T(\varepsilon)| < \frac{9}{2}\sqrt{\rho}$. □

*Application to* $P_\Lambda$. We append a ninth coordinate 0 to the vectors $b_1, \ldots, b_8$ of Appendix B, let $b_9 = e_1 + e_9$, and let $C$ be the matrix with columns $b_1, \ldots, b_9$. Then $P_\Lambda = 2C^{-T}C^{-1}$ and $C^T e_9 = e_9$. By Gram–Schmidt, $\det C^T C$ is $\det G_{E_8} = 1$ times the squared last coordinate of $b_9$, so $\det C^T C = 1$. The lattice $C\mathbb{Z}^9$ contains the 32 vectors $\pm e_i + \sigma e_9$ ($i \leq 8$, $\sigma = \pm 1$): such a vector equals $\sigma b_9 + (\pm e_i - \sigma e_1)$, and $\pm e_i - \sigma e_1$ lies in $E_8$ (Appendix B).
Let $Y = \frac{1}{2}C^{-T}QC^{-1}$, so that $Y = I$ exactly when $Q = P_\Lambda$.

Then $\det Y = d/512$. Next $Y^{-1} = 2CQ^{-1}C^T$, so $\operatorname{tr}(Y^{-1}) = \operatorname{tr}(Q^{-1}P_\Lambda) \le 9$ and $(Y^{-1})_{99} = 2Q^{-1}[e_9] \ge 2u$.
Finally $x^TYx = Q[c]/2 \ge 2$ for every nonzero $x = Cc$ in $C\mathbb{Z}^9$, in particular for $x = e_i \pm e_9$. Lemma 12 gives $d/512 \ge 2u$.

### The form $P_S$.

*The simplex moment condition.* For $P_S$ the lattice $C\mathbb{Z}^9$ contains nine vectors $(h_j, \tau)$ whose first eight coordinates $h_1, \dots, h_9 \in \mathbb{R}^8$ form a regular simplex centred at the origin: $\sum_j h_j = 0$ and $\sum_j h_j h_j^T = I_8$ (the *moment* identities, checked in the application below). Summing the nine hypotheses $Y[(h_j, \tau)] \ge 2$ and using the moments turns them into the single linear condition $\operatorname{tr} A + 10t \ge 18$ on the blocks of $Y$, which replaces the sixteen conditions of Lemma 12. Lemma 13 then runs the same contradiction as Lemma 12, with the threshold $81/100$ in place of 1 and the three estimates (on $q$, $r$ and $\lambda$) redone for the new condition.

**Lemma 13.** *Let* $Y = \begin{pmatrix} A & b \\ b^T & t \end{pmatrix}$ *be positive definite of size* 9*, with* $\operatorname{tr}(Y^{-1}) \le 9$ *and* $\operatorname{tr} A + 10t \ge 18$. *Then* $\det Y > \frac{81}{100}(Y^{-1})_{99}$.

*Proof.* With the notation of Lemma 12, we must show $q^2 \det A > c := 81/100$. Suppose $q^2 \det A \le c$.
$q < 4/5$. AM–GM gives $9 - 1/q \ge U \ge 8(q^2/c)^{1/8}$. So $q > 1/9$ and $h(q) := c(9q-1)^8/(8^8 q^{10}) \ge 1$. As $h'/h = (10 - 18q)/(q(9q-1))$, $h$ decreases for $q \ge 5/9$, and $h(4/5) = (81/64)(31/32)^8 < 1$.
$r \le r_0 := 31441/500000$. As in a above, $r \le 9q - 8c^{-1/8}q^{5/4} - 1$. With $q = (9/10)\zeta^4$ this is $(9/10)(9\zeta^4 - 8\zeta^5) - 1$, which is largest at $\zeta = 9/10$, where it equals $2(9/10)^6 - 1 = r_0$.
$\lambda < 2$. As in b above, $(1+r)^2 \le 4c\,\alpha(1-\alpha/9)^9$.
Since $\varphi$ increases on $(0, 9/10)$, $\alpha \le 1/2$ would give $(1+r)^2 \le 2c(17/18)^9 < 1$.
*Contradiction.* As in c above, $\operatorname{tr} A \le 10 + 3\ln c - 6\ln q - 2(1+r)/q$; and $t = q + \xi^T A \xi \le q + 2r$. So $\operatorname{tr} A + 10t \le F_c(q) + (20 - 2/q)r$ with $F_c(q) = 10 + 3\ln c - 6\ln q - 2/q + 10q$.
Since $q^2 F_c'(q) = 10(q - 3/10)^2 + 11/10 > 0$, $F_c(q) < F_c(4/5) = 31/2 + 6\ln(9/8) < 65/4$, using $\ln(1+x) < x$. Also $20 - 2/q < 35/2$ and $0 \le r \le r_0$, so $(20 - 2/q)r \le (35/2)r_0$. Hence $\operatorname{tr} A + 10t < 65/4 + (35/2)r_0 = 3470087/200000 < 18$. □

*Application to $P_S$.* Let $\beta = \frac{1}{6}(1,1,1,1,1,1,1,5)$, $\tau = \sqrt{10/9}$, and let $C$ have columns $b_1, \dots, b_8$ and $b_9 = (\beta, \tau)$. Then $P_S = 2C^TC$, $C^T e_9 = \tau e_9$, and as before $\det C^TC = \det G_{E_8} \cdot \tau^2 = 10/9$. Consider the nine vectors $h_1, \dots, h_9 \in \mathbb{R}^8$

$$\beta, \qquad \beta - \frac{1}{2}(1, \dots, 1), \qquad \beta - f_k - f_8 \quad (k = 1, \dots, 7).$$

Each $h_j - \beta$ lies in $E_8$ (Appendix B), so $(h_j, \tau) = b_9 + (h_j - \beta, 0)$ lies in $C\mathbb{Z}^9$. A direct computation gives $\sum_j h_j = 0$ and $\sum_j h_j h_j^T = I_8$, so $\sum_j (h_j, \tau)(h_j, \tau)^T = \operatorname{diag}(1, \dots, 1, 10)$.

Let $Y = \frac{1}{2} C^{-T} Q C^{-1}$. Then

$$\begin{aligned} \det Y &= \frac{9d}{5120}, \\ \operatorname{tr}(Y^{-1}) &= \operatorname{tr}(Q^{-1} P_S) \le 9, \\ \operatorname{tr} A + 10t &= \sum_j (h_j, \tau)^T Y (h_j, \tau) \ge 9 \cdot 2 = 18, \\ (Y^{-1})_{99} &= 2\tau^2 Q^{-1}[e_9] \ge \frac{20}{9} u. \end{aligned}$$

Lemma 13 gives $9d/5120 > \frac{81}{100} \cdot \frac{20}{9} u$, that is, $d > 1024\, u$.

### 5.6 Proof of Theorem 3

Suppose a lattice of rank 10 has minimum 4 and determinant below 768. Proposition 6 gives a lattice $L'$ with determinant $D' < 768$ and a sublattice $M$ with $\min M \ge 4$, $d = \det M < c^*$ and $D' \ge 3d/(4u)$. Let $Q$ be a Gram matrix of $M$. Proposition 7 gives a perfect $P$ with $\min P = 4$, $\operatorname{tr}(Q^{-1}P) \le 9$ and $\det P \le d < c^*$. By Proposition 8, $P = U^T P_{\mathrm{ref}} U$ with $U \in GL_9(\mathbb{Z})$ and $P_{\mathrm{ref}} \in \{P_\Lambda, P_S, P_K\}$.
The matrix $Q' = U^{-T} Q U^{-1}$ is another Gram matrix of $M$, so $\min Q' \ge 4$, $\det Q' = d$ and $\min Q'^{-1} = u$; and $\operatorname{tr}(Q'^{-1} P_{\mathrm{ref}}) = \operatorname{tr}(Q^{-1}P) \le 9$. Theorem 10 gives $d \ge 1024\, u$, so $D' \ge 768$, a contradiction. Hence $\gamma_{10} \le 4/\sqrt[10]{768}$, with equality for $\Lambda_{10}$. □

## 6 Dimension 11

**Theorem 14 (announced; proved in [14]).** *Assume Hypothesis 2, Computational result 9, and the global and local computational statements specified in Section 6.2 and [14]. Every lattice of rank* 11 *with minimum* 4 *has determinant at least* 972. *Hence* $\gamma_{11} = 4/\sqrt[11]{972} = 2.1401980\ldots$, *attained by* $K_{11}$.

The proof is assembled in *A Focus on Hermite Constant 11* [14]. Here we connect dimension 11 to dimension 10 by two successive sections, identify the rank-nine forms requiring local arguments, and explain how the companion papers settle them.

### 6.1 The chain of sections

The letters $M$, $d$ and $u$ change rank between Section 5 and this one; Table 2 fixes their meaning here, next to the cutoffs they must respect. Figure 10 places the thresholds on one axis.
Let

$$d_{\mathrm{cut}} = 4\sqrt[11]{972^9} = 1113.048\ldots,$$

$$e_{\mathrm{cut}} = \frac{4\sqrt[10]{d_{\mathrm{cut}}^9}}{\sqrt[10]{768}} = \sqrt[55]{2^{246}3^{197}} = 1136.021\ldots \quad \Rightarrow \quad e_{\mathrm{cut}}^{55} = 2^{246}3^{197}.$$

**Table 2.** The lattices of the chain in dimension 11 and the constants of this section. How each rung is obtained – rank 11: the counterexample, balanced with its dual; rank 10: first cut, $\gamma_{10}$ bounds $u$; rank 9: second cut, compared with the perfect form $P$.

| lattice | rank | determinant | dual minimum |
|---|---|---|---|
| $L$ | 11 | $\Delta < 972$ | $v = \min L^* \le \frac{4}{3}u$ |
| $M = L \cap n_{11}^{\perp}$ | 10 | $768 \le d < d_{\mathrm{cut}}$, $\Delta \ge \frac{3d}{4u}$ | $u = \min M^* \le \frac{4}{\sqrt[10]{768d}}$ |
| $N = M \cap n_{10}^{\perp}$ | 9 | $\det N = ud < e_{\mathrm{cut}}$ | – |

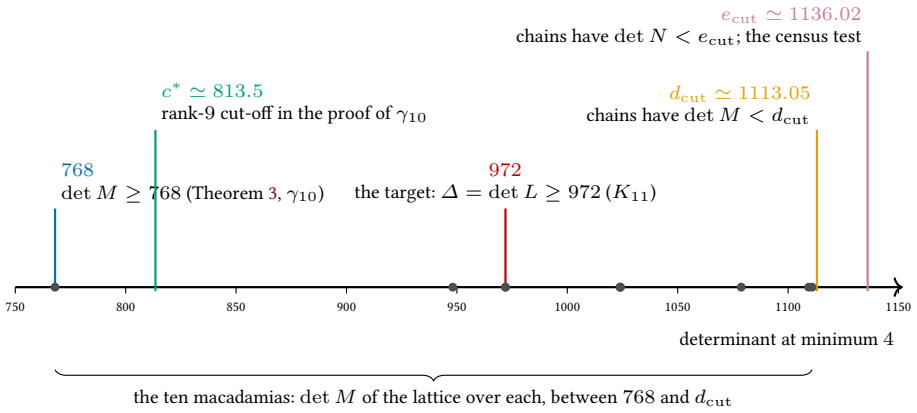



**Fig. 10.** The five thresholds of the series on one axis of determinants at minimum 4: 768 and $c^* = 4 \cdot 768^{4/5}$ for dimension 10; 972, $d_{\mathrm{cut}} = 4 \cdot 972^{9/11}$ and $e_{\mathrm{cut}} = \sqrt[55]{2^{246}3^{197}}$ for dimension 11. The dots are the determinants of the rank-10 lattices $M$ over the ten macadamias (Table 4).

**Proposition 15.** *Assume Theorem 3. Let $L_0$ have rank 11, minimum 4 and determinant $D_0 < 972$. There are a lattice $L$ of rank 11 with minimum 4 and determinant $\Delta < 972$, a shortest vector $n_{11}$ of $L^*$ and a shortest vector $n_{10}$ of $M^*$, where $M = L \cap n_{11}^{\perp}$ and $N = M \cap n_{10}^{\perp}$ are sublattices of ranks 10 and 9, with*

$$\min M \geq 4, \qquad 768 \leq d := \det M < d_{\mathrm{cut}},$$
$$\Delta \geq \frac{3d}{4u} \ \text{ where } u = |n_{10}|^2 = \min M^*,$$
$$\min N \geq 4, \qquad \det N = ud < e_{\mathrm{cut}}.$$

*Proof.* Write $\gamma(K) = \min K/(\det K)^{1/11}$ for lattices of rank 11. As in Proposition 6, let $L$ be the multiple with minimum 4 of whichever of $L_0$, $L_0^*$ has the larger $\gamma$; then $\gamma(L^*) \leq \gamma(L)$ and $\Delta = (4/\gamma(L))^{11} \leq D_0 < 972$.
Let $n_{11}$ be a shortest vector of $L^*$, $v = |n_{11}|^2$, and $M = L \cap n_{11}^{\perp}$.
By Fact 4 a, $M$ has rank 10 and $d = v\Delta$; and $\min M \geq 4$.
From $v\Delta^{1/11} = \gamma(L^*) \leq \gamma(L) = 4\Delta^{-1/11}$ we get $v \leq 4\Delta^{-2/11}$, so $d \leq 4\Delta^{9/11} < d_{\mathrm{cut}}$.
Let $n_{10}$ be a shortest vector of $M^*$, $u = |n_{10}|^2$. Exactly as in Proposition 6, Fact 4 b c gives $v \leq u + v/4$, so $\Delta = d/v \geq 3d/(4u)$.
Now Theorem 3 enters. $M$ has rank 10 and minimum at least 4, so $d \geq 768$; and $M^*$ has rank 10 and determinant $1/d$, so $u\, d^{1/10} = \gamma(M^*) \leq \gamma_{10} = 4/\sqrt[10]{768}$, that is, $u \leq 4/\sqrt[10]{768\, d}$. Let $N = M \cap n_{10}^{\perp}$. By Fact 4 a applied to $M$, $N$ has rank 9 and $\det N = ud$; and $\min N \geq \min M \geq 4$.
Finally $\det N = ud \leq 4d^{9/10}/\sqrt[10]{768} < 4d_{\mathrm{cut}}^{9/10}/\sqrt[10]{768} = e_{\mathrm{cut}}$. The exponents of 2 and 3 in

$$e_{\mathrm{cut}} = \frac{4\left(4 \cdot (2^2 3^5)^{9/11}\right)^{9/10}}{\sqrt[10]{2^8 \cdot 3}}$$

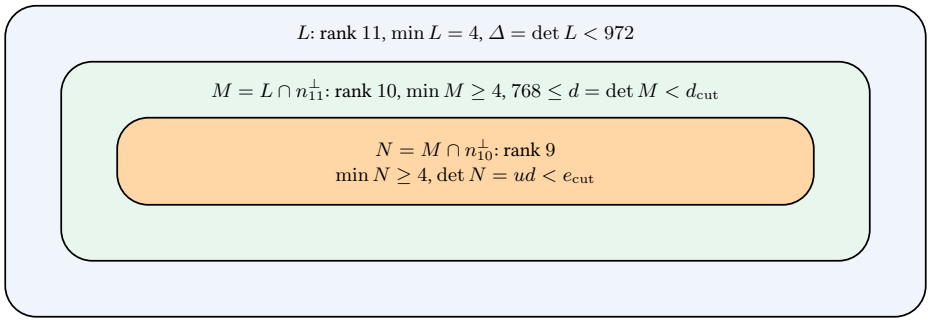
are

$$2 + \frac{9}{10}\left(2 + \frac{18}{11}\right) - \frac{8}{10} = \frac{246}{55} \qquad \text{and} \qquad \frac{9}{10} \cdot \frac{45}{11} - \frac{1}{10} = \frac{197}{55}.$$

□

Figure 11 shows the chain and the marking used below.
Proposition 7 applies *verbatim* to a Gram matrix $Q$ of $N$: there is a perfect form $P$ of rank 9 with $\min P = 4$, $\mathrm{tr}(Q^{-1}P) \leq 9$ and $\det P \leq \det N < e_{\mathrm{cut}}$.
By Hypothesis 2 and the argument of Proposition 8, $P$ is equivalent to $(4/\min G)\, G$ for a matrix $G$ of the data; if $m$ is the least diagonal entry of $G$, then $\min G \leq m$ and $4^9 \det G < e_{\mathrm{cut}} m^9$, a condition tested exactly as $(4^9 \det G)^{55} < 2^{246} 3^{197} m^{495}$.
Exactly 4 487 813 of the 2 237 251 040 matrices pass the test [14].
We call them the *survivors*; for each, the minimum is computed exactly, and the survivor is taken as the form $(4/\min G)\, G$ of minimum 4.

*How a form is named.* A form is identified by `file:row` in the data set of [12] (Zenodo record 15707640). The four-digit `file` selects one of `perfect_0000.nc`, …, `perfect_0063.nc`; the nine-digit `row` is its position, counted from zero. Thus `0030:030301719` denotes row 30 301 719 of `perfect_0030.nc`, scaled to minimum 4. The data layout and exact selection procedure are specified in [14, Section 3].




$n_{11}$ is a shortest vector of $L^*$ and $v = |n_{11}|^2$, so $d = v\Delta$; $n_{10}$ is a shortest vector of $M^*$ and $u = |n_{10}|^2 = \min M^*$; then $\Delta \geq 3d/(4u)$. A Gram matrix $Q$ of $N$ is *admissibly marked* by a perfect form $P$, which may be taken to be a survivor: $Q^{-1} = \sum_c w_c\, cc^T$ with $w_c \geq 0$, $\sum_c w_c \leq 9/4$ and $w_c \leq 1/4$, where $c$ runs over one vector of each opposite pair of $\operatorname{Min} P$ and $\min Q \geq 4$. The obligation: every such chain marked by a survivor has $\Delta \geq 972$, a contradiction.

**Fig. 11.** The chain $L \supset M \supset N$ of Proposition 15, the object on which every companion paper works. Each cut is orthogonal to a shortest dual vector.

### 6.2 What must be proved, and what the certificates prove

*The global computational inputs.* The rank-ten census is the exact check of Computational result 9. The complete rank-eleven selection and preliminary eliminations are computed and checked by the reproduction pack, as specified in [14]. For exact selection, write $\delta_G = \det G$ and $m = \min_i G_{ii}$ for a raw integral form. The rank-ten filter is $4^9\delta_G < 814m^9$; the rank-eleven filter is given in [14]. Since $\min G \leq m$, these tests retain every form below the required normalised cutoff. Integer elimination computes $\delta_G$ exactly; both scans of the reproduction pack decide every matrix this way.

*The obligation.* Let $N$ be the rank-9 member of a chain as in Proposition 15 and $Q$ a Gram matrix of $N$. Proposition 7 gives a perfect $P$ and a marking (2) of $Q$ by $P$, and $P$ is equivalent to a survivor; transporting the marking along the equivalence (the remark after Proposition 7), we may take $P$ to be the survivor itself. We say that $Q$ is *admissibly marked by* $P$ when nonnegative weights $w_c$ with

$$Q^{-1} = \sum_c w_c\, cc^T, \qquad \sum_c w_c \leq \frac{9}{4}, \qquad w_c \leq \frac{1}{4}$$

exist ($c$ running over the opposite-pair representatives of $\operatorname{Min} P$) and $\min Q \geq 4$; then $\operatorname{tr}(Q^{-1}P) = 4\sum_c w_c \leq 9$ automatically. Theorem 14 therefore follows from Proposition 15 once the following is shown for each survivor $P$:

> every chain $L \supset M \supset N$ as in Proposition 15 whose rank-9 member $N$ has a Gram matrix $Q$ admissibly marked by $P$ satisfies $\Delta \geq 972$.

The rank-eleven certificates use the full marking and work at two levels.

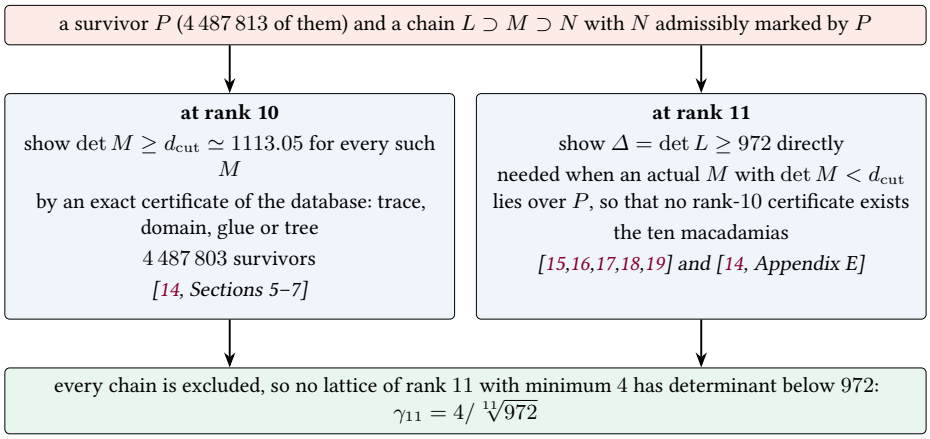



**Fig. 12.** The two levels at which a survivor is excluded. A rank-10 certificate implies the rank-11 statement, because $\det M \geq d_{\mathrm{cut}}$ contradicts the chain of Proposition 15; for the ten macadamias it cannot exist.

*The two levels.* After the two cuts, every possible counterexample is associated with one of the survivors. For most forms, we can already rule out the rank-ten lattice $M$: it would have to have determinant at least $d_{\mathrm{cut}}$. The ten forms in Table 4 are different. The table gives rank-ten examples below $d_{\mathrm{cut}}$, so that stronger exclusion cannot work for them. We must instead show that no admissible extension to rank eleven can have minimum four and determinant below 972. Figure 12 shows the two levels.

*At rank* 10. One shows that $\det M \geq d_{\mathrm{cut}}$ for every lattice $M$ of rank 10 with $\min M \geq 4$ whose section $N$ by a shortest dual vector has a Gram matrix $Q$ admissibly marked by $P$. This is a statement of the same nature as Theorem 10, and it admits exact certificates. A certificate is a finite piece of data from which a short exact computation proves a bound; the $\gamma_{11}$ paper [14] uses three ways of proving the bound $\det M \geq d_{\mathrm{cut}}$ directly, and one way of combining them:

- TRACE CERTIFICATES prove the bound by one exact inequality, in the spirit of Lemma 11, valid on the whole set $\{Q : \mathrm{tr}(Q^{-1}P) \leq 9,\ \min Q \geq 4\}$; trace certificates use only the trace part of the marking;
- DOMAIN CERTIFICATES prove the bound using the full marking: $Q^{-1} = \sum_c w_c\, cc^T$ with the bounds on the weights, that is, the position of $Q^{-1}$ in the Voronoi domain of $P$ (the cone spanned by the $xx^T$, $x \in \mathrm{Min}\, P$);
- GLUE CERTIFICATES prove the bound using, in addition, the position of the tenth basis vector of $M$ relative to $N$ (the *gluing*);
- TREE CERTIFICATES combine such bounds over regions that together cover all possibilities: a branch-and-bound subdivision (40 trees over cases, that is, ranges for a pool of dual vectors, and 30 trees over boxes of gluing parameters) in which every leaf carries an exact certificate for its region or shows that the region holds no gluing, and the checker verifies that the leaves cover the whole region; the largest tree has 763 594 leaves and its weakest leaf clears $d_{\mathrm{cut}}$ by a relative margin of $2.22 \cdot 10^{-8}$

(its least value is $1113.0488306\ldots$ against $d_{\mathrm{cut}} = 1113.0488059\ldots$), compared exactly in integers; the margin is measured by `margin.py`, in the verification report of the Zenodo record (Section 9).

Table 3 gives their counts. The certificates, database and checker are described in [14], which also records the verification run and the reproduction procedure.

*Remark 16.* The survivor `0060:012815567`, whose determinant at minimum $4$ is $22528/27 \simeq 834.4$, is excluded at rank $10$ by a tree certificate of the certificate database (the largest tree, with the $763\,594$ leaves mentioned above), and it is counted with the tree certificates of Table 3.

**Table 3.** The survivors and how they are excluded [14].

| **stage** | **excluded** | **left** |
|---|---|---|
| determinant cut-off $\det P < e_{\mathrm{cut}}$ on the $2\,237\,251\,040$ forms | $2\,232\,763\,227$ | $4\,487\,813$ |
| trace certificates | $4\,486\,439$ | $1\,374$ |
| domain certificates ($1\,215$) and glue certificates ($79$) | $1\,294$ | $80$ |
| tree certificates ($40$ trees of cases, $30$ trees of gluing boxes) | $70$ | $10$ |
| nine cases in five companions; `0052` in [14] (Table 4) | $10$ | $0$ |

*At rank* 11. For ten survivors (the macadamias) no certificate at rank $10$ can exist, because an actual lattice $M$ of rank $10$ with minimum $4$ and determinant below $d_{\mathrm{cut}}$ lies over them: $\Lambda_{10}$ over $\Lambda_9$, sections of $K_{11}$ over two of the forms, and so on. For these the inequality $\Delta \geq 972$ must be proved at the level of the rank-11 lattice, where it is attained ($K_{11}$ itself has determinant $972$, so for the forms under its sections there is no margin at all, and for $\Lambda_9$ under $\Lambda_{10}$ the margin is small).
Table 4 names them (Figure 13 places them against $d_{\mathrm{cut}}$), with the lattice $M$ of rank $10$ that lies over each, its determinant and its dual minimum $u = \min M^*$. Nine macadamias are settled in the five companion papers [15,16,17,18,19], and `0052` in [14]. They are grouped by the method that cracks them (last column of the table): localisation of the extremal configurations, symmetry reductions by the automorphism group of the form, and exact certificates on the regions that remain; the $\gamma_{11}$ paper [14] lists these papers and the theorem that each of them proves.

*What is imported, precisely.* Theorem 14 rests on this paper (Proposition 15, Proposition 7 with its marking, Theorem 3) and on the following two statements proved elsewhere:

– From [14] (Theorem 20): *assume Hypothesis 2, Theorem 3, and Computational results 5 and 19 of [14]; let $P$ be a survivor other than the ten of Table 4, and let $M$ be a lattice of rank $10$ with $\min M \geq 4$ whose section by a shortest dual vector has a Gram matrix admissibly marked by $P$; then* $\det M \geq d_{\mathrm{cut}}$. Its analytic proof applies the proved certificate rules to the global finite verification of the reproduction pack [14, Section 7.1].

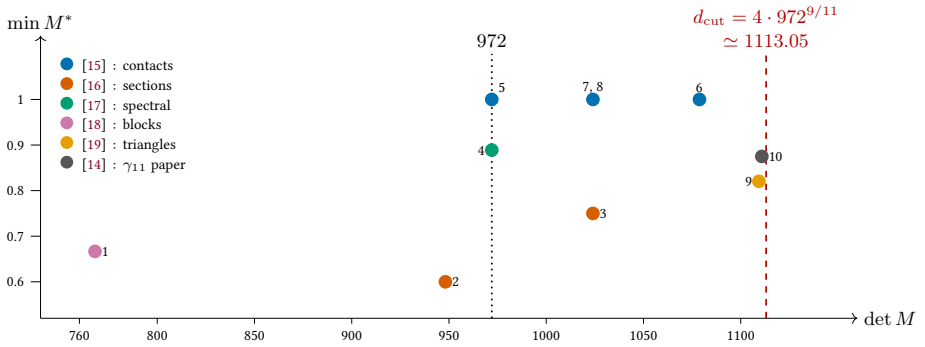



**Fig. 13.** The ten macadamias of Table 4, placed by the determinant (at minimum 4) and the dual minimum of the lattice $M$ of rank 10 that lies over each, and coloured by the paper that settles it. All lie to the left of $d_{\mathrm{cut}}$: an actual lattice of rank 10 with minimum 4 and $\det M < d_{\mathrm{cut}}$ lies over each, so no exclusion at rank 10 can exist and $\det L \geq 972$ must be proved at rank 11. Macadamias 9 and 10 are within 0.4% of the cut; 4 and 5, the two sections of $K_{11}$, are on the line $\det M = 972$.

– From [14, Appendix E] for `0052`, and from [15,16,17,18,19] for the other nine forms: *under the finite reference and cover checks stated in the respective paper, let $L \supset M \supset N$ be a chain as in Proposition 15 whose rank-9 member $N$ has a Gram matrix admissibly marked by $P$; then $\Delta = \det L \geq 972$.* Each of these proofs uses exact certificates of its own.

The $\gamma_{11}$ paper assembles Theorem 14 from these statements and Proposition 15 exactly as Section 5.6 assembles Theorem 3.

**Table 4.** The ten macadamias: the survivors that lie under a lattice $M$ of rank 10 with minimum 4 and $\det M < d_{\mathrm{cut}}$, so that no certificate at rank 10 exists. The last column names the paper that settles each (contacts: contact geometry; sections: reduction by sections; spectral: finite spectral relaxation; blocks: one selected contact, two linked blocks; triangles: unequal triangles).

| no. | form | lattice $M$ over it | $\det M$ | $\min M^*$ | paper |
|---|---|---|---|---|---|
| 1 | `0030:030301719` | $\Lambda_{10}$ (the form is $\Lambda_9 = P_\Lambda$) | 768 | 2/3 | blocks [18] |
| 2 | `0019:030884313` | the lattice over $P_S$; 294 min. vectors | 25600/27 | 3/5 | sections [16] |
| 3 | `0017:001746552` | the lattice over $P_K$; 270 min. vectors | 1024 | 3/4 | sections [16] |
| 4 | `0008:022143116` | section of $K_{11}$; 276 min. vectors | 972 | 8/9 | spectral [17] |
| 5 | `0015:030291998` | section of $K_{11}$; 270 min. vectors | 972 | 1 | contacts [15] |
| 6 | `0001:010488327` | rescaled dual of the 270-vector section | $2^{18}/3^5$ | 1 | contacts [15] |
| 7 | `0014:001749549` | a lattice $M_{260}$ with 260 min. vectors | 1024 | 1 | contacts [15] |
| 8 | `0059:016902802` | $M_{260}$ of macadamia 7 (a second section) | 1024 | 1 | contacts [15] |
| 9 | `0050:024468269` | an unnamed lattice with 180 min. vectors | 3328/3 | 32/39 | triangles [19] |
| 10 | `0052:005247066` | an unnamed lattice with 164 min. vectors | 7776/7 | 7/8 | $\gamma_{11}$ [14] |

A lattice of rank 10 can lie over several survivors, as different sections of it by different shortest dual vectors: macadamias 4 and 5 are two inequivalent perfect forms under two sections of $K_{11}$ (with 276 and 270 minimal vectors), and macadamias 7 and 8 are two inequivalent perfect forms under one and the same lattice $M_{260}$ of rank 10 (minimum 4, determinant 1024, 260 minimal vectors, $\min M_{260}^* = 1$); macadamia 6 is the rescaled dual of the 270-vector section. The lattices over macadamias 7–10 have no name in the literature that we know of; they are described by their invariants. Each form still needs its own proof, since the chains over the two forms are different sets. Two things make the macadamias harder than Section 5. First, $\Delta$ depends on how $M$ is glued over $N$, that is, on the class of the tenth basis vector modulo $N$ and on its component orthogonal to $N$, and the proofs parametrise these *gluings* and certify the bound on regions of the parameter space, every leaf verified in exact arithmetic. Second, the bound is attained for some of the reference configurations, so those configurations must be located exactly and the certificates must remain valid up to them; this is why macadamias 1, 9 and 4 are the longest cases of the series.

## 7 Dimension 12 from dimension 11

**Theorem 17.** *Assume Theorem 14. Then every lattice of rank* 12 *with minimum at least* 4 *has determinant at least* 729*, that is,*

$$\gamma_{12} \le \frac{4}{\sqrt[12]{729}} = \frac{4}{\sqrt{3}} = 2.3094010\ldots$$

*Since the Coxeter–Todd lattice* $K_{12}$ *[10] has minimum* 4 *and determinant* $729 = 3^6$*, in fact* $\gamma_{12} = 4/\sqrt{3}$*.*

The tool is Mordell's inequality $\gamma_n^{n-2} \le \gamma_{n-1}^{n-1}$ ($n \ge 3$), whose proof is recalled in Appendix A, Lemma 19.

*Proof through Mordell's inequality.* With $n = 12$ and Theorem 14, which by (1) says $\gamma_{11}^{11} \le 4^{11}/972$:

$$\gamma_{12}^{10} \le \gamma_{11}^{11} \le \frac{4^{11}}{972} = \frac{4 \cdot 4^{10}}{4 \cdot 243} = \frac{4^{10}}{3^5},$$

since $972 = 2^2 \cdot 3^5$. Hence $\gamma_{12} \le 4/3^{5/10} = 4/\sqrt{3}$. By (1) with $c = 4/\sqrt{3}$, every lattice of rank 12 with minimum at least 4 has determinant at least $(\sqrt{3})^{12} = 729$. Since $K_{12}$ has minimum 4 and determinant 729, $\gamma(K_{12}) = 4/\sqrt[12]{729} = 4/\sqrt{3}$, hence $\gamma_{12} = 4/\sqrt{3}$. □

The identity $4^{11}/972 = 4^{10}/3^5$ makes this application of Mordell's inequality exact. Appendix A gives a direct section proof, the reference data for $K_{12}$, and the other exact jumps between successive dimensions.

*Remark 18 (Equality conditions).* Assume Theorem 14 and let $L$ of rank 12 have $\min L = 4$ and $\det L = 729$. Every inequality in the direct proof of Theorem 17 (Appendix A) is then an equality: for a shortest vector $v$ of $L^*$, $a = |v|^2 = 4/3$, both $L$ and $L^*$ attain $\gamma_{12}$, and the section $M = L \cap v^\perp$ has $\min M = 4$ and $\det M = 972$, so $M$ attains $\gamma_{11}$.

## 8 Lean formalisation

The accompanying `hermite11_lean_proofs.zip` uses Lean 4.19.0 and the pinned mathlib revision `c44e0c8ee63ca166450922a373c7409c5d26b00b`. Figure 14 summarises what it proves and what it assumes. The final theorems are `Hermite11.PaperInputs.hermiteTen`, `hermiteEleven` and `hermiteTwelve` in that namespace. They take mathematical statements as hypotheses: the attributed rank-nine classification, the exact rank-ten census and the preliminary rank-eleven eliminations of the reproduction pack, including `0060`, and the stated finite reference and cover facts of the ten local cases. The classical inputs include the Coxeter–Todd lattice and the rank-three and rank-four Hermite bounds used in [16].

The marking is proved, under the stated classical lattice framework, by `Hermite11.voronoiContactMarkingInput`. The development also checks the explicit rank-ten attaining lattice and proves the local analytic arguments and their assembly. Rank-eleven attainment follows from a section of the attributed Coxeter–Todd lattice and the proved upper bound; dimension twelve follows by Mordell's inequality and the same attributed lattice. Separate assumptions asserting a rank-eleven attaining lattice or the marking are therefore unnecessary.

Some finite checks are internal to Lean: the saved Gram decompositions, eighteen exceptional Sturm rows for `0008`, and thirteen small polynomial certificates across the development (three for `0014`, three for `0030`, seven for `0008`), with 84 Bernstein coefficients in all. The large local covers, full strip tensors and global census are checked outside Lean. The correspondence file accompanying the archive gives the exact finite inputs and conclusion for each case.

From the top directory of the extracted Lean archive, the following commands fetch the pinned dependencies, build the development and its census supplement, and print the axioms on which their theorems depend:

```
$ lake exe cache get
$ lake build Hermite11
$ lake build Hermite11.CensusFilter Hermite11.CensusFilterDiagnostic
$ lake env lean AuditCensus.lean
$ lake env lean Audit.lean
... depends on axioms: [propext, Classical.choice, Quot.sound]
```

The integrated audit of the main development and the separate census audit use only Lean's standard `propext`, `Classical.choice` and `Quot.sound` axioms, with no `sorry`.

## 9 Additional material

The series has three supporting archives, deposited with its verification report and its notation index in one Zenodo record, doi:10.5281/zenodo.23979316. The *reproduction pack* (`hermite11_rank9_reproduction.zip`) contains the program of Computational result 9 (`result9.py`), the certificate database of [14] with its programs, and one driver, `bash reproduce.sh`, which downloads the classification [12], runs the filtering and certificate checks described in [14], and reports their verdicts:

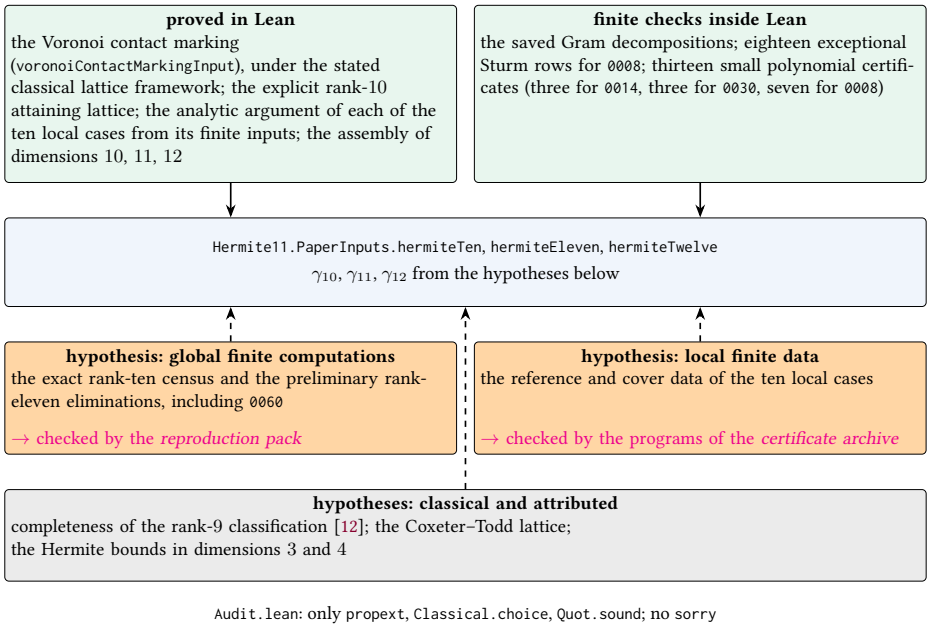



**Fig. 14.** What the Lean development proves and what it assumes. Solid arrows: proved in Lean; dashed arrows: hypotheses of the final theorems, which are precise mathematical propositions, checked outside Lean (orange) or taken from the literature (grey).

```
$ bash reproduce.sh
...
VERDICT: RESULT 9 HOLDS
...
VERDICT: COMPLETE: every survivor of the classification has a row, and ...
RESULT: EVERYTHING PASSED.
```

The *certificate archive* (`hermite11_local_certificates.zip`) contains the finite records on which the ten local proofs rest (the Computational results of the five companion papers and of the case `0052:005247066` in [14]) and the programs that replay them. Its `case_map.json` identifies each principal record, its size, the underlying checker sources and recorded result; `certificate_manifest.json` binds the stored files and compressed logical streams. The Lean development of Section 8 is published as `hermite11_lean_proofs.zip`. The same record contains a *notation index* of the series (`hermite11_notation_index.pdf`): for every symbol of the seven papers, its meaning and the place where it is defined, one part per paper; the script `hermite11_k12_check.py` of Appendix A, which recomputes the invariants of $K_{12}$ and of its section $K_{11}$; and the script `hermite11_printed_forms_check.py`, which checks that the eleven reference forms printed in the series (the ten macadamias and `0060:012815567`) are, entry by entry, the records of the data set as stored in the certificate database of the reproduction pack (`python3 -I hermite11_printed_forms_check.py k11_certificates_v3.sqlite`, about

a minute), so that the chains handed over by [14, Theorem 1] are marked by the printed forms.

The certificate archive has SHA-256 `91723b07084947154712ab9727dc5af1 2501a941ccce35dc07fb8429e36eec14`.

The Lean archive has SHA-256 `b9c5f8165fde5c0d7f7061c0ffe9547a 4e6d66588deb958e240bd52d890f4300`. The reproduction pack `hermite11_rank9_reproduction.zip` has SHA-256 `22a9336ef6677171a1e9d6760f0ca21c11abf3b0cc9e3c62 cfe009e1a6982776`; a complete run of its programs (5–6 November 2026, one node with 160 hardware threads) ended with `RESULT: EVERYTHING PASSED.` It contains the exact rank-ten census of Section 5.4. The logs, transcripts and statistics of that run, with the measurement of the margin of Section 6.2, form the *verification report* `hermite11_verification_report.zip` of the same record.

*Checking the local data.* Extract the certificate archive. Its driver `verify_cases.py` runs the saved checkers of each case in the order fixed by `verification_plan.json` (125 steps for the ten cases): it checks the SHA-256 of every input, decompresses a stored stream when a checker expects a plain file, and never searches for or accepts a new certificate. From the archive's folder, `python3 verify_cases.py --case all --root . --resume` checks the ten cases and ends with `PASS_LOCAL_CHECKING_SCHEDULE`; `--case 0050` (etc.) checks one case and `--plan` prints its steps. The archive's `README.txt` lists the environment. Acceptance requires every terminal inequality and every split or resumed root to verify, with no pending region; a partial replay is insufficient. The corresponding paper states the domain, the leaf rules and the expected complete result; [14, Appendix E] also gives the concrete database replay command for `0052`. A complete run of the ten cases (6–7 November 2026, one node with an Intel Xeon Platinum 8468) passed; its log is in the folder `referee_run/` of the archive.[3]

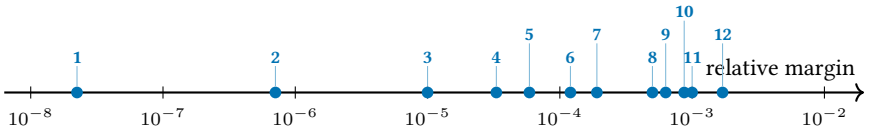
The margins by which the certified inequalities hold are, in places, very small (Figure 15); this is why every test of the series is decided in exact arithmetic.

## 10 Use of computational tools

### 10.1 Use of artificial intelligence

This series of papers used AI assistance from GPT 6 Astra, GPT 6.1 Sol and Terra, GPT Sol 5.6, Claude Fable 5.1, Claude Opus 5.5, and GLM 5.3 heterogeneously orchestrated by *ad hoc* software developed by the authors. The models were used in three ways. First, as research assistants to explore and challenge candidate arguments. Second, as auxiliary programmers: the searches over the classification of [12] and the independent checkers were written upon our instruction and were re-verified by programs written separately. Third, for verbal text improvement: portions of the text were reworked by the models from our notes and rewritten under our full supervision.

[3] Throughout this paper and its archives, a "referee" is a verification run: an independent exact recomputation of a certificate by a checker program, not a human reviewer. The folder and file names of the archives keep this word.




1 weakest leaf of the largest tree vs. $d_{\mathrm{cut}}$ (Section 6.2): $2.2 \cdot 10^{-8}$
2 [15, Section 4.4], band row $a = 1061$, determinant test: $7 \cdot 10^{-7}$
3 [15, Section 4.4], band row $a = 1061$, phase test: $10^{-5}$
4 [18, Section 5], trace ceiling $11.9996 < 12$
5 [17, Section 11], a Bernstein coefficient $5.9 \cdot 10^{-5}$
6 [17, eq. (31)], the pair $d_0, d_1$: $1.2 \cdot 10^{-4}$
7 [19, Lemma 18]: $0.99981 < 1$
8 [14, Lemma 38] at $H = 25/16$: $5 \cdot 10^{-4}$
9 [19, Proposition 25], $F_9(73/40)$: $6 \cdot 10^{-4}$
10 Lemma 13: $1024.9 > 1024$
11 Lemma 12, step (c): $\psi(14/5) = 0.0031$
12 [18, eq. (51)]: $1.29773 < 1.3$

**Fig. 15.** The tightest certified inequalities of the series: the relative margin by which each holds, on a logarithmic axis (values from the exact recomputations of the verification runs of 5–7 November 2026). Every one is on the right side, and none could have been rounded; the weakest leaf of the largest tree clears the cutoff by two parts in a hundred million. Exact arithmetic is not a luxury here.

### 10.2 Computing facilities

Results presented in this series of papers were computed using the National Artificial Intelligence Platform of the Office for Information Technologies and e-Government (НАЦИОНАЛНА ПЛАТФОРМА ЗА ВЕШТАЧКУ ИНТЕЛИГЕНЦИЈУ), Republic of Serbia. Some specialized calculations were run in parallel on the Chameleon testbed [21] supported by the National Science Foundation.

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# A When Mordell's jump is exact

**Lemma 19 (Mordell's inequality [27]).** *For $n \geq 3$: $\gamma_n^{n-2} \leq \gamma_{n-1}^{n-1}$.*

*Proof.* Let $L$ be any lattice of rank $n$, scaled so that $\det L = 1$, and let $v$ be a shortest nonzero vector of $L^*$. Since $L^*$ is a lattice of rank $n$ with $\det L^* = 1$, $|v|^2 = \gamma(L^*) \leq \gamma_n$. The vector $v$ is primitive in $L^*$ (a shortest vector always is), so by Fact 4 a, $M = L \cap v^\perp$ has rank $n-1$ and $\det M = |v|^2 \leq \gamma_n$. Since $M \subset L$,

$$\gamma(L) = \min L \leq \min M \leq \gamma_{n-1}\, \sqrt[n-1]{\det M} \leq \gamma_{n-1}\, \sqrt[n-1]{\gamma_n}.$$

This holds for every $L$. Taking the supremum gives $\gamma_n \leq \gamma_{n-1}\gamma_n^{1/(n-1)}$, and since $0 < \gamma_n < \infty$ this is $\gamma_n^{(n-2)/(n-1)} \leq \gamma_{n-1}$. Raise to the power $n-1$. □

*A direct proof of Theorem 17, in the form of the chain.* The deduction of Section 7 also follows directly by taking one further shortest-dual section.

*Direct proof of Theorem 17.* Let $L$ be any lattice of rank 12, scaled to $\min L = 4$, and let $D = \det L$. Let $v$ be a shortest vector of $L^*$ and $a = |v|^2$. By the definition of $\gamma_{12}$ applied to $L^*$, $a \leq \gamma_{12}(\det L^*)^{1/12} = \gamma_{12} D^{-1/12}$. The section $M = L \cap v^\perp$ has rank 11, $\min M \geq 4$ and $\det M = aD \leq \gamma_{12} D^{11/12}$ (Fact 4 a). By Theorem 14, $972 \leq \gamma_{12} D^{11/12}$, so

$$\gamma(L) = \frac{4}{\sqrt[12]{D}} \leq 4\, \sqrt[11]{\frac{\gamma_{12}}{972}}.$$

This holds for every $L$. Taking the supremum, $\gamma_{12} \leq 4(\gamma_{12}/972)^{1/11}$, that is $\gamma_{12}^{10/11} \leq 4/\sqrt[11]{972}$, or $\gamma_{12}^{10} \leq 4^{11}/972 = 4^{10}/3^5$, and we conclude as in Section 7. □

*Zero slack.* The inequality is tight. $K_{12}$ is 3-modular: $K_{12}^*$ is similar to $K_{12}/\sqrt{3}$ [9, Ch. 4], so $\min K_{12}^* = 4/3$, which is exactly the bound $4D^{-1/6}$ on the dual minimum of a lattice of rank 12 with minimum 4 and determinant $D = 729$ that attains $\gamma_{12}$. The section of $K_{12}$ orthogonal to a minimal vector of $K_{12}^*$ has minimum 4, determinant $\frac{4}{3} \cdot 729 = 972$ and 432 minimal vectors: the minimum, determinant and kissing number (the number of minimal vectors) of $K_{11}$ [28, entry KAPPA11].
We recomputed these data exactly with a short script in exact arithmetic, `hermite11_k12_check.py` of the Zenodo record of Section 9 (Python 3, standard library only, two seconds), in the Eisenstein description of $K_{12}$ [9, Ch. 4, §9]. The ambient module is $\mathcal{E}^6$, where $\mathcal{E} = \mathbb{Z}[\omega] \subset \mathbb{C}$, $\omega = e^{2\pi i/3}$, $\theta = \omega - \omega^2 = \sqrt{-3}$, with the Euclidean form

$$\langle x, y \rangle = \frac{2}{3} \operatorname{Re} \textstyle\sum_{i=1}^{6} x_i \bar{y}_i,$$

so that each copy of $\mathcal{E}$ is a hexagonal lattice of minimum $2/3$ and determinant $(2/3)^2 \cdot 3/4 = 1/3$, and $\det \mathcal{E}^6 = 3^{-6}$. Inside it,

$$K_{12} = \{x \in \mathcal{E}^6 : \; x_1 \equiv \cdots \equiv x_6 \pmod{\theta}, \;\; x_1 + \cdots + x_6 \equiv 0 \pmod{3}\}.$$

Since $K_{12} \supset 3\mathcal{E}^6$, the index is counted on residues modulo 3: $[\mathcal{E}^6 : K_{12}] = 729$, hence $\det K_{12} = 3^{-6} \cdot 729^2 = 729$. The script checked this index, the 756 minimal vectors of norm 4, $3K_{12}^* \subset K_{12}$, the 756 minimal vectors of $K_{12}^*$ of norm $4/3$, and, for each of them, a section of determinant 972 with 432 minimal vectors.

*When the jump is exact.* In the language of determinants at minimum $4$ (equation (1)), Lemma 19 reads: if every lattice of rank $n-1$ with minimum $4$ has determinant at least $D_{n-1}$, then every lattice of rank $n$ with minimum $4$ has determinant at least $(D_{n-1}/4)^{n/(n-2)}$. The jump from $n-1$ to $n$ is *exact* when this number is the determinant $D_n$ of the record lattice of rank $n$, that is, when

$$D_{n-1} = 4\sqrt[n]{D_n^{n-2}}. \tag{6}$$

The following proposition explains when (6) holds.

**Proposition 20.** *Let $L$ be a lattice of rank $n \geq 3$ with minimum $4$ and determinant $D$ which is similar to its dual. Then $\min L^* = 4D^{-2/n}$, and the section $M = L \cap v^\perp$ by a minimal vector $v$ of $L^*$ has rank $n-1$, minimum at least $4$ and determinant exactly $4D^{(n-2)/n}$.*

*Proof.* If $L^* \cong cL$ then $\det L^* = c^{2n}D$, and $\det L^* = 1/D$ gives $c = D^{-1/n}$; hence $\min L^* = c^2 \min L = 4D^{-2/n}$. By Fact 4 a, $\det M = |v|^2 D = 4D^{-2/n}D = 4D^{(n-2)/n}$, and $\min M \geq \min L = 4$. □

The exact jump requires $\min M = 4$. Indeed, optimality of $M$ up to similarity, with $m = \min M \geq 4$, gives $\det M = (m/4)^{n-1} D_{n-1}$, and one needs $m = 4$ before (6) can be inferred. The following statement makes the hypothesis explicit.

**Proposition 21 (A sufficient exact-jump statement).** *Let $L$ have rank $n \geq 3$, minimum four, determinant $D$, and be similar to its dual. Suppose a shortest-dual section $M$ has minimum* exactly *four, and its determinant $B$ is a proved lower bound for every minimum-four rank-$(n-1)$ lattice. Then $L$ is optimal and $\gamma_n = 4/D^{1/n}$.*

*Proof.* The section identity gives $B = 4D^{(n-2)/n}$. Since $M$ attains the bound, $\gamma_{n-1} = 4/B^{1/(n-1)}$. Mordell therefore gives

$$\gamma_n^{n-2} \leq \gamma_{n-1}^{n-1} = \frac{4^{n-1}}{B} = \frac{4^{n-2}}{D^{(n-2)/n}}.$$

Taking the positive $(n-2)$-th root gives the upper bound attained by $L$. □

With $D = D_n$ and $B = D_{n-1}$ the hypotheses give (6), and Lemma 19 transfers the value of $\gamma_{n-1}$ to the exact value $\gamma_n = 4/\sqrt[n]{D_n}$; this is what happens for $n = 12$, where $M$ is the section of $K_{12}$ of Section 7, with minimum exactly $4$ and determinant $972$. Table 5 (plotted in Figure 16) tests (6) against the record determinants $D_n$ of Table 1 for $3 \leq n \leq 24$. Exactness occurs for $n = 4, 8, 12, 16, 24$ and for no other $n$ in this range. In each of the five cases the record of rank $n$ is similar to its dual (in their conventional, unscaled versions, $E_8$ and the Leech lattice are unimodular, $D_4$ and $BW_{16}$ are 2-modular and $K_{12}$ is 3-modular; at minimum $4$ the Gram determinants and levels change, but similarity to the dual does not) and its section by a minimal dual vector is the record of rank $n-1$ ($A_3$, $E_7$, $K_{11}$, $\Lambda_{15}$, $\Lambda_{23}$), with minimum exactly $4$, as Proposition 21 requires. Gauss's $\gamma_3$ gives $\gamma_4$, Blichfeldt's $\gamma_7$ gives $\gamma_8$ (this is Mordell's 1944 observation), and Theorem 14 gives $\gamma_{12}$; the jump $\gamma_{15} \Rightarrow \gamma_{16}$ is available for the day $\gamma_{15}$ is proved, and $\gamma_{23} \Rightarrow \gamma_{24}$ would give a second proof of Cohn and Kumar's theorem if $\gamma_{23}$ were known.

**Table 5.** Mordell's jump between consecutive record lattices: $\rho_n$ is the record lattice of rank $n$ and $D_n$ its determinant at minimum 4. The jump is exact when the last column equals $D_n$; those rows are in green.

| $n$ | $\rho_{n-1}$ | $D_{n-1}$ | $\rho_n$ | $D_n$ | $(D_{n-1}/4)^{n/(n-2)}$ |
|---|---|---|---|---|---|
| 3 | $A_2$ | 12 | $A_3$ | 32 | 27 |
| 4 | $A_3$ | 32 | $D_4$ | 64 | 64 |
| 5 | $D_4$ | 64 | $D_5$ | 128 | 101.59 |
| 6 | $D_5$ | 128 | $E_6$ | 192 | 181.02 |
| 7 | $E_6$ | 192 | $E_7$ | 256 | 225.81 |
| 8 | $E_7$ | 256 | $E_8$ | 256 | 256 |
| 9 | $E_8$ | 256 | $\Lambda_9$ | 512 | 210.01 |
| 10 | $\Lambda_9$ | 512 | $\Lambda_{10}$ | 768 | 430.54 |
| 11 | $\Lambda_{10}$ | 768 | $K_{11}$ | 972 | 617.59 |
| 12 | $K_{11}$ | 972 | $K_{12}$ | 729 | 729 |
| 13 | $K_{12}$ | 729 | $K_{13}$ | 972 | 469.57 |
| 14 | $K_{13}$ | 972 | $\Lambda_{14}$ | 768 | 607.03 |
| 15 | $\Lambda_{14}$ | 768 | $\Lambda_{15}$ | 512 | 431.10 |
| 16 | $\Lambda_{15}$ | 512 | $BW_{16} = \Lambda_{16}$ | 256 | 256 |
| 17 | $\Lambda_{16}$ | 256 | $\Lambda_{17}$ | 256 | 111.43 |
| 18 | $\Lambda_{17}$ | 256 | $\Lambda_{18}$ | 192 | 107.63 |
| 19 | $\Lambda_{18}$ | 192 | $\Lambda_{19}$ | 128 | 75.69 |
| 20 | $\Lambda_{19}$ | 128 | $\Lambda_{20}$ | 64 | 47.03 |
| 21 | $\Lambda_{20}$ | 64 | $\Lambda_{21}$ | 32 | 21.42 |
| 22 | $\Lambda_{21}$ | 32 | $\Lambda_{22}$ | 12 | 9.85 |
| 23 | $\Lambda_{22}$ | 12 | $\Lambda_{23}$ | 4 | 3.33 |
| 24 | $\Lambda_{23}$ | 4 | Leech | 1 | 1 |

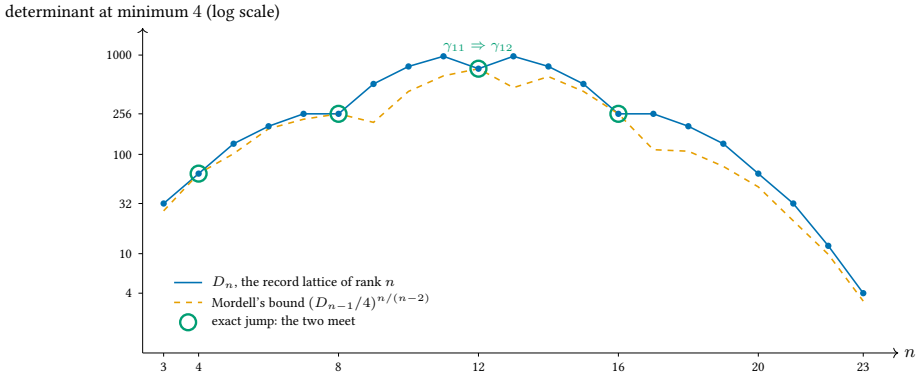


**Fig. 16.** Mordell's inequality $\gamma_n^{n-2} \le \gamma_{n-1}^{n-1}$ along the record lattices (the data of Table 5). Where the lower bound it gives for the determinant at minimum 4 equals the record $D_n$, the value $\gamma_n$ follows from $\gamma_{n-1}$: $n = 4$, 8, 12, 16, and $n = 24$ (outside the plot). The jump at $n = 12$ is the one this series of papers uses.

*Remark 22 (Upper bounds above 12).* Iterating Lemma 19 from $\gamma_{12} = 4/\sqrt{3}$ gives $\gamma_n \le (4/\sqrt{3})^{(n-1)/11}$ for $n \ge 12$, because the exponents telescope: $\prod_{k=13}^{n} \frac{k-1}{k-2} = \frac{n-1}{11}$. Under Theorem 14 this yields $\gamma_{13} \le 2.4920$ and $\gamma_{14} \le 2.6890$, against the conjectured values $\gamma(K_{13}) = 4/\sqrt[13]{972} \approx 2.3563$ and $\gamma(\Lambda_{14}) = 4/\sqrt[14]{768} \approx 2.4886$; for comparison, the three-point bounds of [3], valid for all packings, give $\gamma_{13} \le 2.6334$ and $\gamma_{14} \le 2.7639$ when converted from their density bounds. In the range of dimensions tabulated in [3, Table 1.1], the three-point bounds are the better ones from $n = 15$ on; this comparison concerns the tabulated dimensions.

## B The reference forms of Section 5

Let $b_1 = \frac{1}{2}(1,-1,-1,-1,-1,-1,-1,1)$, $b_2 = f_1 + f_2$ and $b_k = f_{k-1} - f_{k-2}$ for $3 \le k \le 8$ (the simple roots of $E_8$), let $E_8 = \mathbb{Z}b_1 + \cdots + \mathbb{Z}b_8$, and let $G_{E_8}$ be the Gram matrix of $b_1, \ldots, b_8$. Let $D_8$ be the lattice of integer vectors with even coordinate sum, and $h = \frac{1}{2}(1, \ldots, 1)$.

*$E_8 = D_8 \cup (D_8 + h)$, and $\det G_{E_8} = 1$.* The set $D_8 \cup (D_8 + h)$ is a lattice, because $2h \in D_8$. It contains $b_2, \ldots, b_8 \in D_8$ and $b_1 \in D_8 + h$, so it contains $E_8$. Conversely, $b_2, \ldots, b_8$ give all $f_i \pm f_j$ with $i, j \le 7$; then $2b_1 + (f_2 + f_3) + (f_4 + f_5) + (f_6 + f_7) = f_1 + f_8$ gives all $f_i \pm f_8$; these vectors generate $D_8$; and $h = b_1 + (f_2 + \cdots + f_7)$. So $E_8$ contains $D_8$ with index 2.
Since $D_8$ has index 2 in $\mathbb{Z}^8$, $\det D_8 = 4$ and $\det G_{E_8} = \det D_8/2^2 = 1$.

*The three matrices.* $P_\Lambda$ and $P_S$ are the matrices $2C^TC$ of Section 5.5. Computing the inner products $b_i \cdot b_9$ gives

$$P_\Lambda = \begin{pmatrix} 2G_{E_8} & f_1 + 2f_2 - 2f_3 \\ (f_1 + 2f_2 - 2f_3)^T & 4 \end{pmatrix}, \qquad P_S = \begin{pmatrix} 2G_{E_8} & \frac{2}{3}f_2 \\ \frac{2}{3}f_2^T & 4 \end{pmatrix}.$$

The third matrix is the Gram matrix of KAPPA9.2 in [28]:

$$P_K = \begin{pmatrix} 4 & 2 & 0 & 0 & 0 & -2 & -2 & -2 & 0 \\ 2 & 4 & -2 & -2 & 0 & -2 & -1 & -1 & 1 \\ 0 & -2 & 4 & 0 & -2 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 4 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & -2 & 0 & 4 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 0 & 4 & 0 & 2 & 0 \\ -2 & -1 & 1 & 0 & 0 & 0 & 4 & 2 & -1 \\ -2 & -1 & 0 & -1 & 1 & 2 & 2 & 4 & -2 \\ 0 & 1 & 0 & -1 & -1 & 0 & -1 & -2 & 4 \end{pmatrix}.$$

The determinants are $\det P_\Lambda = 2^9 \det C^T C = 512$ and $\det P_S = 2^9 \cdot 10/9 = 5120/9$ (Section 5.5), and, by a direct calculation, $\det P_K = 768$.

## C Bernstein coefficients

Section 5 needs $p > 0$ on $[0, 3/5]$ for the quintic

$$p(x) = 324 - 3357x + 13806x^2 - 26085x^3 + 22704x^4 - 7396x^5.$$

On an interval $[x_0, x_1]$ write

$$p\big(x_0 + (x_1 - x_0)s\big) = \sum_{k=0}^{5} c_k \binom{5}{k} s^k (1-s)^{5-k}, \qquad s \in [0,1];$$

the $c_k$ are the Bernstein coefficients of $p$ on $[x_0, x_1]$, obtained from the coefficients of $p$ by the change of variable and the identity $s^j = \sum_{k \ge j} \binom{k}{j}\binom{5}{j}^{-1}\binom{5}{k} s^k (1-s)^{5-k}$. Since the Bernstein polynomials are nonnegative and sum to 1, $c_k > 0$ for all $k$ implies $p > 0$ on $[x_0, x_1]$, and $\min_k c_k$ is a lower bound for $p$ there. Table 6 gives the coefficients on $[0, 3/10]$ and on $[3/10, 3/5]$, exactly and to two decimals; all twelve are positive, the least being $236331/12500 = 18.906\ldots$ on the first interval and $40392/3125 = 12.925\ldots$ on the second. As a consistency check, $c_5$ on the first interval and $c_0$ on the second are both $p(3/10) = 263439/12500$.

## D The macadamia

The ten forms of Table 4 are named after the macadamia nut, whose shell is famously the toughest of all: it takes up to 300 pounds of pressure per square inch to break open, so that ordinary nutcrackers fail and a specialised heavy-duty cracker, or a vice, is needed. Each of the ten forms resisted every general-purpose certificate of [14] and had to be cracked by a tool built for it, or for the few forms that share its geometry: this is the origin of the five companion papers.

**Table 6.** Bernstein coefficients of $p$ on the two intervals (exact, and rounded to two decimals).

| $k$ | $c_k$ on $[0, 3/10]$ | | $c_k$ on $[3/10, 3/5]$ | |
|---|---|---|---|---|
| 0 | 324 | 324.00 | 263439/12500 | 21.08 |
| 1 | 6129/50 | 122.58 | 290547/12500 | 23.24 |
| 2 | 22707/500 | 45.41 | 1537353/50000 | 30.75 |
| 3 | 8829/400 | 22.07 | 143721/5000 | 28.74 |
| 4 | 236331/12500 | 18.91 | 66258/3125 | 21.20 |
| 5 | 263439/12500 | 21.08 | 40392/3125 | 12.93 |




**Fig. 17.** The macadamia (*Macadamia ternifolia*), with its nut in its shell and husk. Watercolour by Deborah Griscom Passmore, 1904 [8].

## E Where each finite claim is checked

Figure 18 lists every finite claim of the series, the program that checks it and the line that the program prints when the check passes. The three archives are those of Section 9.

| **finite claim** | **checked by** | **verdict to look for** |
|---|---|---|
| this paper, Computational result 9: the three forms | phase **ten**: `result9.py` | `VERDICT: RESULT 9 HOLDS` |
| [14], Computational result 5: the scan | phase **verify B**: `verifier.py` | keys `EQUAL`, 0 missing, 0 extra |
| [14], Computational result 19: the database | phases **verify B** and **C** | `RESULT: EVERYTHING PASSED.` |
| [14], Computational result 46 | `verify_cases.py -case 0052` | `PASS_LOCAL_CHECKING_SCHEDULE` |
| [15], Computational results 16, 21, 22, 24, 27, 29, 31 | `verify_cases.py -case 0001, 0014, 0015, 0059` | |
| [16], Computational results 11, 13 | `verify_cases.py -case 0019, 0017` | |
| [17], Computational result 14 | `verify_cases.py -case 0008` | |
| [18], Computational result 17 | `verify_cases.py -case 0030` | |
| [19], Computational results 14–17 | `verify_cases.py -case 0050` | |
| analytic proofs and assembly (Section 8) | `lake build, Audit.lean, AuditCensus.lean` | only `propext, Classical.choice, Quot.sound` |

**Fig. 18.** Where each finite claim of the series is checked. The colour of a row is the archive that holds its data: red, the reproduction pack `hermite11_rank9_reproduction.zip`; blue, the certificate archive `hermite11_local_certificates.zip`; green, the Lean archive `hermite11_lean_proofs.zip`. All three are in the Zenodo record of Section 9. The case checks are those of `verify_cases.py` and its plan of 125 steps; `-case all` checks the ten cases at once.