Achieving vCCA security from the linear-only homomorphism assumption

Marina Checri, Pierre-Emmanuel Clet, Marc Renard and Renaud Sirdey

Université Paris-Saclay, CEA, List, Palaiseau, France

Abstract. In the wake of Manulis and Nguyen's Eurocrypt'24 paper, new CCA security notions, vCCA and vCCA^D, and associated construction blueprints have been proposed to leverage either CPA or CPA^D secure FHE beyond the CCA1 security barrier. These two notions are the strongest CCA security notions so far achievable, respectively, by correct and approximate homomorphic schemes. However, the only known construction strategies intimately require advanced SNARK machinery, undermining their practicality. In this context, this paper aims to achieve these advanced CCA security notions in the restricted case of linearly homomorphic encryption, without resorting to SNARKs. To do so, we investigate the relationship between the Linear-Only Homomorphism (LOH) assumption, an assumption that has been used for more than a decade at the core of several proof-of-knowledge constructions, and these two recent security notions (vCCA and vCCA^D). On the bright side, when working under the correctness assumption, we establish that the LOH property is sufficient to achieve vCCA security in both the private and public-key settings. In the public-key setting, we further show that a surprisingly simple and previously known Paillier-based construction also achieves this level of security, at only twice the cost of the baseline scheme. We then turn our attention to LWE-based schemes for which the Pandora box of decryption errors opens up. In the private-key setting, we are only able to achieve CPA^D and vCCA^D security in a fairly restrictive non-adaptive setting, in which vCCA^D collapses onto a weak relaxation of CCA1. Finally, we eventually achieve adaptive vCCA^D security provided that the number of ciphertexts given to the adversary is suitably restricted. While bridging the gap towards credible practicality requires further work, this is a first step towards obtaining linear homomorphic schemes achieving these recent CCA security notions by means only of relatively lightweight machinery.

1 Introduction

Since its inception more than ten years ago, Fully Homomorphic Encryption (FHE) has been the subject of much research towards more efficiency and better practicality. However, from a security perspective, FHE still raises several questions and challenges. In particular, all the FHE usable in practice, BFV [Bra12, FV12], BGV [BGV14], CKKS [CKKS17] and TFHE [CGGI16], achieve only CPA security and are also known to be trivially CCA1 insecure. Although it is well-known that malleability is contradictory with CCA2 security, building efficient FHE constructions achieving some degree of CCA security (e.g. CCA1) remains a very important open challenge. Recently, this topic is the subject of renewed attention at least from a theoretical viewpoint [MN24, BCF+25].

On the one hand, a number of correct linearly homomorphic schemes are known to achieve CCA1 security under standard or almost standard assumptions. This is the case,

E-mail: marina.checri@cea.fr (Marina Checri), pierre-emmanuel.clet@cea.fr (Pierre-Emmanuel Clet), marc.renard@cea.fr (Marc Renard), renaud.sirdey@cea.fr (Renaud Sirdey)

as recently established in [Lib25], for a simple variant of the Paillier-ElGamal scheme where λ plaintext bits are forced to be zeroes and the Dåmgard-ElGamal scheme (which is now proven CCA1 secure solely under the DDH assumption), or the Cramer-Shoup Lite scheme [CS03]. On the other hand, recent works investigating CCA1 or beyond-CCA1 security for *compact* FHE from a more theoretical perspective, do so by starting either from a CPA/correct [MN24] or CPA^D [BCF$^+$25] secure scheme, and augment it with the machinery necessary for proving some form of plaintext awareness on fresh ciphertexts as well as correctness of evaluated ciphertexts derivation. Unfortunately, these generic construction blueprints are not easily amenable to efficient implementations in their full generality, essentially because they require powerful SNARK machinery.

A natural question is then the following: is it possible to achieve vCCA or vCCA^D security by resorting only to lightweight machinery? In particular, without using SNARKs? In this paper, we provide a first round of answers to this question in the case of *linearly* homomorphic encryption, using the Linear-Only Homomorphism (LOH) assumption [BCI$^+$13] as a yardstick. We first establish a connection between the LOH assumption and the vCCA security notion recently introduced by Manulis and Nguyen [MN24] (CCA1<vCCA<CCA2), by showing that any CPA/correct linear HE satisfying this assumption also achieves vCCA security in both the private and public-key settings. In the public-key setting, we then show that a surprisingly simple (and previously known) construction based on the Paillier encryption scheme [Pai99] is vCCA secure under the reasonable assumption that it has the LOH property. We then investigate LWE-based schemes which are notoriously prone to decryption errors. For such schemes, we have to craft a hierarchy of constructions in order to first achieve CPA^D security and, then, leverage on this to further achieve vCCA^D security (a strictly stronger generalization of vCCA security when the correctness assumption is relaxed [BCF$^+$25]) in the private-key setting. We, however, only formally achieve this in a fairly restrictive *non-adaptive* setting (in the sense of [LM21]), in which vCCA^D collapses onto a weak relaxation of CCA1. Finally, we extend our line of LWE-based schemes to eventually achieve adaptive vCCA^D security under the conjectured (adaptive) CPA^D security of our first construction and the restriction that the number of ciphertexts given to the adversary is suitably restricted.

Although generalizing our techniques to the FHE setting remains largely open, this work is a first step towards obtaining *compact* (linearly) homomorphic schemes that achieve the strongest CCA security notions so far known to be achievable by homomorphic schemes using relatively lightweight machinery.

1.1 Summary of contributions

The contributions of this paper are as follows:

- We first prove the general results that any (perfectly or statistically) correct CPA-secure linearly homomorphic scheme satisfying the Linear-Only Homomorphism assumption achieves vCCA security (hence also achieves CCA1 security) in both the private and public-key settings.

- In the public-key setting, we show that the (well-known) "two-ciphertexts" blueprint, which consists of sparsifying the ciphertext domain of a correct CPA-secure linearly homomorphic scheme by following a Knowledge-of-Exponent (KEA) template, achieves vCCA security under the assumption that it has the LOH property. We do so by explicitly considering the Paillier cryptosystem as the base scheme, but the approach can be expected to apply starting from any (statistically) correct CPA-secure linearly HE scheme.

- In the case of LWE-based linear HE, where we cannot operate under the correctness assumption, we focus on the *private-key setting*:

  - Starting from the Regev scheme, we propose a multi-secret variant, i.e. where the same a vector is reused over several LWE pairs under independent keys, with a ciphertext-dependent variance estimation procedure which essentially consists in using several message-less slots to obtain an estimate for the noise variance in the ciphertext. This estimate is then used to derive a bound on the true ciphertext variance that has a negligible probability (in the security parameter) of being exceeded. This bound then allows proper smudging to occur within the decryption function of the scheme. We refer to this first baseline scheme as FS_1 and prove its CPA^D security *in the non-adaptive setting*, where the adversary specifies all its requests right after seeing the public material of the scheme. In doing so, we positively answer an open question from [LMSS22] regarding the existence of schemes with dynamic error estimation achieving at least a weak form of CPA^D security. Furthermore, FS_1 is provably immune to the attacks in [LMSS22] on a "natural" class of schemes with dynamic error estimation.
  - We then build on that latter scheme to leverage the (well-known) approach consisting of applying the Knowledge-of-Exponent template to a multiple-secret variant of Regev (using independent key material for each slot) and show that it also is non-adaptive CPA^D secure. We refer to this scheme as FS'_1 due to its connection to the previous one. Although we show that this scheme does not satisfy the LOH property (by exhibiting a concrete attack), we show that it achieves non-adaptive vCCA^D security under the credible assumption that it has a much weaker form of the LOH property. However, in the non-adaptive setting, vCCA^D security collapses onto a weaker-than-CCA1 CCA security notion, which we refer to as CCA0 (Sect. 4.3.3). In this latter, in the private-key setting, the adversary specifies all its decryption requests before seeing any ciphertext. As weak as this relaxation is, we give concrete CCA0 attacks that are indeed thwarted by our scheme.
  - We then study the adaptive setting under the conjectured (adaptive) CPA^D security of FS'_1 (a conjecture that is presently not disproved by any concrete attacks, including those in [LMSS22], provided that the number of ciphertexts given to the adversary is suitably restricted). In that setting, we have to augment the latter scheme with a linearly homomorphic (keyed) hash function to credibly achieve a weak variant of LOH in which the adversary is allowed to see only a limited number of ciphertexts (with a number $n + K - O(1)$ of ciphertexts, where n is the LWE dimension and $K = O(\lambda)$ is the number of slots in a multi-secret LWE ciphertext). This further gives a generic pattern whereby any CPA^D secure scheme with *linear* linearly homomorphic operators (should such schemes exist[1]) can be leveraged into a vCCA^D secure scheme (still under the above restriction on the number of ciphertexts).

- Finally, this paper also contains a number of side contributions:

  - We revisit the definition of vCCA security (and related notions) for avoiding the use of an extractor in the second step decryption oracle. This new definition is based on a formalization of admissible adversaries by means of a $\mathsf{Finalize}$ procedure, a formalism that has recently been used for reasoning on security notions in the field of Functional Encryption. As a consequence, extractors only reappear *within security proofs* when building vCCA secure schemes *from Knowledge Assumptions*, which is the strategy we pursue in this paper.
  - We provide a non-asymptotic analysis for the mean of squares of centered discrete Gaussian deviates that allows us to obtain new confidence intervals on

[1] We negatively answer this question in a follow-up work [CCRS26].

this quantity. These results are then used in our candidate schemes to get "safe" variance bounds suitable for smudging.

– We provide a new CPA^D attack path on the vanilla Regev scheme, its RLWE variant and the other mainstream FHE schemes which are based on them in Sect. 8.4.4. This attack is of independent interest.

1.2 Paper organization

This paper is organized as follows. First, Sect. 2 covers preliminary discussions on the strength of the LOH assumption and motivates its use in this paper. Second, Sect. 3 positions our contribution with respect to other works on CCA security for FHE and the assumptions they rely on. Sect. 4 then covers the preliminaries. In Sect. 5 we revisit the formal definitions of vCCA and vCCA^D security. Then Sect. 6 focuses on the results and constructions we obtain under the correctness assumption of the underlying linear HE scheme, and Sect. 8 addresses our LWE-based constructions after a technical section, Sect. 7, that provides a non-asymptotic analysis of the mean of squares of discrete Gaussian deviates. Lastly, Sect. 10 concludes the paper by considerations towards practicality and perspectives.

2 Preliminary discussion on the linear-only homomorphism assumption

In this paper we work under the Linear-only Homomorphism (LOH) assumption introduced in [BCI+13, BCI+12, BCI+22] (and formally recalled in Sect. 4.5). This assumption is a distant sibling of the Knowledge-of-Exponent assumption introduced and used by Dåmgard in the early nineties to prove the CCA1 security of a variant of ElGamal [Dam92, BP04c], known since then as Dåmgard-ElGamal or DEG. In a nutshell, this latter assumption (sometimes referred to as KEA1 or DHK0) states that, given a pair $(g, h = g^\alpha)$ of elements in a cyclic group $\mathbb{G}_p$ in which the discrete log problem is hard, the *only* way to build a pair (X, Y) such that $Y = X^\alpha$, without knowing α, is to proceed the "natural" way, by picking some value $x \in \mathbb{Z}_p$ and letting $X = g^x$ and $Y = h^x$. In security proofs, the assumption materializes itself through the existence of an *extractor* that retrieves x when the adversary outputs such a pair, given auxiliary data forming the trace of execution of the adversary and including the randomness it used. As such it is a non-standard assumption which is non-falsifiable (following Naor's challenge-based categories of falsification [Nao03]) in the sense that doing so requires proving that there are adversaries for which there exists no extractor i.e. such that the "infinite" number of possible extraction algorithms fail[2]. By contrast, more standard assumptions can be falsified by exhibiting concrete attacks, e.g. a PPT algorithm solving Search-LWE with a non-negligible probability of success.

In a similar spirit, the LOH assumption states that the only way for building a valid ciphertext under a linearly (and no more) homomorphic scheme is to homomorphically evaluate a linear combination by applying the homomorphic operators of the scheme over ciphertexts that have been outputted by the encryption function. In exchange, the assumption materializes via the existence of an extractor which, given a valid ciphertext generated by the adversary, retrieves the linear combination that "explains" the ciphertext. Additionally, the LOH assumption is related to the Knowledge-of-Exponent assumption as it is heuristically achieved (e.g. [BCI+13, GGPR13]) following a "two-ciphertexts" blueprint which consists in sparsifying the ciphertext domain of a linearly homomorphic scheme

[2]Still, this does not mean that this kind of statement cannot be proved and, indeed, a "degree 2" variant of KEA1, KEA2, was eventually proven false in [BP04b] (though the techniques do not provide any insights on KEA1).

(e.g. Paillier) by following an approach analogous to Dåmgard's original pattern: encrypt a message m as a pair of ciphertexts $(\mathsf{Enc}(m), \mathsf{Enc}(\alpha \cdot m))$ for a secret multiplier α and check that the linear relation holds upon decryption.

Nowadays, following Naor's seminal criticism regarding using non-falsifiable assumptions in the design of cryptographic schemes [Nao03], it may thus however seem that anyone using the LOH assumption to do so is "in a state of sin". The situation is however different in the realm of Proof-of-Knowledge where the KEA (or variants) and LOH assumptions are used in numerous works, e.g. [BP04a, HT98, BP04b, BCI+13, BISW17, GMNO18, Nit19, BHI+24a]. This state of affairs is also explained by Gentry&Wichs' no-go theorem which establishes that no black-box reduction exists to prove a SNARG or SNARK construct secure based on a falsifiable assumption [GW11]. This thus motivates the use of such assumptions when building Proof-of-Knowledge primitives. Furthermore, we also emphasize that the CCA1 security of Dåmgard-ElGamal has recently been proven *solely* under the DDH assumption [Lib25]. Although this does not easily carry over to the two-ciphertexts heuristic of Bitansky et al. [BCI+13, GGPR13, BCI+22] that we use in our paper, it hints that such constructions may have stronger foundations than previously considered.

In this paper, we investigate the (non-straightforward) relationship between the LOH assumption and two recent CCA security notions for FHE, vCCA and vCCA^D [MN24, BCF+25]. It turns out that these two notions are intimately connected to Proof-of-Knowledge approaches as the decryption oracles in their respective security games embed a PPT witness extractor allowing to retrieve both the input ciphertexts and the function homomorphically applied when a well-formed evaluated ciphertext is submitted by the adversary. As such, the spirit of these two notions is to model construction blueprints which embed proof material in their ciphertexts and rely on a SNARK to enforce genuine homomorphic evaluations over some well-formed input ciphertexts. Thus, because of the powerful witness extractor involved in the security notion definition, we may speculate (also following [YYS25, FRSS26]) that achieving vCCA security under falsifiable assumptions is also prevented by Gentry&Wichs' result. Although this may not rule out that more standard assumptions may eventually be used, it may thus not seem unreasonable to rely on non-falsifiable assumptions for designing vCCA or vCCA^D secure cryptographic schemes, at least in a first attempt to do so. To some extent, this is the goal that we successfully achieve in this paper.

3 Positioning of the paper

In light of the previous discussion, it may be interesting to highlight that the large majority of works investigating either or both CCA security notions achievable by FHE or FHE constructions achieving some degree of CCA security [BSW12, CRRV17, MN24, YYS25] do so under some correctness assumption. This is to the notable exception of [LMSV11][3] and [BCF+25], which investigates how to leverage CPA^D secure approximate FHE to build CCA secure ones.

Then, all the constructions in these papers require either the random oracle model or some non falsifiable assumption due to their reliance on general zk-SNARKs [GW11] or even stronger primitives such as iO. This is to the exception of the recent (Crypto'25) paper of Yang et. al [YYS25] which achieves the tour de force of giving a construction that is IV-CCA secure (a notion they define and study in the paper, such that CCA1$\prec$IV-CCA$\prec$vCCA) in the standard model, based solely on the LWE assumption (and the additional assumption that *perfectly* correct FHE can be built from LWE[4]). On the downside, their construction is

[3] However, the assumption underlying the CPA security of the construction in [LMSV11] was eventually broken [CDPR16, BEF+17].

[4] Indeed, [YYS25] proposes an LWE-based construction instantiating the Naor-Yung paradigm, which explicitly requires *perfect* correctness from the underlying (FHE) scheme [DNR04]. Although no explicit

non-compact, in the sense that, in IV-CCA, the second step decryption oracle specification relies on a verification algorithm, which is part of the FHE scheme and which takes as input a ciphertext to be decrypted and *a set of input ciphertexts* that explains the former (via some legit homomorphic evaluation).

The present paper thus departs from the above ones for several reasons. In the first part of the paper (Sect. 6), we work under the correctness assumption and we provide a *compact* vCCA secure construction under a *previously known* non-falsifiable assumption (LOH). However, the construction itself is *simple* in the sense that it relies only on the *previously known* "two ciphertexts" heuristic (introduced in the previous section and duly detailed in Sect. 4.5 and 6.2) and, as such, does not require general SNARK machinery. Then, in the second part of the paper (Sect. 8), we investigate how the results in the first part of the paper may be adapted when we *relax the correctness assumption*. On the downside, our techniques are tightly coupled to linearly homomorphic encryption and difficult to generalize beyond that capability.

Finally, we emphasize that all the constructions in the present paper are very simple compared to the general blueprints considered in the two papers [MN24, BCF+25] which rely on general purpose SNARK machinery. This is particularly so when working over the correctness assumption where our concrete two-slots Paillier construction achieves vCCA security at "just" twice the cost of achieving CPA security, when general purpose SNARK constructions are known to be computationally heavy. In terms of cryptographic assumptions, there is however no particular benefit since the LOH assumption yields SNARK constructs (at the cost of additional non lightweight machinery beyond the linear-only CPA-secure scheme). So our construction approach shows how to turn a particular non-falsifiable assumption into simple constructions achieving vCCA security.

4 Preliminaries

4.1 Basic notations

Given $l, u \in \mathbb{Z}^2$, we use $[\![l, u]\!]$ to denote the set $\{l, l+1, ..., u-1, u\}$. Reduction modulo q is denoted as $[.]_q$. We use this notation explicitly only when it avoids possible ambiguities.

Let $\lambda > 0$, recall that $\mathrm{neg}(\lambda) = 2^{-\lambda}$. Given two discrete random variables X and Y we write $X \overset{i}{=} Y$ when the distribution of X and that of Y are such that $d(f_X, f_Y) \leq \mathrm{neg}(\lambda)$ where

$$d(f_X, f_Y) = \frac{1}{2} \sum_{k=-\infty}^{+\infty} |P(X = k) - P(Y = k)|$$

is the usual statistical distance. In this case, X and Y are said to be *statistically indistinguishable* or, for short, *indistinguishable* from one another. For simplicity sake, we write λ to denote both the computational security parameter *and* the statistical security parameter.

4.2 Basic definitions

We define an encryption scheme $\mathcal{E} = (\mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec})$ over key space $\mathcal{K}$, plaintext domain $\mathcal{P}$ and ciphertext domain $\mathcal{C}$ as a triplet of PPT algorithms:

- KeyGen: on input 1^λ, outputs an encryption key ek and a decryption key sk.
- Enc: on input $m \in \mathcal{P}$ and ek, outputs an encryption $c \in \mathcal{C}$ of m.

construction is provided in [YYS25], their intent is to achieve this strong property by using truncated discrete Gaussians for the noise distribution in order to get exact L_∞ norm bounds [Yan25].

- Dec: on input $c \in \mathcal{C}$ and sk, outputs a decryption[5] $m \in \mathcal{P} \cup \{\perp\}$ of c.

Let COIN denote the randomness space of $\mathcal{E}$. We sometimes externalize the randomness used in the encryption function by means of the notation $\mathsf{Enc}(\mathsf{ek}, m, r)$, with $m \in \mathcal{P}$ and $r \in \mathsf{COIN}$. In this latter case, the function $\mathsf{Enc} : \mathcal{K} \times \mathcal{P} \times \mathsf{COIN} \longrightarrow \mathcal{C}$ is deterministic[6]. When ek is public, we say that $\mathcal{E}$ is a *public-key* encryption scheme *and use* pk *to denote* ek. When for all $(\mathsf{ek}, \mathsf{sk}) \in \mathcal{K}$, all $m \in \mathcal{P}$ and $r \xleftarrow{\$} \mathsf{COIN}$ we have that,

$$\Pr(\mathsf{Dec}(\mathsf{sk}, \mathsf{Enc}(\mathsf{ek}, m, r)) \neq m) \leq \mathrm{neg}(\lambda), \tag{1}$$

we say that $\mathcal{E}$ is *statistically correct* or simply *correct*. When the above probability is zero, we talk of *perfect* correctness.

Let $\mathcal{O}$ denote a set of operations and let $\mathcal{F}(\lambda)$ denote the set of circuits over $\mathcal{O}$ of polynomially bounded size in the security parameter λ. We define a homomorphic encryption (HE) scheme $\mathcal{E}_H$ over $\mathcal{F}(\lambda)$ as an encryption scheme augmented by a *deterministic*[7] polynomial-time algorithm Eval which, on input $f \in \mathcal{F}(\lambda)$ and $c_1, ..., c_L \in \mathcal{C}^L$, where L denotes the arity of function f, outputs a new *evaluated* ciphertext. When $\mathcal{E}_H$ satisfies condition (1) and when Eval is such that for all $(\mathsf{ek}, \mathsf{sk}) \in \mathcal{K}$, all $f \in \mathcal{F}(\lambda)$, all $m_0, ..., m_{L-1} \in \mathcal{P}^L$ and $(r_0, ..., r_{L-1}) \xleftarrow{\$} \mathsf{COIN}^L$,

$$\Pr(\mathsf{Dec}(\mathsf{sk}, c_e) \neq f(m_1, ..., m_{L-1})) \leq \mathrm{neg}(\lambda), \tag{2}$$

with,

$$c_e = \mathsf{Eval}(f, \mathsf{Enc}(\mathsf{ek}, m_0, r_0), ..., \mathsf{Enc}(\mathsf{ek}, m_{L-1}, r_{L-1})),$$

we say that $\mathcal{E}_H$ is a *correct* HE scheme[8]. Additionally, when we say that we are working under the *correctness assumption* for a scheme $\mathcal{E}_H$, we mean that it achieves this latter property. When this is not the case, we say that $\mathcal{E}_H$ is an *approximate* HE scheme. Consistently with [LMSS22], to avoid arbitrary schemes with unreliable Eval to be marketed as approximate HE schemes, we add an additional condition that, for some (small) $\varepsilon \geq 0$, the following holds,

$$\Pr(\|\mathsf{Dec}(\mathsf{sk}, c_e) - f(m_0, ..., m_{L-1})\|_\infty \leq \varepsilon) \geq \mu, \tag{3}$$

with[9] $\mu \geq \frac{3}{4}$. Lastly, a scheme such that $\varepsilon = 0$ and $\frac{3}{4} \leq \mu < 1 - \mathrm{neg}(\lambda)$ is said to be *somewhat correct*. Note that it is a common misconception that only the approximate CKKS scheme does not satisfy correctness. Although schemes such as BGV/BFV and TFHE are usually said to be "exact", they are all only somewhat correct (following the above definition) and prone to decryption errors. This raises security issues also for those schemes in the CPA^D model resulting in practical full key recovery attacks [CSBB24, CCP+24] (even, for some parameters, in cases where bootstrapping is systematically used [CCP+24]).

4.3 Basic security notions

In this section, we give the formal definitions of the main security notions that we use in this paper. Unless otherwise stated, we reproduce the original definitions. Some of the definitions in this section are revisited in Sect. 5.

[5] Decryption may not be deterministic.

[6] In this case we assume that r is uniformly picked in COIN to provide a seed that the encryption function then uses to sample from the proper distributions.

[7] As is the case for the mainstream FHE schemes such as BFV, BGV, TFHE and even CKKS.

[8] Remark that the definition of correctness is independent of whether the scheme is single or multiple hops. Indeed, a call to Eval can be arbitrarily partitioned between several hops (which essentially means that intermediary ciphertexts are transferred without alteration in-between hops). So, for every multi-hop FHE evaluation there is an equivalent single-hop calculation that eventually takes fresh well-formed ciphertexts as inputs.

[9] In practice μ is typically chosen above $1 - 2^{-40}$. In some contexts, e.g. [ABMP24], μ even has to be at least $1 - \mathrm{neg}(\lambda)$.

4.3.1 CPA^D (and CPA$_0^D$) security.

The CPA^D game has been introduced in the context of approximate FHE [LM21]. CPA^D security is a slight extension of CPA security defined by the following Left-Or-Right multiple challenges security game.

Given a homomorphic encryption scheme $\mathcal{E}_H = (\mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec}, \mathsf{Eval})$, an adversary $\mathcal{A}$ and value λ for the security parameter, the game is parameterized by a bit $\gamma \xleftarrow{\$} \{0,1\}$, unknown to $\mathcal{A}$, and an initially empty state S of message-message-ciphertext triplets. The adversary is then granted access to the following oracles:

- Encryption oracle (E): given $m \in \mathcal{P}$ and ek compute $c = \mathsf{Enc}(\mathsf{ek}, m)$, return c after updating $S := [S; (m, m, c)]$.
- Challenge oracle (C): given $m_0, m_1 \in \mathcal{P}$ $(m_0 \neq m_1)$ and ek, compute $c = \mathsf{Enc}(\mathsf{ek}, m_\gamma)$, return c after doing $S := [S; (m_0, m_1, c)]$.
- Evaluation oracle (H): given $f \in \mathcal{F}(\lambda)$ and $l_0, \ldots, l_{L-1}$ $(l_i \in [\![0, |S|-1]\!], \forall i)$, compute,
$$m'_0 = f(S[l_0].m_0, \ldots, S[l_{L-1}].m_0),$$
and,
$$m'_1 = f(S[l_0].m_1, \ldots, S[l_{L-1}].m_1),$$
as well as,
$$c' = \mathsf{Eval}(f, S[l_0].c, \ldots, S[l_{L-1}].c),$$
return c' after doing,
$$S := [S; (m'_0, m'_1, c')].$$
- Decryption oracle (D): given $l \in [\![0, |S|-1]\!]$ and sk, proceed as follows. If $S[l].m_0 \neq S[l].m_1$, return $\perp$. Otherwise return $\mathsf{Dec}(\mathsf{sk}, S[l].c)$.

With these oracles, the CPA^D experiment is then defined as follows.

Definition 1 (CPA^D experiment [LM21])**.**

$$\begin{aligned}
&(\mathsf{ek}, \mathsf{sk}) \leftarrow \mathcal{E}_H.\mathsf{KeyGen}(1^\lambda),\\
&\gamma \xleftarrow{\$} \{0,1\},\\
&\gamma' \leftarrow \mathcal{A}^{\mathsf{E}(\mathsf{ek},.),\mathsf{C}(\mathsf{ek},.,.),\mathsf{H}(\ldots),\mathsf{D}(\mathsf{sk},.)}([\mathsf{ek}]),\\
&\gamma = \gamma'.
\end{aligned}$$

Remark that the decryption oracle accepts only ciphertexts from the game state which are necessarily *well-formed*, i.e. either produced by a call to the encryption or challenge oracle, or derived by means of the evaluation oracle i.e., derived by correctly applying homomorphic operators to ciphertexts from the game state (hence well-formed). As such, the above game does not capture any CCA aspects. Let us also emphasize that, in the above game, $\mathcal{A}$ controls the homomorphic calculations performed as f is included in the evaluation oracle request parameters.

As defined in [LM21], CPA^D security admits a strictly weaker *non-adaptive* variant of interest to us in this paper. In this variant, the adversary specifies all its requests straight after seeing the public material of the scheme (if any). *We will refer to this variant as CPA$_0^D$*. Other variants have been defined and studied in [BCF+25], where it is in particular shown that the single challenge variants of CPA^D security are strictly weaker than the multiple-challenge notion.

For completion, the definitions for CPA and CCA1 security are given in Sect. 5.1 (Def. 9).

4.3.2 vCCA and vCCA^D security.

As introduced in [MN24], vCCA security is a "CCA2-style" security notion for FHE. As such, the vCCA experiment has two decryption oracles where the second step (post-challenge generation) oracle assumes the existence of a PPT witness extractor defined as follows.

Definition 2 ($\mathcal{H}$-extraction algorithm, reproduced from [MN24])**.** Given an FHE scheme $\mathcal{E}_H = (\mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec}, \mathsf{Eval})$ with plaintext domain $\mathcal{P}$, ciphertext domain $\mathcal{C}$ and function domain $\mathcal{F}(\lambda)$, let $\mathcal{H} \subseteq \mathcal{F}(\lambda)$, an $\mathcal{H}$-extraction algorithm is a PPT algorithm $\mathsf{extract} : \mathcal{C} \times \Omega \longrightarrow \mathcal{H} \times \mathcal{C}^L \cup \{\perp\}$, where Ω is a set of auxiliary data and where, for simplicity sake, all functions in $\mathcal{H}$ are assumed to have the same arity L. Additionally, a $\mathcal{H}$-extraction algorithm $\mathsf{extract}$ is sound if for any PPT adversary $\mathcal{A}$, any $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathcal{E}_H.\mathsf{KeyGen}(1^\lambda)$, any $c \leftarrow \mathcal{A}(\mathsf{pk})$ and any $f \in \mathcal{H}$, $(f, c_0, ..., c_{L-1}) \leftarrow \mathsf{extract}(c, \mathsf{aux})$ is such that,

- c_i is a fresh ciphertext, $\forall i \in [\![0, L-1]\!]$.
- The distribution of $\mathcal{E}_H.\mathsf{Dec}(\mathsf{sk}, c)$ is indistinguishable from that of

$$f(\mathcal{E}_H.\mathsf{Dec}(\mathsf{sk}, c_0), ..., \mathcal{E}_H.\mathsf{Dec}(\mathsf{sk}, c_{L-1})).$$

With the conventions that, $f(\mathcal{E}_H.\mathsf{Dec}(\mathsf{sk}, c_0), ..., \mathcal{E}_H.\mathsf{Dec}(\mathsf{sk}, c_{L-1})) = \perp$, if $\exists i \in [\![0, L-1]\!] : \mathcal{E}_H.\mathsf{Dec}(\mathsf{sk}, c_i) = \perp$ and that $\mathsf{extract}(c, \mathsf{aux}) = (\mathsf{id}, c)$ for any fresh ciphertext c.

Following this, the vCCA experiment is defined as follows.

Definition 3 (vCCA experiment [MN24])**.**

$$\begin{aligned}
&(\mathsf{ek}, \mathsf{sk}) \leftarrow \mathcal{E}_H.\mathsf{KeyGen}(1^\lambda),\\
&(m_0, m_1, S) \leftarrow \mathcal{A}_1^{[\mathcal{E}_H.\mathsf{Enc}(\mathsf{ek},.)], \mathcal{E}_H.\mathsf{Dec}(\mathsf{sk},.)}([\mathsf{ek}], \mathsf{aux}),\\
&\gamma \xleftarrow{\$} \{0,1\},\\
&c^* \leftarrow \mathcal{E}_H.\mathsf{Enc}(\mathsf{ek}, m_\gamma),\\
&\gamma' \leftarrow \mathcal{A}_2^{[\mathcal{E}_H.\mathsf{Enc}(\mathsf{ek},.)], \mathcal{O}_{\mathrm{vCCA}}(\mathsf{sk},.,c^*,\mathsf{aux})}(S, c^*, [\mathsf{ek}], \mathsf{aux}),\\
&\gamma = \gamma',
\end{aligned}$$

with, letting $(f, c_0, ..., c_{L-1}) \leftarrow \mathsf{extract}(c, \mathsf{aux})$,

$$\mathcal{O}_{\mathrm{vCCA}}(\mathsf{sk}, c^*, c) = \begin{cases} \mathcal{E}_H.\mathsf{Dec}(\mathsf{sk}, c) & \text{if } c^* \notin \{c_0, ..., c_{L-1}\},\\ \perp & \text{otherwise.} \end{cases}$$

Definition 4 ($\mathcal{H}$-IND-vCCA security [MN24])**.** A scheme $\mathcal{E}_H$ is $\mathcal{H}$-IND-vCCA secure with respect to $\mathcal{H} \subseteq \mathcal{F}(\lambda)$ if there exists an $\mathcal{H}$-extraction algorithm $\mathsf{extract}$ such that for any PPT adversary $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$, the advantage of $\mathcal{A}$ winning the vCCA experiment is negligible in the security parameter λ.

When $\mathcal{H} = \mathcal{F}(\lambda)$, we omit the $\mathcal{F}(\lambda)$ prefix, i.e. refer to $\mathcal{F}(\lambda)$-IND-vCCA simply as IND-vCCA (and likewise for other related notions).

The initial spirit of the vCCA security notion is to model construction blueprints which embed proof material in their ciphertexts and rely on a SNARK to enforce correct homomorphic evaluations over some input ciphertexts. In this context, the above $\mathsf{extract}$ thus corresponds to the extractor of that underlying SNARK which allows to retrieve a witness from the proven statement as well as auxiliary data. The nature of the auxiliary data depends on whether extractability is black-box in some ROM [GKO+23] or white-box,

as is usually the case for knowledge assumptions[10]. Still, because of the involvement of an extractor in the definition, the security notion appears tight to SNARK constructs. In Sect. 5 (Def. 10, p. 18) we provide another definition that does not involve any extractor and that thus appears more consistent with the philosophy that security notions should be defined independently of any scheme property (as most schemes do not yield any natural extractors).

In [MN24], vCCA security is defined and studied under the correctness assumption of the underlying FHE scheme, and then further studied in [BCF+25] when that assumption is not satisfied. Then, [BCF+25] also defines the notion of vCCA^D security, which, in a nutshell, is a "CPA^D-style" multiple-challenge variant of vCCA *in which the decryption oracle further accepts byproducts of the challenge ciphertexts as long as the associated left and right cleartext evaluations coincide*. The original definition of vCCA^D security in [BCF+25] is quite close to that of vCCA security and also involve an extractor in the decryption oracle. We defer the formal definition of vCCA^D security to Sect. 5 (Def. 11 and 15, p. 19 and 20) where we provide a consistent definitional framework that avoids the pitfall of relying on extractor to define these notions.

Finally, following [BCF+25], vCCA and vCCA^D security are related as follows. *Under the correctness assumption*, the two notions are equivalent. In the general regime where approximate or somewhat correct FHE are allowed, we have that

$$\text{vCCA} \prec \text{vCCA}^D_{\text{SC}} \prec \text{vCCA}^D,$$

where vCCA$^D_{\text{SC}}$ is the single-challenge variant of vCCA^D. Additionally, still in that latter regime, vCCA security remains equivalent to its multiple-challenge variant. In terms of which notion should be targeted, the bottom line is then to target (single-challenge) vCCA security when working under the correctness assumption (and it will be our focus in Sect. 6) and to target (multiple-challenge) vCCA^D security when working in the somewhat correct setting (in Sect. 8).

4.3.3 CCA0 security

Lastly, we will also consider a non-adaptive variant of CCA(2) security where the adversary specifies all its requests straight after seeing the public material of the scheme. We will refer to this variant, defined by the following experiment, as CCA0.

Definition 5 (CCA0 experiment)**.**

$$\begin{aligned}
&(\mathsf{ek}, \mathsf{sk}) \leftarrow \mathcal{E}_H.\mathsf{KeyGen}(1^\lambda),\\
&(M^{(0)}, M^{(1)}, C, S) \leftarrow \mathcal{A}_1([\mathsf{ek}], \mathsf{aux}), \text{ with } |M^{(0)}| = |M^{(1)}|,\\
&\gamma \xleftarrow{\$} \{0,1\},\\
&C' = (\mathcal{E}_H.\mathsf{Enc}(\mathsf{ek}, M_0^{(\gamma)}), ..., \mathcal{E}_H.\mathsf{Enc}(\mathsf{ek}, M_{|M^{(\gamma)}|-1}^{(\gamma)})),\\
&M' = (\mathcal{E}_H.\mathsf{Dec}(\mathsf{sk}, C_0), ..., \mathcal{E}_H.\mathsf{Dec}(\mathsf{sk}, C_{|C|-1})),\\
&\gamma' \leftarrow \mathcal{A}_2(M', C', S, [\mathsf{ek}], \mathsf{aux}),\\
&\gamma = \gamma',
\end{aligned}$$

For consistency with other notions, we have given above the multiple challenge definition. We highlight that, in the above experiment, the adversary is given *no* oracles. The situation is similar in the single-challenge case, which implies that in this non-adaptive setting all CCA security notions (non-adaptive CCA2, single-challenge non-adaptive vCCA^D, non-adaptive vCCA and non-adaptive CCA1) collapse onto the above. We however leave

[10] See [MN24] for more details. The spirit of the original vCCA definition was arguably to favor the former option [Ngu26].

open the question of the equivalence between single and multiple challenge CCA0, although we think the two notions are equivalent since the equivalence holds for CCA2 in the usual adaptive setting [BDJR97, BDPR98]. Although this variant may not appear too restrictive in the public-key setting, where the adversary can generate ciphertexts on its own and also perform homomorphic evaluations (or any other treatments) over them before specifying its set of requests, it is much more restrictive in the private-key setting. Indeed, in that case, the adversary has to specify all of its decryption requests before seeing any ciphertext and therefore can ask for the decryption only of a priori chosen ciphertexts (in essence, in the private key setting the KeyGen step can be performed *after* running $\mathcal{A}_1$ in the above experiment).

For sanity checking, we now establish a few easy separation results to position the CCA0 security notion with respect to CCA1 and CPA_0^D (we do so in the private-key setting and implicitly assuming single challenge). We do not claim that we are the first to consider this rather weak (yet intuitive) relaxation of CCA security.

Proposition 1. *If there exist a correct private-key scheme S which is CCA1 secure, then there exists a scheme S' which is CCA0 secure and CCA1 insecure.*

Proof. The proof works under the mild assumption that $|\mathcal{P}| \geq O(2^\lambda)$. Let $S = (\mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec})$ be a CCA1 secure private-key correct scheme. We then consider $S' = (\mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec}')$ such that:

$$\mathsf{Dec}'(\mathsf{sk}, c) = \begin{cases} \mathsf{sk} & \text{if } \mathsf{Dec}(\mathsf{sk}, c) = 0 \\ \mathsf{Dec}(\mathsf{sk}, c) & \text{otherwise} \end{cases}$$

S' is CCA0 secure. CCA1 security implies CCA0 security, hence S is CCA0 secure. Then S''s CCA0 security follows from a trivial reduction to that of S with the reduction transferring all decryption requests consisting of a priori chosen ciphertext from the CCA0 adversary as is to the CCA0 challenger against S. Finally, under the assumption that $|\mathcal{P}| \geq O(2^\lambda)$, $\mathcal{A}$ has a negligible probability of submitting an encryption of 0.

S' is CCA1 insecure. The CCA1 adversary simply asks an encryption of 0 to get $c = \mathsf{Enc}(\mathsf{ek}, 0)$ and then asks the decryption of c to get $\mathsf{sk} = \mathsf{Dec}'(\mathsf{sk}, c)$. Then, once the challenge ciphertext is obtained, $\mathcal{A}$ simply decrypts it on its own. □

We also show that CCA0 is independent from non-adaptive CPA^D security from [LM21, Def. 5], which we refer to as CPA_0^D (Sect. 4.3.1).

Proposition 2 ($\text{CCA0} \not\Longrightarrow \text{CPA}_0^D$)**.** *If there exists a correct scheme S scheme that is CCA0 secure, then there exists a scheme S' which is CCA0 secure but CPA_0^D insecure.*

Proof. Because S is correct and CPA0 secure ($\text{CCA0} \implies \text{CPA0}$), it is also CPA_0^D secure (the proof of [LM21, Lem. 1] works as is in the non-adaptive setting). The claim then follows using the same counter-example to that in [BCF+25, Prop. 2] (we highlight that the CPA_2^D attack in that proof is a CPA_0^D one). □

Proposition 3 ($\text{CPA}_0^D \not\Longrightarrow \text{CCA0}$)**.** *If there exists a correct scheme S that is CPA0 (hence CPA_0^D) secure, then there exists a scheme S' which is CPA_0^D secure but CCA0 insecure.*

Proof. The claim follows using the same counter-example to that in [BCF+25, Prop. 1], we highlight that the vCCA^D attack in that proof is a vCCA_0^D (i.e. a CCA0) one. □

For convenience, we provide in Fig. 1 a summary of the relationships among the security notions considered in this paper.

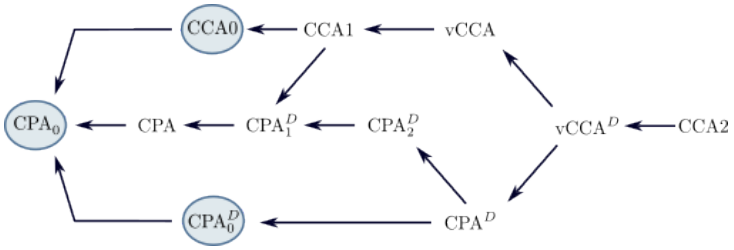

Figure 1: Overview of the relations among the security notions considered in this paper. The notions in grey are the non-adaptive variants introduced or discussed at the end of Sect. 4.3.2.

4.4 Smudging

Smudging is a technique that consist in "hiding" a small noise by flooding it in a much larger noise such that the effect of the small noise becomes negligible. Smudging was first introduced in [BD10] in the context of threshold PKE and later, in the context of threshold FHE in [AJL+12, BGG+18] (essentially to make sure that a decryption oracle over well-formed ciphertexts can be simulated without actually decrypting in the relevant security reductions) and has been used, since then in several constructions proposals, e.g. [MTPBH21]. Beyond threshold FHE, smudging has also been suggested as a countermeasure to CPA^D attacks against CKKS [LMSWS22] (where it is referred to as noise flooding) or the other "exact" FHE schemes [CSBB24]. Smudging comes into different flavors depending on whether the statistical distance or the Rényi divergence is considered [CSS+22, BS23, MS23] or whether worst-case/non-worst-case smudging should be performed [BCDS+24, GNSJ24].

In this work, for simplicity sake, we consider *worst-case* smudging based on the *statistical distance* (as such we do not claim that this simple approach is optimal and more advanced approaches, e.g. [PS24], may yield smaller smudging noise bounds or variances, eventually leading to smaller LWE parameters). More specifically, we rely only on simple "Smudging Lemmas" such as the following from [AJL+12] (Lemma 1 in that paper), which we reproduce below.

Lemma 1 (Smudging Lemma [AJL+12])**.** *Let B_0 and B_1 be two positive integers and let $e_0 \in [\![-B_0, B_0]\!]$ be a fixed integer. Let e_1 be chosen uniformly at random in $[\![-B_1, B_1]\!]$. Then, if $B_1 \geq 2^\lambda B_0$ the statistical distance between the distribution of e_1 and that of $e_0 + e_1$, $d(f_{e_1}, f_{e_0+e_1})$, is bounded by $\mathrm{neg}(\lambda)$.*

The above Lemma is useful as it shows that a centered uniform noise with an appropriately large support can "smudge out" a constant value and, more generally, any random variable following a distribution with a bounded support. We can further extend it in order to "smudge out" a discrete Gaussian noise, as we do just below.

Definition 6 (Centered discrete Gaussian)**.** Let $\sigma > 0$, we denote by χ_σ the centered discrete Gaussian distribution over $\mathbb{Z}$ with parameter σ, i.e. a random variable $X \xleftarrow{\$} \chi_\sigma$ satisfies, $\forall x \in \mathbb{Z}$,

$$f_X(x) = P(X = x) = \frac{1}{Z} e^{-\frac{x^2}{2\sigma^2}} = \frac{1}{Z} e^{-\frac{\pi x^2}{w^2}},$$

with $Z = \sum_{k\in\mathbb{Z}} e^{-\frac{k^2}{2\sigma^2}}$. The parameter σ is a scale parameter, and σ^2 is referred to as the variance proxy. The parameter $w = \sqrt{2\pi}\sigma$ is called the width, following the convention commonly used in lattice-based cryptography.

Let X denote a centered discrete Gaussian random variable with parameter $\sigma > 0$, following [CKS20, Corollary 17] we have that $P(|X| \geq B) \leq 2e^{-\frac{B^2}{2\sigma^2}}$ (Chernoff bound).

Lemma 2. *Let ε be a centered discrete Gaussian (or subgaussian, Sect. 7.3) random variable with variance proxy σ_0^2. Further let $B_0 = \sigma_0\sqrt{2(\lambda+1)\log 2}$, then $P(\varepsilon \notin [-B_0, B_0]) \leq 2^{-\lambda}$.*

Proof. Recall that the above Chernoff bound for the (centered) discrete Gaussian distribution tells that $P(|\varepsilon| \geq B) \leq 2e^{-\frac{B^2}{2\sigma_0^2}}$. Then, consider B_0 such that $2e^{-\frac{B_0^2}{2\sigma_0^2}} = 2^{-\lambda}$ i.e. $B_0 = \sigma_0\sqrt{2(\lambda+1)\log 2}$. □

If we now choose

$$B_1 = 2^{\lambda} B_0 = 2^{\lambda}\sigma_0\sqrt{2(\lambda+1)\log 2} \tag{4}$$

then Lemma 1 applies, directly leading the following Lemma.

Lemma 3. *Let ε be a centered discrete Gaussian (or subgaussian) random variable with variance proxy σ_0^2 and let $B_1 \geq 2^{\lambda}\sigma_0\sqrt{2(\lambda+1)\log 2}$, then $d(f_v, f_{v+\varepsilon}) \leq 2^{-\lambda}$, where v is picked uniformly in $[\![-B_1, B_1]\!]$.*

Alternatively, a small (centered) discrete Gaussian (or subgaussian) noise can also be smudged out by a discrete Gaussian noise of much larger variance proxy. This is implied by the following lemmas which are part of folklore but which we prove for completeness.

Lemma 4. *Let X denote a centered* discrete *Gaussian random variable with variance proxy σ^2 and $\varepsilon > 0$ an integer, then the statistical distance between the distribution of X and that of $X + \varepsilon$ is bounded by $\frac{\varepsilon}{\sqrt{2\pi}\sigma}$.*

Proof. Let $f_X(j) = P(X = j) = \frac{1}{Z}e^{-\frac{j^2}{2\sigma^2}}$ and $f_{X+\varepsilon}(j) = \frac{1}{Z}e^{-\frac{(j-\varepsilon)^2}{2\sigma^2}}$, with $Z = \sum_{k=-\infty}^{+\infty} e^{-\frac{k^2}{2\sigma^2}}$. We have,

$$\begin{aligned} f_X(j) - f_{X+\varepsilon}(j) &= \frac{1}{Z}e^{-\frac{j^2}{2\sigma^2}} - \frac{1}{Z}e^{-\frac{(j-\varepsilon)^2}{2\sigma^2}} \\ &= \frac{1}{Z}\left(e^{-\frac{j^2}{2\sigma^2}} - e^{-\frac{j^2-2j\varepsilon+\varepsilon^2}{2\sigma^2}}\right) \\ &= \frac{e^{-\frac{j^2}{2\sigma^2}}}{Z}\left(1 - e^{\frac{2j\varepsilon-\varepsilon^2}{2\sigma^2}}\right). \end{aligned}$$

Then $f_X(j) - f_{X+\varepsilon}(j) \geq 0$ when $e^{\frac{2j\varepsilon-\varepsilon^2}{2\sigma^2}} \leq 1$ i.e., for $\frac{2j\varepsilon-\varepsilon^2}{2\sigma^2} \leq 0$, so

$$j \leq \frac{\varepsilon}{2} = \varepsilon_0.$$

Then,

$$\begin{aligned} d(f_X, f_{X+\varepsilon}) &= \frac{1}{2}\sum_{j=-\infty}^{+\infty} |f_X(j) - f_{X+\varepsilon}(j)| \\ &= \frac{1}{2}\sum_{j=-\infty}^{\lfloor\varepsilon_0\rfloor} f_X(j) - f_{X+\varepsilon}(j) + \frac{1}{2}\sum_{j=\lfloor\varepsilon_0\rfloor+1}^{+\infty} f_{X+\varepsilon}(j) - f_X(j) \\ &= F_X(\lfloor\varepsilon_0\rfloor) - F_{X+\varepsilon}(\lfloor\varepsilon_0\rfloor) \\ &= F_X(0) + \sum_{j=0}^{\lfloor\varepsilon_0\rfloor} f_X(j) - \left(F_{X+\varepsilon}(\varepsilon) - \sum_{j=\lfloor\varepsilon_0\rfloor+1}^{\varepsilon} f_{X+\varepsilon}(j)\right) \\ &= F_X(0) + \sum_{j=0}^{\lfloor\varepsilon_0\rfloor} f_X(j) - F_X(0) + \sum_{j=\lfloor\varepsilon_0\rfloor+1}^{\varepsilon} f_{X+\varepsilon}(j) \\ &= 2\sum_{j=0}^{\lfloor\varepsilon_0\rfloor} f_X(j) \\ &\leq 2 f_X(0)\varepsilon_0. \end{aligned}$$

Since $2f_X(0)\varepsilon_0 \leq \frac{2}{Z}\frac{\varepsilon}{2} = \frac{\varepsilon}{Z}$ and, from the Poisson summation formula[11],

$$Z = \sqrt{2\pi}\sigma\left(1 + 2\sum_{k=1}^{+\infty} e^{-2\pi^2\sigma^2k^2}\right),$$

$Z > \sqrt{2\pi}\sigma$, we get $\frac{\varepsilon}{Z} \leq \frac{\varepsilon}{\sqrt{2\pi}\sigma}$. Hence the claim. □

Lemma 5. *Let $\varepsilon \in [-B, B]$ be some fixed value and X denote a centered discrete Gaussian random variable with variance proxy $\sigma^2 = \frac{2^{2\lambda}B^2}{2\pi}$ then $d(f_X, f_{X+\varepsilon}) \leq neg(\lambda)$.*

Proof. Following Lemma 4, choosing σ^2 such that,

$$2^{-\lambda} = \frac{|\varepsilon|}{\sqrt{2\pi}\sigma},$$

i.e. $\sigma = \frac{2^\lambda B}{\sqrt{2\pi}}$, leads to,

$$d(f_X, f_{X+\varepsilon}) \leq d(f_X, f_{X+B}) \leq 2^{-\lambda}.$$

□

The above lemma is useful as it shows that a discrete Gaussian noise with an appropriately large variance proxy can "smudge out" a constant value and, more generally, any random variable following a distribution with a bounded support. We can further extend it in order to "smudge out" a discrete Gaussian noise, as we do just below.

Lemma 6. *Let ε be a centered subgaussian variable and X be a centered discrete Gaussian random variable, with respective variance proxy σ_0^2 and σ_1^2, with $\sigma_1^2 = \frac{2^{2\lambda}\sigma_0^2(\lambda+1)\log 2}{\pi}$, then $d(f_X, f_{X+\varepsilon}) \leq neg(\lambda)$.*

Proof. Again, using the Chernoff bound for a subgaussian distribution yields,

$$P(|\varepsilon| \geq B) \leq 2e^{-\frac{B^2}{2\sigma_0^2}}.$$

Let us consider B_0 such that $2e^{-\frac{B_0^2}{2\sigma_0^2}} = 2^{-\lambda}$ i.e.,

$$B_0 = \sigma_0\sqrt{2(\lambda+1)\log 2}.$$

Then $\varepsilon \in [-B_0, B_0]$ with probability $1 - \mathrm{neg}(\lambda)$. The claim then follows from Lemma 5. □

4.5 The Linear-Only Homomorphism (LOH) assumption

Given an encryption scheme $\mathcal{E}_H = (\mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec}, \mathsf{Eval})$, let ImVer denote a function that verifies if a ciphertext is in the image of the encryption function with the knowledge of sk (it is essentially the "verification" part of $\mathcal{E}_H.\mathsf{Dec}$ and $\mathcal{E}_H.\mathsf{ImVer}(\mathsf{sk}, c) = \mathsf{True} \Leftrightarrow \mathcal{E}_H.\mathsf{Dec}(\mathsf{sk}, c) \neq \perp$[12]). Then, informally, the Linear-Only Homomorphism (LOH) property [BCI+13, BCI+22] states that given polynomially-many ciphertexts $(c_0, ..., c_{m-1})$ under $\mathcal{E}_H$ it is infeasible for an adversary to create a new ciphertext c', which is in the image of the encryption function (as verified by ImVer) and cannot be expressed by (homomorphically) evaluating an affine combination of the ciphertexts in the previous list. The LOH property has been introduced in [BCI+13] to serve as the basis for several SNARK constructions in that paper and other subsequent works [BISW17, GMNO18, Nit19].

[11] Recall that the Fourier transform of $f(x) = e^{-\frac{x^2}{2\sigma^2}}$ is $\hat{f}(\xi) = \sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2\xi^2}$.

[12] However, Def. 7 does not stricto sensu imply this. We will further comment on this in Sect. 6.1.

The LOH property has been introduced in [BCI+13] to serve as the basis for several SNARK constructions in that paper and other subsequent works [BISW17, GMNO18, Nit19].

Formally, following [BCI+13], we have the following definition.

Definition 7 (LOH property, reproduced from [BCI+13])**.** An encryption scheme $\mathcal{E}_H = (\mathsf{KeyGen}, \mathsf{Enc}, \mathsf{ImVer}, \mathsf{Dec}, \mathsf{Eval})$ (with $\mathcal{P} = \mathbb{Z}_t$[13]) satisfies the *Linear-only Homomorphism property* if for every PPT adversary $\mathcal{A}$, there is a PPT extractor $\mathsf{extract}$ such that for any auxiliary input $\mathsf{aux} \in \{0,1\}^{\mathrm{poly}(\lambda)}$ and any plaintext generator $\mathcal{M}$,

$$P\left[\begin{array}{l|l} \exists i \in [\![0, k-1]\!] \text{ s. t.} & (\mathsf{ek}, \mathsf{sk}) := \mathcal{E}_H.\mathsf{KeyGen}(1^\lambda) \\ \mathsf{ImVer}(\mathsf{sk}, c'_i) = \mathsf{True} & (a_0, ..., a_{m-1}) := \mathcal{M}([\mathsf{ek}]) \\ \text{and} & (c_0, ..., c_{m-1}) := (\mathcal{E}_H.\mathsf{Enc}(\mathsf{ek}, a_0), ..., \mathcal{E}_H.\mathsf{Enc}(\mathsf{ek}, a_{m-1})) \\ \mathcal{E}_H.\mathsf{Dec}(\mathsf{sk}, c'_i) \neq a'_i & (c'_0, ..., c'_{k-1}) := \mathcal{A}(c_0, ..., c_{m-1}, [\mathsf{ek}]; \mathsf{aux}) \\ & (\Pi, b) := \mathsf{extract}(c_0, ..., c_{m-1}, [\mathsf{ek}]; \mathsf{aux}) \\ & (a'_0, ..., a'_{k-1})^T := \Pi \cdot (a_0, ..., a_{m-1})^T + b^T \end{array}\right] \leq \mathrm{neg}(\lambda). \tag{5}$$

where $\Pi \in \mathbb{Z}_t^{k \times m}$ and $b \in \mathbb{Z}_t^k$, and with the convention that row i of Π is left empty (i.e. $(\Pi_i, b_i) = \varnothing$) and $a'_i = \perp$ when c'_i was not generated by homomorphically evaluating an affine combination over the c_j's. The notation $[\mathsf{ek}]$ indicates that the encryption key ek is optionally provided, depending on whether the setting is private or public-key.

Remark that the c'_i do not have to be explicitly included as inputs to the extractor since they are built by the adversary and therefore it can implicitly recover them from aux.

In summary, whenever $\mathcal{A}$ builds c'_i by doing "something equivalent to",

$$c'_i = \mathcal{E}_H.\mathsf{Add}(\mathcal{E}_H.\mathsf{Eval}(\mathtt{lincomb}_\pi, c_0, ..., c_{m-1}), \underbrace{\mathcal{E}_H.\mathsf{Eval}(\mathtt{lincomb}_{\pi'}, c''_0, ..., c''_{l-1})}_{\text{pub. key case only}}), \tag{6}$$

where $c''_0 = \mathcal{E}_H.\mathsf{Enc}(\mathsf{ek}, \mu_0), ..., c''_{l-1} = \mathcal{E}_H.\mathsf{Enc}(\mathsf{ek}, \mu_{l-1})$, then $\Pi_i = \pi$ and $b_i = \sum_{j=0}^{l-1} \pi'_j \mu_j$. Conversely, $(\Pi_i, b_i) = \varnothing$ when this is not the case.

In our security proofs, we will use the following more convenient one-ciphertext notation for the above extractor,

$$(\pi, \beta) = \mathsf{extract}(c, \mathsf{aux}), \tag{7}$$

as a shortcut for $\Pi_{i:c'_i=c}$ and $b_{i:c'_i=c}$ (i.e. π, respectively β, is the row of Π, respectively the component of b, associated to ciphertext c). While a more rigorous notation would be to use $(\pi, \beta) = \mathsf{extract}(c, \{c_0, ..., c_{m-1}\}, \mathsf{aux})$, we use Eq. (7) in the sequel as the set of ciphertexts given to the LOH adversary will always be clear from the nearby context. Remark that affine rather than linear combinations are considered in the above definition to account for the fact that, in the public-key setting, the adversary can create (from scratch) additional fresh well-formed ciphertexts on its own (a case that does not happen in the private-key setting) and homomorphically add them to homomorphic evaluations of linear combinations over the c_i's. When operating in the private-key setting, we will thus omit β and simply use the notation,

$$\pi = \mathsf{extract}(c, \mathsf{aux}). \tag{8}$$

On top of the above definition, [BCI+13] further proposes several heuristic approaches to build schemes satisfying the LOH property, starting from a *correct* CPA secure linear homomorphic scheme. An example, which has been considered in several works (e.g. [BCI+13, GGPR13]), is the "two-ciphertexts" blueprint which consists in sparsifying the ciphertext domain of a correct linearly homomorphic scheme (e.g. Paillier) by following a

[13] The definition still extends to the case where the plaintext domain is a polynomial ring [BCI+13].

Knowledge-of-Exponent (KEA) template[14]. To illustrate this approach with the Paillier scheme, one may consider that the encryption of a message m consists of a pair of ciphertexts $(\mathsf{Enc}(\mathsf{ek}, m), \mathsf{Enc}(\mathsf{ek}, \alpha \cdot m))$ under the same key material, for a secret random value $\alpha \in \mathbb{Z}_n$ (with n the RSA modulus of the scheme), and with the ImVer algorithm checking this linear relation by decrypting both ciphertexts. It is then assumed that this scheme satisfies Definition 7 as the underlying Paillier scheme only exhibits linear homomorphic properties (to the best of the research community's knowledge), although the baseline Paillier scheme does not have the LOH property as also argued in [BCI+13] ([BCI+12], p. 33). We will further explore the security properties of this construction in Sect. 6.2.

In the present paper (Sect. 8), we also consider LWE-based candidate schemes[15]. However, when considering lattice-based constructions following the above "two-ciphertexts" blueprint, some additional precautions need to be taken. First, as already noted in [BCI+13], it is well-known that LWE or RLWE schemes can be turned into Fully Homomorphic Schemes. So if one proceeds directly as above by encrypting a message m as a pair of independent ciphertexts $(\mathsf{Enc}(\mathsf{ek}, m), \mathsf{Enc}(\mathsf{ek}, \alpha \cdot m))$ under the same key, an adversary may eventually compute two ciphertexts $\mathsf{Enc}(\mathsf{ek}, f(m))$ and $\mathsf{Enc}(\mathsf{ek}, \alpha \cdot f(m))$, for some nonlinear function f, from an encryption of m and an encryption of 1 (giving the pair $(\mathsf{Enc}(\mathsf{ek}, 1), \mathsf{Enc}(\mathsf{ek}, \alpha))$). In practice, this issue can be dealt with by having the two (or more, see below) ciphertexts be under *independent* keys, so that one cannot (homomorphically) obtain a consistent encryption of $\alpha \cdot f(m)$ from an encryption of m (say under $\mathsf{sk}^{(0)}$) and an encryption of α (say under $\mathsf{sk}^{(1)}$). Furthermore, all known FHE constructions require issuance of evaluation keys such as relinearization or bootstrapping keys. When no such keys are provided, these schemes are (to the best of the community's knowledge) stuck with linear-only homomorphic properties[16]. The second pitfall, is that such schemes tend to use a plaintext modulus t of small size. Then, given an arbitrary first ciphertext in a "two-ciphertexts" pair, the adversary can randomly sample the second ciphertext and then succeed with probability $\frac{1}{t}$ to forge a valid ciphertext pair and thus violate the LOH property. This difficulty can easily be worked around either by choosing $t = O(2^\lambda)$ (but this option would result in very large ciphertext modulus) or to follow a "multiple-ciphertexts" blueprint using $1 + \lceil \lambda / \log_2 t \rceil$ ciphertexts rather than just 2 in order to ensure that an adversary "obliviously sampling" valid ciphertexts without knowing the corresponding plaintext succeeds only with $\mathrm{neg}(\lambda)$ probability. The last but not least pitfall is that LWE-based schemes are prone to decryption errors. Although the occurrence of such errors does not necessarily contradict the LOH assumption (for example if ImVer consistently returns False when such an error occurs), credibly assuming the LOH property for LWE-based schemes leads to delicate issues to which a large part of this paper is devoted (Sect. 8). Note that prior works (e.g. [BISW17, GMNO18, Nit19]) essentially workaround this issue by assuming a weaker LOH property in which the adversary is restricted to evaluate linear combinations of bounded L_2 norm and choosing the LWE parameters such that (statistical) correctness is achieved under that bound constraint[17].

[14] This is a slight abuse of terminology. Yet, we will use it in this paper as the "two-ciphertexts" blueprint is a distant sibling of Dåmgard original heuristic [Dam92] and similar in spirit. Recall the discussion in Sect. 2.

[15] Note that the more recent instantiations of linear-only encryption, e.g. [BHI+24a, BHI+24b], require perfect correctness ([BHI+24b], Def. D.3, p. 71) and as such seem to exclude LWE-based schemes. Although this assumption is added without discussions, our results in Sect. 8 clearly illustrate that it is much harder to achieve linear-only HE for such schemes.

[16] Although this is not necessarily true for "old" constructs such as, for example, DGHV. It is the case for the mainstream efficient xLWE-based schemes such as BFV, BGV, CKKS in which the post-multiplication ciphertext structure is different from the pre-multiplication one and thus these ciphertexts cannot be decrypted by the original decryption function, i.e. without extending it to the post-multiplication ciphertext structure, unless some relinearization is done (which then requires this relinearization key).

[17] These works essentially use such LWE-based schemes as building blocks in the context of proof-of-knowledge constructions in which the L_2 bound constraint ends up satisfied "by construction".

We further highlight that, stricto sensu, the aforementioned prior works do *not* consider the above "multiple-ciphertexts" blueprint but rather simply assume that vanilla LWE encryption (i.e. the Regev scheme of Sect. 8.1) has the LOH property.

It also follows from the recent results in [DAFS24a] on quantum oblivious LWE sampling, that the LOH assumption for LWE-based schemes does not hold against quantum adversaries. Indeed, [DAFS24b, Sect. 6.3] presents a very clever attack in which the set of ciphertexts $c_0, ..., c_{m-1}$ is inputted (in matrix form) to a quantum oblivious (knM)LWE sampler, with the effect of applying a linear combination which is the secret of an obliviously sampled instance. As a result, this linear combination is *not extractable* unless one solves that obliviously sampled (knM)LWE instance, which cannot be done in quantum polynomial time under the LWE assumption. Hence, a violation of the LOH property results. Remark that this attack works regardless of whether or not the "multiple ciphertexts" blueprint is considered. However, this attack strategy applies in a parameter regime where m is quite large (typically 2^{20}) and, in particular, does not apply to the degenerate variant of LOH with $m = 0$ that we will later consider and refer to as LOH_0.

As a last remark, let us also emphasize that the LOH property requires that the ciphertexts for which $\mathcal{E}_H.\mathsf{ImVer}(\mathsf{sk}, c) = \mathsf{True}$ decrypt correctly (with overwhelming probability) as, otherwise, spurious violations of the property would occur.

5 Revisiting vCCA security definition and others

Advanced cryptographic primitives such as Functional Encryption (FE) and FHE, to name only two, yield subtle security notions. Following [CDG+18], a recent trend in FE has been to consider attack admissibility conditions in security games in order to sort attacks against the functionality (trivial attacks) from attack against the scheme (admissible attacks).

In this context, security games are defined the usual way by a scene made up of an adversary, modelled as non-uniform PPT algorithm, and a challenger, modelled as a uniform PPT algorithm, with the challenger providing a number of oracles [NPP23], along with a$\mathsf{Finalize}$ procedure (initially introduced in [BR06] to the best of our knowledge). When the adversary terminates, this latter procedure then checks a set of conditions, *with full knowledge of all data involved in the experiment*, and returns either the adversary's guess when the adversary's execution conforms to these conditions or a random bit otherwise[18]. For simplicity's sake we consider only the public-key setting (in the private key setting the encryption key ek is not communicated to the adversary and an additional encryption oracle, $\mathcal{E}.\mathsf{Enc}(\mathsf{ek}, .)$, is made available).

5.1 Indistinguishability (IND) notions

Definition 8 (Generic indistinguishability experiment, admissible adversaries)**.** Given an encryption scheme $\mathcal{E} = (\mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec}, [...])$ over keyspace $\mathcal{K}$, as well as plaintext and ciphertext domains $\mathcal{P}$ and $\mathcal{C}$, and a PPT adversary $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$, with internal state ω,

[18] Note that several other conventions exist in the literature

we define the following generic indistinguishability experiment, $\mathsf{Exp}^{\mathrm{IND}}_{\mathcal{E},\mathcal{A}}(\lambda)$,

$$
\begin{aligned}
&(\mathsf{pk},\mathsf{sk}) \leftarrow \mathcal{E}.\mathsf{KeyGen}(1^\lambda),\\
&\mathsf{cc} \xleftarrow{\$} \{0,1\}^{O(\mathrm{poly}(\lambda))}\\
&(m_0,m_1,\omega,\tau_1) \leftarrow \mathcal{A}_1^{\mathcal{E}.\mathsf{Dec}(\mathsf{sk},.)}(\mathsf{pk},\mathsf{cc}),\\
&\gamma \xleftarrow{\$} \{0,1\},\\
&c^* \leftarrow \mathcal{E}.\mathsf{Enc}(\mathsf{pk},m_\gamma),\\
&(\gamma',\tau_2) \leftarrow \mathcal{A}_2^{\mathcal{E}.\mathsf{Dec}(\mathsf{sk},.)}(\mathsf{pk},\omega,c^*,\mathsf{cc}),\\
&\gamma'' \leftarrow \mathsf{Finalize}(\mathsf{cc},\tau_1,\tau_2,\gamma',(\mathsf{pk},\mathsf{sk}),m_0,m_1,c^*),\\
&\gamma = \gamma''
\end{aligned}
$$

where $\mathsf{Finalize} : \{0,1\}^{O(\mathrm{poly}(\lambda))} \longrightarrow \{0,1\}$ is such that $\mathsf{Finalize}(\mathsf{cc},\tau_1,\tau_2,\gamma',(\mathsf{pk},\mathsf{sk}),m_0,m_1,c^*)$ returns the adversary's guess for admissible ones and a random bit for others, where cc denotes $\mathcal{A}$'s random coins and where τ_1 and τ_2 are the transcripts of $\mathcal{A}_1$'s and $\mathcal{A}_2$'s interactions with the oracle $\mathcal{E}.\mathsf{Dec}(\mathsf{sk},.)$.

In this paper, $\mathsf{Finalize}(\mathsf{cc},\tau_1,\tau_2,\gamma',(\mathsf{pk},\mathsf{sk}),m_0,m_1,c^*)$ is always of the form,

$$(\gamma' \wedge \mathsf{Verify}(\mathsf{cc},\tau_1,\tau_2,(\mathsf{pk},\mathsf{sk}),m_0,m_1,c^*)) \vee (\beta \wedge \neg\mathsf{Verify}(\mathsf{cc},\tau_1,\tau_2,(\mathsf{pk},\mathsf{sk}),m_0,m_1,c^*)),$$

with $\beta \xleftarrow{\$} \{0,1\}$ and $\mathsf{Verify} : \{0,1\}^{O(\mathrm{poly}(\lambda))} \longrightarrow \{\mathsf{True},\mathsf{False}\}$. Then an execution of $\mathcal{A} = (\mathcal{A}_1,\mathcal{A}_2)$ is said to be *admissible* if and only if,

$$\mathsf{Verify}(\mathsf{cc},\tau_1,\tau_2,(\mathsf{pk},\mathsf{sk}),m_0,m_1,c^*) = \mathsf{True}.$$

The standard IND-CPA, IND-CCA1 and IND-CCA2 security game can be naturally casted into this framework as follows (following [CDG+18, Def. 2]):

Definition 9 (IND-CPA, IND-CCA1, IND-CCA2 security)**.** A scheme $\mathcal{E} = (\mathsf{KeyGen},\mathsf{Enc},\mathsf{Dec},\ldots)$ is said to be IND-CPA secure (respectively IND-CCA1 and IND-CCA2 secure) if any PPT adversary $\mathcal{A} = (\mathcal{A}_1,\mathcal{A}_2)$ wins the generic indistinguishability experiment $\mathsf{Exp}^{\mathrm{IND}}_{\mathcal{E},\mathcal{A}}(\lambda)$ (Def. 8) only with negligible probability with respect to a $\mathsf{Finalize}$ procedure involving the following (respectively):

- IND-CPA: $\mathsf{Verify}(\mathsf{cc},\tau_1,\tau_2,(\mathsf{pk},\mathsf{sk}),m_0,m_1,c^*)$ returns True if and only if $\mathcal{A}$ did not invoke the $\mathcal{E}.\mathsf{Dec}(\mathsf{sk},.)$ oracle.
- IND-CCA1: $\mathsf{Verify}(\mathsf{cc},\tau_1,\tau_2,(\mathsf{pk},\mathsf{sk}),m_0,m_1,c^*)$ returns True if and only if $\mathcal{A}_2$ did not invoke the $\mathcal{E}.\mathsf{Dec}(\mathsf{sk},.)$ oracle.
- IND-CCA2: $\mathsf{Verify}(\mathsf{cc},\tau_1,\tau_2,(\mathsf{pk},\mathsf{sk}),m_0,m_1,c^*)$ returns True if and only if $\mathcal{A}_2$ did not invoke the $\mathcal{E}.\mathsf{Dec}(\mathsf{sk},.)$ oracle over c^*.

In the above, the Verify procedure can trivially be efficiently implemented respectively as $\tau_1 = \tau_2 = \varnothing$ (IND-CPA), $\tau_2 = \varnothing$ (IND-CCA1) and $(c^*,.) \notin \tau_2$ (IND-CCA2).

Definition 10 ($\mathcal{H}$-IND-vCCA security)**.** An FHE scheme $\mathcal{E}_H = (\mathsf{KeyGen},\mathsf{Enc},\mathsf{Dec},\mathsf{Eval})$ over plaintext domain $\mathcal{P}$, ciphertext domain $\mathcal{C}$ and function domain $\mathcal{F}(\lambda)$ is said to be $\mathcal{H}$-IND-vCCA secure, for some $\mathcal{H} \subseteq \mathcal{F}(\lambda)$, if any PPT adversary $\mathcal{A} = (\mathcal{A}_1,\mathcal{A}_2)$ wins the generic indistinguishability experiment $\mathsf{Exp}^{\mathrm{IND}}_{\mathcal{E},\mathcal{A}}(\lambda)$ (Def. 8) only with negligible probability with respect to a $\mathsf{Finalize}$ procedure involving the following:

- $\mathsf{Verify}(\mathsf{cc},\tau_1,\tau_2,(\mathsf{pk},\mathsf{sk}),m_0,m_1,c^*)$ returns True if and only if $\mathcal{A}$ did not invoke the $\mathcal{E}_H.\mathsf{Dec}(\mathsf{sk},.)$ oracle either over c^* or over a ciphertext it obtained by doing,

  $$c = \mathcal{E}_H.\mathsf{Eval}(f,c_0,\ldots,c_{L-1}),$$

  with $f \in \mathcal{H}$, $(c_0,\ldots,c_{L-1}) \in \mathcal{C}^L$ and $c^* \in \{c_0,\ldots,c_{L-1}\}$.

For simplicity's sake, the above definition (and the next ones) assume one-hop homomorphic evaluations. In the multi-hop case, one has to consider a tree of evaluated ciphertexts rooted in c (see also [MN24, Remark 2]) and with fresh (not necessarily well-formed) ciphertexts as leafs.

Contrary to the original definition in [MN24], the above one does not involve any extractor. Although this does not change the spirit of the original definition, it appears more consistent with the philosophy that security notions should be defined independently of any scheme property, while the existence of an extractor for some knowledge is a property that emerges (sometimes only heuristically) from some of the specific characteristics of a concrete cryptographic scheme. In particular, most schemes do not yield any natural extractors. On the downside, the above definition provides no indications regarding how the $\mathsf{Finalize}$ procedure may be efficiently implemented. Consistently with the initial spirit of vCCA security, extractors will naturally reappear *within security proofs* when building vCCA secure schemes *from Knowledge Assumptions*, which is a natural strategy to do so.

Definition 11 ($\mathcal{H}$-IND-vCCA^D security)**.** An FHE scheme $\mathcal{E}_H = (\mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec}, \mathsf{Eval})$ over plaintext domain $\mathcal{P}$, ciphertext domain $\mathcal{C}$ and function domain $\mathcal{F}(\lambda)$ is said to be $\mathcal{H}$-IND-vCCA^D secure, for some $\mathcal{H} \subseteq \mathcal{F}(\lambda)$, if any PPT adversary $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$ wins the generic indistinguishability experiment $\mathsf{Exp}^{\mathrm{IND}}_{\mathcal{E},\mathcal{A}}(\lambda)$ (Def. 8) only with negligible probability with respect to a $\mathsf{Finalize}$ procedure involving the following:

- $\mathsf{Verify}(\mathsf{cc}, \tau_1, \tau_2, (\mathsf{pk}, \mathsf{sk}), m_0, m_1, c^*)$ returns True if and only if and $\mathcal{A}$ did not invoke the $\mathcal{E}_H.\mathsf{Dec}(\mathsf{sk}, .)$ oracle either over c^* or over a ciphertext it obtained by doing[19],

$$c = \mathcal{E}_H.\mathsf{Eval}(f, c_0, ...c_{j-1}, c^*, c_{j+1}, ..., c_{L-1}), \tag{9}$$

  with $f \in \mathcal{H}$, $(c_0, ..., c_{L-1}) \in \mathcal{C}^L$, $c_i = \mathcal{E}.\mathsf{Enc}(\mathsf{pk}, \mu_i)$, $\forall i \in [\![0, L-1]\!] \setminus \{j\}$ and

$$f(\mu_0, ..., \mu_{j-1}, m_0, \mu_{j+1}, ..., \mu_{L-1}) \neq f(\mu_0, ..., \mu_{j-1}, m_1, \mu_{j+1}, ..., \mu_{L-1}).$$

From [BCF+25, Prop. 8], it is known that IND-vCCA^D security is strictly stronger than IND-vCCA security.

5.2 Left-or-Right (LOR) notions

Definition 12 (Generic left-or-right experiment)**.** Given an encryption scheme $\mathcal{E} = (\mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec}, [...])$ over keyspace $\mathcal{K}$, as well as plaintext and ciphertext domains $\mathcal{P}$ and $\mathcal{C}$, and a PPT adversary $\mathcal{A}$, we define the following generic left-or-right experiment, $\mathsf{Exp}^{\mathrm{LOR}}_{\mathcal{E},\mathcal{A}}(\lambda)$,

$$\begin{aligned}
&(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathcal{E}.\mathsf{KeyGen}(1^\lambda),\\
&\mathsf{cc} \xleftarrow{\$} \{0,1\}^{O(\mathrm{poly}(\lambda))}\\
&S \leftarrow [],\ \gamma \xleftarrow{\$} \{0,1\},\\
&(\gamma', \tau) \leftarrow \mathcal{A}^{\mathsf{C}(\mathsf{pk},.,.),\mathcal{E}.\mathsf{Dec}(\mathsf{sk},.)}(\mathsf{pk}, \mathsf{cc}),\\
&\gamma'' \leftarrow \mathsf{Finalize}(\mathsf{cc}, \tau, \gamma', (\mathsf{pk}, \mathsf{sk}), S),\\
&\gamma = \gamma''
\end{aligned}$$

where $\mathsf{Finalize} : \{0,1\}^{O(\mathrm{poly}(\lambda))} \longrightarrow \{\mathsf{True}, \mathsf{False}\}$ is as previously defined and where τ is the transcript of $\mathcal{A}$'s interactions with the oracle $\mathcal{E}.\mathsf{Dec}(\mathsf{sk}, .)$. In the above, $\mathsf{C}(\mathsf{pk}, ., .)$ is a left-or-right encryption oracle which given $m_0, m_1 \in \mathcal{P}$ returns $c = \mathcal{E}.\mathsf{Enc}(\mathsf{pk}, m_\gamma)$ after updating the state S by doing $S \leftarrow [S; (m_0, m_1, c)]$ (so S contains the transcript of $\mathcal{A}$'s interactions with the $\mathsf{C}(\mathsf{pk}, ., .)$ oracle).

[19] Assuming wlog that c^* occurs once in Eq. (9).

Definition 13 (LOR-CPA, LOR-CCA security)**.** A scheme $\mathcal{E} = (\mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec}, ...)$ is said to be LOR-CPA secure (respectively LOR-CCA secure) if any PPT adversary $\mathcal{A}$ wins the generic left-or-right experiment $\mathsf{Exp}^{\mathrm{LOR}}_{\mathcal{E},\mathcal{A}}(\lambda)$ (Def. 12) only with negligible probability with respect to the finalization procedures involving the following (respectively):

- LOR-CPA: $\mathsf{Verify}(\mathsf{cc}, \tau, (\mathsf{pk}, \mathsf{sk}), S)$ returns True if and only if $\mathcal{A}$ did not invoke the $\mathcal{E}.\mathsf{Dec}(\mathsf{sk}, .)$ oracle.
- LOR-CCA: $\mathsf{Verify}(\mathsf{cc}, \tau, (\mathsf{pk}, \mathsf{sk}), S)$ returns True if and only if $\mathcal{A}$ did not invoke the $\mathcal{E}.\mathsf{Dec}(\mathsf{sk}, .)$ oracle over a ciphertext c for which $\exists i \in [\![0, |S| - 1]\!]$ such that $S[i].c = c$ and $S[i].m_0 \neq S[i].m_1$.

It is well-known that LOR-CPA (respectively LOR-CCA) is equivalent to IND-CPA (respectively IND-CCA2) in both the public [BDPR98] and private [BDJR97] settings.

Definition 14 ($\mathcal{H}$-LOR-vCCA security)**.** A FHE scheme $\mathcal{E}_H = (\mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec}, \mathsf{Eval})$ over plaintext domain $\mathcal{P}$, ciphertext domain $\mathcal{C}$ and function domain $\mathcal{F}(\lambda)$ is said to be $\mathcal{H}$-LOR-vCCA secure, for some $\mathcal{H} \subseteq \mathcal{F}(\lambda)$, if any PPT adversary $\mathcal{A}$ wins the generic left-or-right experiment $\mathsf{Exp}^{\mathrm{LOR}}_{\mathcal{E},\mathcal{A}}(\lambda)$ (Def. 12) only with negligible probability, with respect to a $\mathsf{Finalize}$ procedure involving the following:

- $\mathsf{Verify}(\mathsf{cc}, \tau, (\mathsf{pk}, \mathsf{sk}), S)$ returns True if and only if, letting $C^* = \{S[i].c : i \in [\![0, |S|-1]\!]\}$, $\mathcal{A}$ did not invoke the $\mathcal{E}_H.\mathsf{Dec}(\mathsf{sk}, .)$ oracle either over a ciphertext in C^* or over a ciphertext obtained by doing,

$$c = \mathcal{E}_H.\mathsf{Eval}(f, c_0, ..., c_{L-1}),$$

  with $f \in \mathcal{H}$, $(c_0, ..., c_{L-1}) \in \mathcal{C}^L$ and $C^* \cap \{c_0, ..., c_{L-1}\} \neq \varnothing$.

It shown in [BCF+25, Prop. 12] that IND-vCCA and LOR-vCCA are equivalent.

Definition 15 ($\mathcal{H}$-LOR-vCCA^D security)**.** A FHE scheme $\mathcal{E}_H = (\mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec}, \mathsf{Eval})$ over plaintext domain $\mathcal{P}$, ciphertext domain $\mathcal{C}$ and function domain $\mathcal{F}(\lambda)$ is said to be $\mathcal{H}$-LOR-vCCA^D secure, for some $\mathcal{H} \subseteq \mathcal{F}(\lambda)$, if any PPT adversary $\mathcal{A}$ wins the generic left-or-right experiment $\mathsf{Exp}^{\mathrm{LOR}}_{\mathcal{E},\mathcal{A}}(\lambda)$ (Def. 12) only with negligible probability, with respect to a $\mathsf{Finalize}$ procedure involving the following:

- $\mathsf{Verify}(\mathsf{cc}, \tau, (\mathsf{pk}, \mathsf{sk}), S)$ returns True if and only if $\mathcal{A}$ did not invoke the $\mathcal{E}_H.\mathsf{Dec}(\mathsf{sk}, .)$ oracle either over a ciphertext c such that[20] $\exists i \in [\![0, |S| - 1]\!] : S[i].c = c$ and $S[i].m_0 \neq S[i].m_1$, or over a ciphertext it obtained by doing,

$$c = \mathcal{E}_H.\mathsf{Eval}(f, c_0, ..., c_{K-1}, S[i_0].c, ..., S[i_{L-1}].c),$$

  given $f \in \mathcal{H}$, $c_k = \mathcal{E}_H.\mathsf{Enc}(\mathsf{pk}, \mu_k)$, $k \in [\![0, K-1]\!]$, and $\exists (i_0, ..., i_{L-1}) \in [\![0, |S| - 1]\!]^L$ such that,

$$f(\mu_0, ..., \mu_{K-1}, S[i_0].m_0, ..., S[i_{L-1}].m_0) \neq f(\mu_0, ..., \mu_{K-1}, S[i_0].m_1, ..., S[i_{L-1}].m_1).$$

6 Results under the correctness assumption

In this section, we investigate the relationship between the LOH property and the vCCA security notion for *correct* linearly homomorphic encryption schemes and establish that, for these schemes, LOH implies vCCA in both the private and public key setting. Because, for vCCA security, the single challenge notion is equivalent to the multiple challenges one [BCF+25], we focus our proofs only on the former notion.

[20] Remark that since S contains only well-formed ciphertexts generated by the challenger, a given ciphertext appears at most once in S (with overwhelming probability over the encryption randomness).

6.1 General black-box results

Private key setting. To show that LOH implies vCCA for the private key setting, our proof strategy consists of a first game hop, in which the vCCA challenger of the original game is changed such that the finalization procedure in Def. 10 is implemented by means of the LOH extractor. Then, in a second game hop, we further modify the challenger such that it uses a modified decryption function which is stripped from the call to ImVer and uses the LOH extractor to replace this call (assuming ImVer is the "verification" part of $\mathcal{E}_H.\mathsf{Dec}$, see the discussion just below). The indistinguishability between these two games is then directly implied by the LOH assumption. Then the proof is finalized by a reduction towards a CPA challenger against $\mathcal{E}_H$ that emulates the vCCA decryption oracle by means of the LOH extractor.

The result below is established under two mildly restrictive technical assumptions which are quite natural but *not* stricto sensu implied by the LOH property. Indeed, in this paper, we stick to the LOH definition of Bitansky et al. which defines ImVer as a function that "tests, using the secret key sk, whether a given candidate ciphertext c is in the image of $\mathsf{Enc}(\mathsf{ek}, .)$" (e.g. [BCI+22, p. 32] for the latest instantiation) without any further explicit connection to the decryption function of the scheme. As a consequence, it is easy to conceive schemes for which some well-formed evaluated ciphertexts are *not* in the image of the encryption function (i.e. over which ImVer returns False) yet for which the decryption function does not return $\perp$[21]. We highlight that this does not contradict Def. 7 which cares only for ciphertexts yielding ImVer returning True and therefore, for LOH schemes in general, the fact that ImVer returns False over a ciphertext *does not* imply that the decryption function returns $\perp$ over the same. In a similar spirit, without contradicting Def. 7, it is also easy to build artificial schemes for which ImVer returns False when a norm bound is exceeded on the evaluated linear combination, yet let the resulting ciphertexts yield a non-$\perp$ answer from the decryption function. However, in the proof of the next proposition, our first step towards a simulation of a vCCA decryption oracle is to use the LOH extractor to detect non-well formed ciphertexts such that the simulation can reply $\perp$ whenever the extractor fails (i.e. returns $\varnothing$), without the adversary noticing. For this step to work, we need that $\mathsf{extract}(c, \mathsf{aux}) = \varnothing \Leftrightarrow \mathsf{Dec}(\mathsf{sk}, c) = \perp$ and conditions (10) and (11) in the statement below precisely aim to imply this property.

Proposition 4. *Let $\mathcal{E}_H = (\mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec}, \mathsf{ImVer}, \mathsf{Eval})$ be a private-key correct CPA secure linearly homomorphic scheme that satisfies the LOH property. Further assume that,*

$$\mathcal{E}_H.\mathsf{Dec}(\mathsf{sk}, c) = \perp \Leftrightarrow \mathcal{E}_H.\mathsf{ImVer}(\mathsf{sk}, c) = \mathsf{False}, \tag{10}$$

and, with overwhelming probability[22]*,*

$$\mathcal{E}_H.\mathsf{ImVer}(\mathsf{sk}, c) = \mathsf{False} \Rightarrow \mathsf{extract}(c, \mathsf{aux}) = \varnothing \tag{11}$$

then $\mathcal{E}_H$ is vCCA secure.

Proof. First game hop. Let G_0 be the vCCA game against $\mathcal{E}_H$ and G_1 be the same game as G_0 where we modify the challenger such that, the Verify procedure of the IND-vCCA experiment is implemented by means of the LOH extractor. Let $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$ denote the adversary. Recall from Def. 10 that the Verify procedure requires that,

[21] A concrete example is to start from the two-ciphertext heuristic that we consider in this paper and to add a third element such that the overall encryption function gives $(c_0, c_1, c_2) = (\mathsf{Enc}(\mathsf{ek}, m), \mathsf{Enc}(\mathsf{ek}, \alpha \cdot m), \mathsf{Enc}(\mathsf{ek}, 1))$ as an encryption for message m, with ImVer returning False whenever $\mathsf{Dec}(\mathsf{sk}, c_2) > 1$ and the overall decryption function ignoring c_2 (while still replying $\perp$ when the expected linear relation between $\mathsf{Dec}(\mathsf{sk}, c_0)$ and $\mathsf{Dec}(\mathsf{sk}, c_1)$ is not satisfied).

[22] Note that condition (11) stricto sensu requires LOH with interactive extraction [BCI+22, 9.6].

- $\mathsf{Verify}(\mathsf{cc}, \tau_1, \tau_2, (\mathsf{pk}, \mathsf{sk}), m_0, m_1, c^*)$ returns True if and only if $\mathcal{A}$ did not invoke the $\mathcal{E}_H.\mathsf{Dec}(\mathsf{sk}, .)$ oracle either over c^* or over a ciphertext of the form,

$$c = \mathcal{E}_H.\mathsf{Eval}(f, c_0, ..., c_{L-1}),$$

with $f \in \mathcal{F}(\lambda)$, $(c_0, ..., c_{L-1}) \in \mathcal{C}^L$ and $c^* \in \{c_0, ..., c_{L-1}\}$.

To efficiently implement this procedure, the new challenger proceeds as follows. First the transcript of $\mathcal{A}$'s interaction with the encryption oracle is parsed in order to retrieve all ciphertexts c_i returned by the latter. We denote by m the number of ciphertexts retrieved and assume w.l.o.g. that c^* is appended in the last position.

The new challenger subsequently invokes $\mathsf{extract}(c_0, ..., c_{m-2}, c^*, \mathsf{aux})$ to get Π according to Eq. (5) in the private-key setting.

Then τ_2 is iterated over in order to retrieve the input ciphertext to each of $\mathcal{A}_2$'s calls to the $\mathcal{E}_H.\mathsf{Dec}(\mathsf{sk}, .)$ oracle and store them in the list ℓ_2 (since IND-vCCA's Verify procedure places no constraint on $\mathcal{A}_1$'s decryption requests, τ_1 is ignored). We stress out that ℓ_2 is a subset of $\{c'_0, ..., c'_{k-1}\}$ in Eq. (5). Finally, for each ciphertext c'_i in ℓ_2, the row i of Π is retrieved getting either $\varnothing$ or π consistently with Eq. (8). In the latter case ($\Pi_i = \pi \neq \varnothing$), whenever $\pi_{m-1} \neq 0$ (π_{m-1} is the coefficient of the linear combination π corresponding to the ciphertext c^*), False is returned. At the end of the iteration over ℓ_2, True is eventually returned meaning that the adversary is admissible. We emphasize that the new challenger does not use sk for this.

The indistinguishability between G_0 and G_1 then follows by definition of the LOH property (Def. 7 and Eq. 5). Hence,

$$\left|\mathsf{Adv}^{G_0}_{\mathcal{E}_H,\mathcal{A}}(\lambda) - \mathsf{Adv}^{G_1}_{\mathcal{E}_H,\mathcal{A}}(\lambda)\right| \leq \mathrm{neg}(\lambda).$$

Second game hop. Let G_2 be the same game as G_1 where we modify the challenger such that, when handling a decryption request on ciphertext ct, the new challenger invokes the LOH extractor to verify that

$$\mathsf{extract}(\mathsf{ct}, \mathsf{aux}) \neq \varnothing,$$

rather than checking $\mathsf{ImVer}(\mathsf{sk}, \mathsf{ct}) = \mathsf{True}$. Indeed, the two games cannot be distinguished since the case where $(\mathsf{extract}(c) = \varnothing) \wedge (\mathsf{ImVer}(\mathsf{sk}, c) = \mathsf{True})$ contradicts the LOH property in conjunction with property (10); and the case where $(\mathsf{extract}(c) \neq \varnothing) \wedge (\mathsf{ImVer}(\mathsf{sk}, c) = \mathsf{False})$ contradicts property (10) in conjunction with property (11). Hence, $\mathsf{extract}(c) = \varnothing \Leftrightarrow \mathsf{ImVer}(\mathsf{sk}, c) = \mathsf{False}$ with overwhelming probability and,

$$\left|\mathsf{Adv}^{G_1}_{\mathcal{E}_H,\mathcal{A}}(\lambda) - \mathsf{Adv}^{G_2}_{\mathcal{E}_H,\mathcal{A}}(\lambda)\right| \leq \mathrm{neg}(\lambda).$$

Final reduction. To finalize the proof, we show that, from an adversary $\mathcal{A}$ against G_2, we can build an adversary $\mathcal{B}$ against the CPA security of $\mathcal{E}_H$ which uses $\mathcal{A}$ as a subroutine and achieves the same advantage as the latter. For the reduction to work, we assume that $\mathcal{A}$ and $\mathcal{B}$ agree on a consistent numbering of the ciphertexts output by the encryption oracle. The reduction then starts by initializing an empty state $S := []$ which will contain message-ciphertext pairs and setting i^* to -1 (here, i^* will serve to store the index of the unique challenge ciphertext in the game state), and then proceeds as follows.

- When receiving an encryption request over message $m \in \mathcal{P}$ from $\mathcal{A}$, it first transfers it as is to the CPA challenger to get ciphertext $\mathsf{ct} = \mathcal{E}_H.\mathsf{Enc}(\mathsf{ek}, m)$ (for unknown ek since we are in the private-key setting) which it sends back to $\mathcal{A}$ after updating its internal state as $S := [S; (m, \mathsf{ct})]$.

- When receiving the *single* challenge request over messages $m_0 \neq m_1 \in \mathcal{P}$ from $\mathcal{A}$, it transfers it as is to the CPA challenger to get ciphertext $\mathsf{ct}^* = \mathcal{E}_H.\mathsf{Enc}(\mathsf{ek}, m_\gamma)$ (for unknown ek and γ), which it sends back to $\mathcal{A}$ after updating its internal state as $S := [S; (m_0, \mathsf{ct}^*)]$ (or, equivalently, $S := [S; (m_1, \mathsf{ct}^*)]$) and setting $i^* = |S| - 1$.

- When $\mathcal{A}$ issues a decryption request over ciphertext $\mathsf{ct} \in \mathcal{C}$, then $\mathcal{B}$ runs the LOH extractor to get $\pi = \mathsf{extract}(\mathsf{ct}, \mathsf{aux})$. When $\pi = \varnothing$, it returns $\perp$ to $\mathcal{A}$. Otherwise, $\mathcal{B}$ returns $\perp$ whenever $\pi_{i^*} \neq 0$ (with the convention that $\pi_{-1} = 0$) or,

$$\sum_{i:\pi_i \neq 0} \pi_i S[i].m, \tag{12}$$

  when this is not the case (i.e. when $\pi_{i^*} = 0$). Recall that, following Eq. (8), $\beta = 0$ in the private-key setting.

Finally, in order to generate G_2's final guess, $\mathcal{B}$ first simulates the verification procedure as in G_1. If it outputs True, $\mathcal{B}$ outputs the same guess γ' as $\mathcal{A}$. Otherwise $\mathcal{B}$ picks a random guess as $\mathsf{Finalize}$ would have done for a non-admissible adversary.

As $\mathcal{B}$ does not make any call to the decryption oracle, $\mathcal{B}$ is a valid CPA adversary.

The claim follows from the facts that, when $\mathsf{ImVer}(\mathsf{sk}, \mathsf{ct}) = \mathsf{True}$ (or, equivalently, when $\mathsf{extract}(\mathsf{ct}, \mathsf{aux}) \neq \varnothing$), the correctness of $\mathcal{E}_H$ implies that (12) cannot be distinguished from $\mathcal{E}_H.\mathsf{Dec}(\mathsf{sk}, \mathsf{ct})$ and that the reduction duly replies $\perp$ for all decryption requests on ciphertexts which are byproducts of the challenge ciphertexts (i.e. for which $c^* \in \{c_0, ..., c_{L-1}\}$ in the notations of the vCCA game 2nd step decryption oracle definition, p. 9, and $\pi_{i^*} \neq 0$), consistently with the Verify procedure since, when this occurs, the adversary's run is not admissible.

Thus we have

$$\left|\mathsf{Adv}^{G_2}_{\mathcal{E}_H,\mathcal{A}}(\lambda) - \mathsf{Adv}^{\mathrm{CPA}}_{\mathcal{E}_H,\mathcal{B}}(\lambda)\right| \leq \mathrm{neg}(\lambda).$$

and finally

$$\left|\mathsf{Adv}^{\mathrm{vCCA}}_{\mathcal{E}_H,\mathcal{A}}(\lambda) - \mathsf{Adv}^{\mathrm{CPA}}_{\mathcal{E}_H,\mathcal{B}}(\lambda)\right| \leq \mathrm{neg}(\lambda).$$

□

As vCCA security is the strongest (so far known) CCA security notion achievable by FHE *under the correctness assumption*, this is a strong implication. Remark also that we get CCA1 security as a corollary to Prop. 4 since vCCA security implies CCA1 security [MN24].

Public-key setting. We now also establish a similar general result in the public-key setting. To do so, our proof strategy first consists in showing that, in the vCCA security game against $\mathcal{E}_H$, the LOH extractor (running over auxiliary data that include the trace of execution of the adversary $\mathcal{A}$) provides the vCCA challenger with a plaintext extractor for ciphertexts generated by $\mathcal{A}$ as well as a mean to detect challenge dependent ciphertexts. As in the private key case, the proof then proceeds with a first game hop in which the vCCA challenger of the original game is similarly changed such that the $\mathsf{Finalize}$ procedure is implemented by means of the LOH extractor and a second one in which the challenger is further changed to use a modified decryption function which is stripped from the call to ImVer and uses the LOH extractor to replace this call. The indistinguishability between these two games is then implied by the LOH assumption. Then the proof is finalized by a reduction towards a CPA challenger against $\mathcal{E}_H$ where the decryption oracle is emulated using the output of the LOH extractor. The main difference with the private key key case, is that the relationship between the LOH extractor and the vCCA extractor is more subtle.

Proposition 5. *Let* $\mathcal{E}_H = (\mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec}, \mathsf{ImVer}, \mathsf{Eval})$ *be a* public-key *correct CPA secure linearly homomorphic scheme that satisfies the LOH assumption. Further assume conditions* (10) *and* (11)*, then* $\mathcal{E}_H$ *is vCCA secure.*

Proof. We first show how to turn a LOH extractor into a plaintext extractor in the public-key setting.

Plaintext extraction using the LOH extractor. In the public-key setting, remark that there are two types of ciphertexts that the adversary may *not* generate on its own: a (possibly empty) set of ciphertexts $\mathsf{pk}_0, ..., \mathsf{pk}_{N-1}$, which are part of the public key, as well as the unique challenge ciphertext ct^*. Assume that the vCCA challenger stores them in a state S containing message-ciphertext pairs i.e. such that

$$S = [(m_0, \mathsf{pk}_0), ..., (m_{N-1}, \mathsf{pk}_{N-1}), (m_0^*, \mathsf{ct}^*)] \tag{13}$$

where m_0^* is the first of the two messages that served for creating ct^* (S might equivalently contain (m_1^*, ct^*) in the last position, as it is expected to return $\perp$ for decryption requests over ciphertexts depending on ct^*). Further remark that the LOH extractor may return a non-empty vector only on ciphertexts generated by the vCCA adversary and *not* over those generated by the challenger.

Thus, following Def. 7 (p. 15), we have $m = N + 1$ and the set $(c_0, ..., c_{m-1})$ in that definition is then reduced to $(\mathsf{pk}_0, ..., \mathsf{pk}_{N-1}, \mathsf{ct}^*)$.

Then, let us consider a ciphertext ct submitted as part of decryption request such that $\mathsf{extract}(\mathsf{ct}, \mathsf{aux}) \neq \varnothing$. Let $(\pi, \beta) = \mathsf{extract}(\mathsf{ct}, \mathsf{aux})$ (from Eq. 7), we have the two following properties:

- ct depends on ct^* (i.e. $c^* \in \{c_0, ..., c_{L-1}\}$ in the notations of Def. 10) if and only if $\pi_N \neq 0$.
- If ct is independent of ct^* (i.e. $c^* \notin \{c_0, ..., c_{L-1}\}$ in the vCCA game 2nd step decryption oracle as well as $\pi_N = 0$) then, from the correctness of $\mathcal{E}_H$,

$$\mathcal{E}_H.\mathsf{Dec}(\mathsf{sk}, \mathsf{ct}) = \sum_{i=0}^{N-1} \pi_i S[i].m + \beta. \tag{14}$$

First game hop. Let G_0 be the vCCA game against $\mathcal{E}_H$ and G_1 be the same game as G_0 where we modify the challenger such that, the Verify procedure of the IND-vCCA experiment is efficiently implemented by means of the LOH extractor. This is done in a similar way as in the proof of Prop. 4.

First recall, as mentioned just above, that in the public-key setting, the only ciphertexts that are not generated by the adversary are those that are part of the public key (if not empty, they are denoted as $\mathsf{pk}_0, ..., \mathsf{pk}_{N-1}$), and the challenge ciphertext c^*. Thus, in the LOH definition Eq. (5), we have,

$$(c_0, ..., c_{m-1}) = (\mathsf{pk}_0, ..., \mathsf{pk}_{N-1}, c^*)$$

The new challenger then proceeds identically as in the proof of Prop. 4 over the above list of ciphertexts rather than the transcript of an encryption oracle (which is not provided to $\mathcal{A}$ in the public-key case).

Second game hop (G_1 to G_2). Identical to that of the proof of Prop. 4.

Final reduction. To finalize the proof, we then show that, from an adversary $\mathcal{A}$ against G_2, we can build an adversary $\mathcal{B}$ against the CPA security of $\mathcal{E}_H$ which uses $\mathcal{A}$ as a subroutine and achieves the same advantage. For the reduction to work, we assume that $\mathcal{A}$ and $\mathcal{B}$ agree on a consistent numbering of the ciphertexts generated by the reduction, which it stores in an internal state S containing message-ciphertext pairs following Eq. (13). Thus, after getting pk (which contains $\mathsf{pk}_0, ..., \mathsf{pk}_{N-1}$) and transferring it to $\mathcal{A}$, the reduction proceeds as follows[23]:

- When receiving the *single* challenge request over messages $m_0^* \neq m_1^* \in \mathcal{P}$ from $\mathcal{A}$, it transfers it as is to the CPA challenger to get ciphertext $\mathsf{ct}^* = \mathcal{E}_H.\mathsf{Enc}(\mathsf{pk}, m_\gamma^*)$

[23] In the public-key setting, the reduction does not have to handle encryption requests.

(for unknown γ), which it sends back to $\mathcal{A}$ after updating its internal state as $S \coloneqq [S; (m_0^*, \mathsf{ct}^*)]$ (or, equivalently, $S \coloneqq [S; (m_1^*, \mathsf{ct}^*)]$). *Remark that following Eq. (13), ct^* is stored in the $N+1$-th position in S.*

- When $\mathcal{A}$ issues a decryption request over ciphertext $\mathsf{ct} \in \mathcal{C}$, it first checks that $\mathsf{extract}(\mathsf{ct}, \mathsf{aux}) \neq \varnothing$ and returns $\perp$ when this is not the case. Let $(\pi, \beta) = \mathsf{extract}(\mathsf{ct}, \mathsf{aux})$, then $\mathcal{B}$ returns $\perp$ whenever $\pi_N \neq 0$ (case of a challenge-dependent ciphertext). Lastly, when this is not the case ($\pi_N = 0$), $\mathcal{B}$ runs the plaintext extractor given by Eq. (14), i.e. simply returns $\sum_{i=0}^{N-1} \pi_i S[i].m + \beta$.

Finally, in order to generate G_2's final guess, $\mathcal{B}$ first simulates the verification procedure as in G_1. If it outputs True, $\mathcal{B}$ outputs the same guess γ' as $\mathcal{A}$. Otherwise $\mathcal{B}$ picks a random guess as $\mathsf{Finalize}$ would have done for a non-admissible adversary.

As $\mathcal{B}$ does not make any call to the decryption oracle, $\mathcal{B}$ is a valid CPA adversary.

As in the proof of Prop. 4, the claim follows from the facts that, when $\mathsf{ImVer}(\mathsf{sk}, \mathsf{ct}) = \mathsf{True}$ (or, equivalently, when $\mathsf{extract}(\mathsf{ct}, \mathsf{aux}) \neq \varnothing$), the correctness of $\mathcal{E}_H$ implies that (14) cannot be distinguished from $\mathcal{E}_H.\mathsf{Dec}(\mathsf{sk}, \mathsf{ct})$ and that the reduction duly replies $\perp$ for all decryption requests on ciphertexts which are byproducts of the challenge ciphertexts (i.e. for which $c^* \in \{c_0, ..., c_{L-1}\}$ in the notations of the vCCA game 2nd step decryption oracle definition, p. 9, and $\pi_N \neq 0$), consistently with the Verify procedure since, when this occurs, the adversary's run is not admissible.

We thus have

$$\left|\mathsf{Adv}^{G_1}_{\mathcal{E}_H,\mathcal{A}}(\lambda) - \mathsf{Adv}^{\mathrm{CPA}}_{\mathcal{E}_H,\mathcal{B}}(\lambda)\right| = \mathrm{neg}(\lambda).$$

and finally

$$\left|\mathsf{Adv}^{\mathrm{vCCA}}_{\mathcal{E}_H,\mathcal{A}}(\lambda) - \mathsf{Adv}^{\mathrm{CPA}}_{\mathcal{E}_H,\mathcal{B}}(\lambda)\right| \leq \mathrm{neg}(\lambda).$$

□

6.2 A public-key vCCA secure construction based on Paillier

6.2.1 Paillier encryption scheme

This section briefly presents the well-known Paillier's original cryptosystem [Pai99], denoted $\mathcal{E}_P$, which security is grounded in the Composite Residuosity Class Problem hardness assumption. The scheme is partially homomorphic allowing additions, or multiplications by a constant, but does not support multiplications between two ciphertexts. Note that a relatively simple modification of this scheme, described in [CF15], allows to perform one level of multiplications.

Let n be an RSA modulus. The plaintext space is $\mathbb{Z}_n$ and the ciphertext space is $\mathbb{Z}^\times_{n^2}$. Let $\mathcal{S}_n$ be the set $\mathcal{S}_n = \{u \in \mathbb{Z}^\times_{n^2} \mid u \equiv 1 \pmod n\}$, which is a multiplicative subgroup of $\mathbb{Z}^\times_{n^2}$. For all $u \in \mathcal{S}_n$ we define the function $L : \mathcal{S}_n \to \mathbb{Z}_n$, such that $L(u) = \frac{u-1}{n}$.

- $\mathcal{E}_P.\mathsf{KeyGen}$: sample a RSA modulus $n = pq$ such that p and q are distinct large prime numbers and such that $\gcd(pq, (p-1)(q-1)) = 1$. Let $\varphi(n) = (p-1)(q-1)$ and $\omega \coloneqq \omega(n) = \mathrm{lcm}(p-1, q-1)$. Choose uniformly at random an integer $g \in \mathbb{Z}^\times_{n^2}$, such that $L(g^\omega \pmod{n^2}) \wedge n = 1$. Set the public key $\mathsf{pk} = (n, g)$ and the secret key $\mathsf{sk} = \omega(n)$.

- $\mathcal{E}_P.\mathsf{Enc}$: given $m \in \mathbb{Z}_n$ and pk, sample uniformly at random $r \xleftarrow{\$} (\mathbb{Z}_n)^\times$ and return $c = g^m r^n \pmod{n^2}$.

- $\mathcal{E}_P.\mathsf{Dec}$: given $c \in \mathbb{Z}_{n^2}$ and sk, return $\dfrac{L(c^{\mathsf{sk}} \pmod{n^2})}{L(g^{\mathsf{sk}} \pmod{n^2})} \pmod n$.

- $\mathcal{E}_P.\mathsf{Add}$: given $c, c' \in \mathbb{Z}^2_{n^2}$, compute and return $[c \cdot c']_{n^2}$.

- $\mathcal{E}_P.\mathsf{Mulc}$: given $\alpha \in \mathbb{Z}_n$ and $c \in \mathbb{Z}_{n^2}$, compute and return $[c^\alpha]_{n^2}$.

As a notable property with respect to the present work, the Paillier scheme achieves perfect correctness.

6.2.2 A public-key vCCA secure construction based on Paillier

Let $\mathcal{E}_P = (\mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec}, \mathsf{Add}, \mathsf{Mulc})$ denote the Paillier encryption scheme. We now consider the scheme $\mathcal{E}_P^{(2)}$, built from $\mathcal{E}_P$ as follows[24].

- $\mathcal{E}_P^{(2)}.\mathsf{KeyGen}$: run $\mathcal{E}_P.\mathsf{KeyGen}$ to get n, g and ω, then pick ξ uniformly at random in $\mathbb{Z}_n$. Generate ciphertext $\mathsf{ct}^\triangle = (\mathcal{E}_P.\mathsf{Enc}(\mathcal{E}_P.\mathsf{pk}, 1), \mathcal{E}_P.\mathsf{Enc}(\mathcal{E}_P.\mathsf{pk}, \xi))$. The public key is set to $\mathsf{pk} = (n, \mathsf{ct}^\triangle)$, while all the other parameters remain private (including the generator g). As in the original scheme, the secret key is $\mathsf{sk} = \omega(n)$.

- $\mathcal{E}_P^{(2)}.\mathsf{Enc}$: given $m \in \mathbb{Z}_n$ and $\mathsf{pk} = (n, \mathsf{ct}^\triangle)$, pick r_0, r_1 uniformly in $\mathbb{Z}_{n^2}$. When $m = 0$, return
$$\mathsf{ct} = (c_0, c_1) = ([r_0^n]_{n^2}, [r_1^n]_{n^2}). \tag{15}$$
Otherwise, compute and return
$$\mathsf{ct} = (c_0, c_1) = ([(\mathsf{ct}^\triangle.c_0)^m r_0^n]_{n^2}, [(\mathsf{ct}^\triangle.c_1)^m r_1^n]_{n^2}). \tag{16}$$

- $\mathcal{E}_P^{(2)}.\mathsf{ImVer}$: given $\mathsf{ct} \in \mathbb{Z}_{n^2}^2$ and sk, let μ_0 denote $\mathcal{E}_P.\mathsf{Dec}(\mathsf{sk}, \mathsf{ct}.c_0)$ (and respectively so for μ_1). Then return True if $[\xi\mu_0]_n = \mu_1$ and False otherwise.

- $\mathcal{E}_P^{(2)}.\mathsf{Dec}$: given $\mathsf{ct} \in \mathbb{Z}_{n^2}^2$ and sk, if $\mathcal{E}_P^{(2)}.\mathsf{ImVer}(\mathsf{sk}, \mathsf{ct}) = \mathsf{True}$, then return $\mathcal{E}_P.\mathsf{Dec}(\mathsf{sk}, \mathsf{ct}.c_0)$. Otherwise, return $\perp$.

- $\mathcal{E}_P^{(2)}.\mathsf{Add}$ and $\mathcal{E}_P^{(2)}.\mathsf{Mulc}$ are straightforwardly derived from those of $\mathcal{E}_P$.

We highlight that:

- $\mathsf{ct}^\triangle$ is a *privately generated* encryption of 1 under $\mathcal{E}_P^{(2)}$.

- Since g and ξ are *not* public, only encryptions of 0 under $\mathcal{E}_P^{(2)}$ can be generated without using $\mathsf{ct}^\triangle$ by picking r_0, r_1 uniformly in $\mathbb{Z}_{n^2}$ and following Eq. (15) above.

- $\mathcal{E}_P^{(2)}.\mathsf{Enc}$ (i.e. Eq. 16) can equivalently be written as
$$\mathsf{ct} = \mathcal{E}_P^{(2)}.\mathsf{Add}(\mathcal{E}_P^{(2)}.\mathsf{Mulc}_m(\mathsf{ct}^\triangle), \mathcal{E}_P^{(2)}.\mathsf{Enc}(\mathsf{pk}, 0)). \tag{17}$$

Proposition 6. *$\mathcal{E}_P^{(2)}$ is CPA secure.*

Proof. The proof is done via a simple reduction to the CPA security of $\mathcal{E}_P$. The reduction picks ξ and, since the scheme is public-key, only has to generate a proper challenge ciphertext from the one it obtains from its CPA challenger against $\mathcal{E}_P$, which is denoted by $c^* = \mathcal{E}_P.\mathsf{Enc}(\mathsf{pk}, m_\gamma)$ (for unknown bit γ). This is simply done by picking r uniformly in $\mathbb{Z}_n$ and returning, $\mathsf{ct}^* = (c^*, [c^{*\xi} r^n]_{n^2})$, i.e. $(c^*, [\mathcal{E}_P.\mathsf{Mulc}_\xi(c^*) r^n]_{n^2})$, to the CPA adversary against $\mathcal{E}_P^{(2)}$. □

[24] Following Sect. 4.5, we emphasize that this scheme is not new and has already been proposed and used in several works, e.g. [BCI+13, GGPR13]

Lemma 7. *Let* $\mathsf{ct} = (c_0, c_1)$ *denote a well-formed ciphertext under* $\mathcal{E}_P^{(2)}$ *encrypting a linear combination* $\sum_i \alpha_i m_i$*, then there exists* r *and* r' *such that*

$$c_0 = (\mathsf{ct}^{\triangle}.c_0)^{\sum_i \alpha_i m_i} r^n$$

and

$$c_1 = (\mathsf{ct}^{\triangle}.c_1)^{\sum_i \alpha_i m_i} r'^n$$

Proof. If ct is a fresh ciphertext, the result directly follows from the definition of $\mathcal{E}_P^{(2)}.\mathsf{Enc}$. Otherwise, asumme that ct is obtained by homomorphically evaluating some linear combination $\mathcal{E}_P^{(2)}.\mathsf{Eval}(\mathtt{lincomb}_\alpha, \mathsf{ct}_0, ..., \mathsf{ct}_{l-1})$ with $\alpha = (\alpha_0, \cdots, \alpha_{l-1})$ over the ct_i's, being well-formed ciphertexts respectively encrypting messages m_i's. Hence, there exist r_i, r'_i's such that $\mathsf{ct}_i.c_0 = (\mathsf{ct}^{\triangle}.c_0)^{m_i} r_i^n$ and $\mathsf{ct}_i.c_1 = (\mathsf{ct}^{\triangle}.c_1)^{m_i} r'^n_i$. We thus have,

$$\begin{aligned}
\mathsf{ct} &= \left(\prod_{i=0}^{l-1} \left((\mathsf{ct}^{\triangle}.c_0)^{m_i} r_i^n \right)^{\alpha_i} , \prod_{i=0}^{l-1} \left((\mathsf{ct}^{\triangle}.c_1)^{m_i} r'^n_i \right)^{\alpha_i} \right) \\
&= \left(\prod_{i=0}^{l-1} \left((\mathsf{ct}^{\triangle}.c_0)^{m_i \alpha_i} (r_i{}^{\alpha_i})^n \right) , \prod_{i=0}^{l-1} \left((\mathsf{ct}^{\triangle}.c_1)^{m_i \alpha_i} (r'_i{}^{\alpha_i})^n \right) \right) \\
&= \left((\mathsf{ct}^{\triangle}.c_0)^{\sum_{i=0}^{l-1} \alpha_i m_i} \left(\prod_{i=0}^{l-1} r_i{}^{\alpha_i} \right)^n , (\mathsf{ct}^{\triangle}.c_1)^{\sum_{i=0}^{l-1} \alpha_i m_i} \left(\prod_{i=0}^{l-1} r'_i{}^{\alpha_i} \right)^n \right)
\end{aligned}$$

Letting $r = \prod_{i=0}^{l-1} r_i{}^{\alpha_i}$ and $r' = \prod_{i=0}^{l-1} r'_i{}^{\alpha_i}$ completes the proof. □

Putting $\mathcal{E}_P^{(2)}$ into perspective with Def. 7, remark that given a well-formed ciphertext ct under $\mathcal{E}_P^{(2)}$, built by some adversary $\mathcal{A}$, the above lemma (and the perfect correctness of $\mathcal{E}_P$) implies that π_0, with

$$(\pi, \beta) = \mathsf{extract}(\mathsf{ct}, \mathsf{aux}) \tag{18}$$

gives a plaintext extractor (i.e. $\pi_0 = \mathcal{E}_P^{(2)}.\mathsf{Dec}(\mathsf{sk}, \mathsf{ct})$). Furthermore, since the only ciphertexts that an adversary can generate from scratch are encryptions of 0, $\beta = 0$ in Eq. (18).

We now prove that $\mathcal{E}_P^{(2)}$ achieves vCCA security.

Proposition 7. $\mathcal{E}_P^{(2)}$ *is vCCA secure under the assumption that it has the LOH property.*

Proof. This follows from Propositions 5 and 6. In the notations of the proof of Proposition 5, we have $N = 1$ and $\mathsf{pk}_0 = \mathsf{ct}^{\triangle}$. Lemma 7 (and Eq. 18) further gives a plaintext extractor which is consistent with Eq. (14) in that proof. Also note that $\mathcal{E}_P^{(2)}$ satisfies property (10). □

It is interesting that, under the assumption that it has the LOH property, such a simple construction eventually achieves vCCA security. As a byproduct, the previous proposition also implies that $\mathcal{E}_P^{(2)}$ achieves CCA1 security. Further note that the recent theoretical barriers revealed in [Sch24, SV25] against the provable CCA1 or even PCA1 security of vanilla ElGamal and Paillier[25] (as well as alike semi-homomorphic PKE schemes) do not apply here. Indeed, we establish CCA1 security for a "two-ciphertexts" variant of Paillier which is not covered by these results (essentially as this construction lacks the required property that the validity of ciphertexts can be publicly verified). The same remark applies to the modified Paillier-ElGamal scheme, where λ plaintext bits are forced to be zeroes,

[25] Despite of the fact that CCA1 security proofs do exist either under non-standard assumptions [AKP13] or in the idealized Generic as well as Algebraic Group Models [Lip10, FKL18].

that is shown CCA1 secure under the DCR assumption[26] in [Lib25][27]. Consistently with these remarks, vanilla Paillier further does not have the LOH property as also argued in [BCI+13] ([BCI+12], p. 33).

7 A non-asymptotic analysis for the mean of squares of centered discrete Gaussian deviates

In this section, we provide one-sided confidence intervals for the mean of squares of a number of centered discrete Gaussian deviates and their linear combinations. We do so via an explicitly non-asymptotic analysis, i.e. without resorting to the CLT. This toolbox will be used in the next section, where we use these confidence intervals to get "safe" bounds for smudging, i.e. bounds that are exceeded only with negligible probability in the security parameter, on the variance of some LWE noise from an *estimation* of its variance. Sect. 7.4 gives the result used later in our constructions: although relatively coarse, it relies only on subgaussianity and is therefore stable under linear combinations. Sect. 7.5 derives a significantly tighter bound for genuine discrete Gaussian samples; this result is of independent interest but does not directly apply to evaluated ciphertext noises. Finally, Sect. 7.6 compares these rigorous bounds with the classical heuristic based on the χ^2 distribution. We decided to include this technical section as part of the paper body rather than in appendix. For a reader more interested in the constructions, we recommend to skip it at first and return to it when needed.

7.1 Additional notations

In this section, given two values $x, y \in \mathbb{R}$, if $|x - y| \leq \mathrm{neg}(\lambda)$ we write $x \simeq_\lambda y$. Similarly, when $x \leq y(1 + \mathrm{neg}(\lambda))$, we write $x \lesssim_\lambda y$. These notations are used only in this section.

7.2 Some properties of the discrete Gaussian

Recall the definition of the centered discrete Gaussian distribution (Def. 6, Sect. 4.4).

Let X denote a centered discrete Gaussian random variable with parameter $\sigma > 0$, i.e. such that $f_X(x) = P(X = x) = \frac{1}{Z(\sigma)} e^{-\frac{x^2}{2\sigma^2}}$, with $Z(\sigma) = \sum_{k=-\infty}^{+\infty} e^{-\frac{k^2}{2\sigma^2}}$.

Following [CKS20, Corollary 17] we have that $\mathsf{Var}[X] \leq \sigma^2$ and $P(|X| \geq B) \leq 2e^{-\frac{B^2}{2\sigma^2}}$.

For all $\sigma > 0$, some known facts on Z are [CKS20, Facts 19,20]:

- $Z(\sigma) = \sum_{k\in\mathbb{Z}} e^{-\frac{k^2}{2\sigma^2}} = \sqrt{2\pi}\sigma \left(1 + 2\sum_{k\geq 1} e^{-2\pi^2\sigma^2 k^2}\right)$ (Poisson summation formula).
- $\max(\sqrt{2\pi}\sigma, 1) \leq Z(\sigma) \leq \sqrt{2\pi}\sigma + 1$.
- $\sqrt{2\pi}\sigma(1 + 2e^{-2\pi^2\sigma^2}) \leq Z(\sigma) \leq \sqrt{2\pi}\sigma(1 + 2e^{-2\pi^2\sigma^2}) + e^{-2\pi^2\sigma^2}$.
- $1 + 2e^{-\frac{1}{2\sigma^2}} \leq Z(\sigma) \leq 1 + 2e^{-\frac{1}{2\sigma^2}} + \sqrt{2\pi}\sigma e^{-\frac{1}{2\sigma^2}}$.

For all $\sigma > 0$, some known facts on $\mathsf{Var}[X]$ are:

- $\mathsf{Var}[X] \leq \sigma^2 \left(1 - \frac{4\pi^2\sigma^2}{e^{4\pi^2\sigma^2} - 1}\right) < \sigma^2$ [CKS20, Prop. 21].
- $\mathsf{Var}[X] \geq (e^{1/\sigma^2} - 1)^{-1}$ [CKS20, Corr. 24].

[26] Stricto sensu, it has been established under standard Decision Composite Residuosity (DCR) and an additional Composite Non-Invertibility assumption which is falsifiable. Furthermore, reliance on this latter assumption may be avoided under some conditions [Lib25].

[27] For the same reasons, Schäge's result does not apply to the Dåmgard-ElGamal scheme [Dam92, BP04c] which is now proven CCA1 solely under the DDH assumption [Lib25].

Remark that, since we consider only centered discrete Gaussians, $\mathsf{Var}[X] = E[X^2]$.

The next lemma conveniently establishes a lower-bound on the variance proxy σ, such that the variance of the resulting discrete Gaussian distribution is negligibly close to σ in some security parameter. Interestingly, some numerical experiments show that this lower bound is quite small and, in fact, already satisfied by the typical values used in FHE parameters.

Lemma 8. *Let $\lambda > 0$ and $X \xleftarrow{\$} \chi_\sigma$, with $\sigma \geq 1$. If,*

$$\sigma \geq \sqrt{\frac{\lambda \log 2 + \log(208\pi^2)}{2\pi^2 - 4}}, \tag{19}$$

we have that $0 \leq \sigma^2 - E[X^2] \leq 2^{-\lambda}$, i.e. $\sigma^2 \simeq_\lambda E[X^2]$.

Proof. Recall that $E[X^2] = \frac{1}{Z(\sigma)} \sum_{x\in\mathbb{Z}} x^2 e^{-\frac{x^2}{2\sigma^2}}$. First remark that,

$$\frac{dZ}{d\sigma} = \frac{1}{\sigma^3} \sum_{x\in\mathbb{Z}} x^2 e^{-\frac{x^2}{2\sigma^2}}.$$

Hence, $E[X^2] = \sigma^3 Z'(\sigma)/Z(\sigma)$. From the Poisson summation formula, we have that,

$$Z(\sigma) = \sqrt{2\pi}\sigma \left(1 + 2 \sum_{k\geq 1} e^{-2\pi^2\sigma^2 k^2}\right), \tag{20}$$

and so,

$$\begin{aligned} Z'(\sigma) &= \sqrt{2\pi}\left(1 + 2 \sum_{k\geq 1} e^{-2\pi^2\sigma^2 k^2}\right) + \sqrt{2\pi}\sigma \left(2 \sum_{k\geq 1} (-4\pi^2\sigma k^2) e^{-2\pi^2\sigma^2 k^2}\right) \\ &= \frac{Z(\sigma)}{\sigma} - 8\pi^2\sqrt{2\pi}\sigma^2 \sum_{k\geq 1} k^2 e^{-2\pi^2\sigma^2 k^2}. \end{aligned}$$

Thus,

$$E[X^2] = \sigma^2 - \frac{8\pi^2\sqrt{2\pi}\sigma^5}{Z(\sigma)} \sum_{k\geq 1} k^2 e^{-2\pi^2\sigma^2 k^2}.$$

Let us now consider, the approximation error

$$\begin{aligned} \sigma^2 - E[X^2] &= \frac{8\pi^2\sqrt{2\pi}\sigma^5}{Z} \sum_{k\geq 1} k^2 e^{-2\pi^2\sigma^2 k^2}, \\ &\leq 8\pi^2\sigma^4 \underbrace{\sum_{k\geq 1} k^2 e^{-2\pi^2\sigma^2 k^2}}_{S}, \end{aligned}$$

since, from Eq. (20), $Z(\sigma) \geq \sqrt{2\pi}\sigma$. Let $\alpha = 2\pi^2\sigma^2 > 0$ so that $S = \sum_{k\geq 1} k^2 e^{-\alpha k^2}$. We have, for $k \geq 1$,

$$\begin{aligned} k^2 e^{-\alpha k^2} &= e^{-\alpha} k^2 e^{-\alpha(k^2-1)}, \\ &\leq e^{-\alpha} k^2 e^{-\alpha(k-1)}. \end{aligned}$$

Hence, letting $\ell = k-1$ and $r = e^{-\alpha} < 1$, yields,

$$\begin{aligned} S &\leq e^{-\alpha} \sum_{k\geq 1} k^2 e^{-\alpha(k-1)}, \\ &\leq e^{-\alpha} \sum_{\ell\geq 0} (\ell+1)^2 e^{-\alpha\ell}, \\ &\leq e^{-\alpha} \left(\sum_{\ell\geq 0} \ell^2 r^\ell + 2\sum_{\ell\geq 0} \ell r^\ell + \sum_{\ell\geq 0} r^\ell \right), \\ &\leq e^{-\alpha} \left(\frac{r(1+r)}{(1-r)^3} + \frac{2r}{(1-r)^2} + \frac{1}{1-r} \right), \end{aligned}$$

since, for $r < 1$,

$$\sum_{\ell\geq 0} \ell^2 r^\ell = \frac{r(1+r)}{(1-r)^3}, \ \sum_{\ell\geq 0} \ell r^\ell = \frac{r}{(1-r)^2} \text{ and } \sum_{\ell\geq 0} r^\ell = \frac{1}{1-r}.$$

We now assume $\sigma \geq 1$ so that $r = e^{-\alpha} = e^{-2\pi^2\sigma^2} \leq e^{-2\pi^2} << 1$. Also $1 - r \geq 1/2$, so that,

$$\frac{1}{1-r} \leq 2, \ \frac{1}{(1-r)^2} \leq 4 \text{ and } \frac{1}{(1-r)^3} \leq 8,$$

yielding,

$$S \leq 26 e^{-2\pi^2\sigma^2}.$$

Putting it all together, and without any optimization attempt, we thus get,

$$\sigma^2 - E[X^2] \leq 208\pi^2\sigma^4 e^{-2\pi^2\sigma^2}. \tag{21}$$

Letting $208\pi^2\sigma^4 e^{-2\pi^2\sigma^2} \leq 2^{-\lambda}$, yields,

$$\log(208\pi^2) + 4\log\sigma - 2\pi^2\sigma^2 \leq -\lambda\log 2.$$

Constant optimization be damned, we have, still under the assumption that $\sigma \geq 1$,

$$\log(208\pi^2) + 4\log\sigma - 2\pi^2\sigma^2 \leq \log(208\pi^2) + 4\sigma^2 - 2\pi^2\sigma^2 \leq -\lambda\log 2.$$

thus,

$$\sigma \geq \sqrt{\frac{\lambda\log 2 + \log(208\pi^2)}{2\pi^2 - 4}}.$$

□

We give a few numerical examples in Table 1.

Table 1: Numerical illustration of Lemma 8. Note that $\sigma = 3.19$ is a typical value used in FHE parameterization.

λ	σ (RHS Eq. 19)	$\log_2$ (RHS Eq. 21)
80	2.00	-98.91
128	2.47	-157.52
192	2.99	-237.27
220	3.19	-272.09
256	3.42	-314.99

7.3 Sub-Gaussian, sub-exponential and almost sub-exponential random variables

Definition 16 (Def. 2.2 in [Wai19])**.** A random variable X is said to be sub-Gaussian with parameters $\sigma > 0$, denoted $\mathsf{subG}(\sigma)$, if $E[X] = 0$ and its MGF is such that,

$$E[e^{sX}] \leq e^{s^2\sigma^2/2}, \ \forall s \in \mathbb{R}.$$

Definition 17 (Def. 1.11 in [Rig15])**.** A random variable Y is said to be sub-exponential with parameter α, denoted $\mathsf{subE}(\alpha)$, if $E[Y] = 0$ and its MGF is such that

$$E[e^{sY}] \leq e^{s^2\alpha^2/2}, \ \forall |s| \leq \frac{1}{\alpha}.$$

Definition 18 (Def. 2.7 in [Wai19])**.** A random variable Y is said to be sub-exponential with parameters $\nu > 0, \alpha > 0$, which we denote $Y \sim \mathsf{subE}(\nu, \alpha)$, if $E[Y] = 0$ and its MGF is such that

$$E[e^{sY}] \leq e^{s^2\nu^2/2}, \ \forall |s| \leq \frac{1}{\alpha}.$$

Essentially, $\mathsf{subE}(\alpha)$ in Def. 17 means $\mathsf{subE}(\alpha, \alpha)$ in Def. 18. We mainly use this latter definition in this section.

Lemma 9 ([Wai19], p. 29)**.** *Let $Y_0, ..., Y_{K-1}$ denote K iid $\mathsf{subE}(\nu, \alpha)$ random variables, then $\sum_{k=0}^{K-1} Y_k$ is $\mathsf{subE}(\nu\sqrt{K}, \alpha)$.*

Lemma 10 (Subexponential tail-bounds ([Wai19], Prop. 2.9))**.** *Let Y be $\mathsf{subE}(\nu, \alpha)$ and $c > 0$, define,*

$$U(c, \nu, \alpha) = \begin{cases} e^{-\frac{c^2}{2\nu^2}}, & 0 \leq c \leq \frac{\nu^2}{\alpha}, \\ e^{-\frac{c}{2\alpha}}, & c > \frac{\nu^2}{\alpha}. \end{cases} \tag{22}$$

Then the following holds,

- $P(Y > c) \leq U(c, \nu, \alpha)$.
- $P(Y < -c) \leq U(c, \nu, \alpha)$.
- $P(|Y| > c) \leq 2U(c, \nu, \alpha)$.

Definition 19 (Almost sub-exponential random variable)**.** Let $\lambda > 0$, a random variable Y is said to be *almost* sub-exponential with parameters $\nu > 0, \alpha > 0$, which we denote $Y \sim_\lambda \mathsf{subE}(\nu, \alpha)$, if $E[Y] = 0$ and its MGF is such that

$$E[e^{sY}] \leq e^{s^2\nu^2/2}(1 + \mathrm{neg}(\lambda)), \ \forall |s| \leq \frac{1}{\alpha}. \tag{23}$$

Lemma 11. *Let $Y \sim_\lambda \mathsf{subE}(\nu, \alpha)$ and $c > 0$, then, recalling Eq. (22), the following holds,*

- $P(Y > c) \lesssim_\lambda U(c, \nu, \alpha)$.
- $P(Y < -c) \lesssim_\lambda U(c, \nu, \alpha)$.
- $P(|Y| > c) \lesssim_\lambda 2U(c, \nu, \alpha)$.

Proof. The proof is very close from that of [Wai19, Prop. 2.9]. We first start with Markov inequality, $P(Y > c) \leq E[Y]/c$ $(c > 0)$, which we can apply to the random variable e^{sY} to get,

$$P(Y > c) = P(e^{sY} > e^{sc}) \leq e^{-sc}E[e^{sY}].$$

Using Eq. (23) then yields,

$$P(Y > c) \leq e^{-sc+s^2\nu^2/2}(1+\mathrm{neg}(\lambda)),\ \forall |s| \leq \frac{1}{\alpha}.$$

Let $g(s,c) = -sc + s^2\nu^2/2$, following the proof of [Wai19, Prop. 2.9] we have,

$$\inf_{s\in[0,\alpha^{-1})} g(s,c) \leq \begin{cases} -\frac{c^2}{2\nu^2}, & 0 \leq c \leq \nu^2/\alpha, \\ -\frac{c}{2\alpha}, & c > \nu^2/\alpha. \end{cases}$$

The claim follows for the first inequality, $P(Y > c) \lesssim_\lambda U(c,\nu,\alpha)$, since,

$$\inf_{s\in[0,\alpha^{-1})} e^{g(s,c)}(1+\mathrm{neg}(\lambda)) = e^{\inf_{s\in[0,\alpha^{-1})} g(s,c)}(1+\mathrm{neg}(\lambda)).$$

The other two inequalities in the statement admit similar proofs. □

Lemma 12. *Let $\lambda > 0$ and $Y_0, ..., Y_{K-1}$ denote K iid random variables, with $2^{-\lambda+1}K < 1$, such that $Y_i \sim_\lambda \mathsf{subE}(\nu,\alpha)$, then $\sum_{k=0}^{K-1} Y_k \sim_{\lambda'} \mathsf{subE}(\nu\sqrt{K},\alpha)$ with $\lambda' = \lambda - 1 - \log_2 K$.*

Proof. From Def. 19 and Eq. (23), $\forall |s| \leq \frac{1}{\alpha}$,

$$\begin{aligned} E[e^{s\sum_{k=0}^{K-1} Y_k}] &= \prod_{k=0}^{K-1} E[e^{sY_k}], \\ &\leq e^{s^2K\nu^2/2}(1+\mathrm{neg}(\lambda))^K. \end{aligned}$$

Since $2^{-\lambda}K \leq 2^{-\lambda+1}K < 1$, we have [28] that $(1+2^{-\lambda})^K \leq 1+2K2^{-\lambda}$. Thus letting, $1+2K2^{-\lambda} = 1+2^{-\lambda'}$ yields $\lambda' = \lambda - 1 - \log_2 K > 0$ (since $2^{-\lambda+1}K < 1$). □

7.4 A one-sided confidence interval for the mean of squares of linearly combined discrete Gaussian deviates

In this section, we derive an *exact* one-sided confidence interval for the mean of squares of discrete Gaussian deviates of the form,

$$P(E[X^2] > \phi(S^2, K, \lambda)) \leq 2^{-\lambda},$$

for some function ϕ and, given $\{X_k \xleftarrow{\$} \chi_\sigma\}_{k=0}^{K-1}$, with $S^2 = \frac{1}{K}\sum_{i=0}^{K-1} X_i^2$. This result is given in Prop. 8. The analysis in this section is explicitly non-asymptotic and uses generic properties from sub-Gaussian variables rather specific properties of the discrete Gaussian distribution. For this reason, as we shall see, the analysis in this section is stable under linear combinations.

Lemma 13 (Lemma 1.12 in [Rig15])**.** *If a random variable X is $\mathsf{subG}(\sigma)$ then $Y = X^2 - E[X^2]$ is $\mathsf{subE}(16\sigma^2)$.*

Lemma 14. *Let $\lambda \geq 5$ and $X \xleftarrow{\$} \chi_\sigma$, with $\sigma \geq 1$ satisfying inequality (19). Then, $\frac{X^2}{E[X^2]} - 1$ is $\mathsf{subE}(17)$.*

Proof. It is easy to check that if Y is $\mathsf{subE}(\alpha)$ then, for any a, aY is $\mathsf{subE}(|a|\alpha)$. Since, from Lemma 13, $X^2 - E[X^2]$ is $\mathsf{subE}(16\sigma^2)$ then $\frac{X^2}{E[X^2]} - 1$ is $\mathsf{subE}(16\sigma^2/E[X^2])$. Following Lemma 8, $E[X^2] \geq \sigma^2 - 2^{-\lambda}$. Hence[29],

$$\frac{\sigma^2}{E[X^2]} \leq \frac{1}{1-2^{-\lambda}/\sigma^2} \leq 1 + 2\cdot(2^{-\lambda}/\sigma^2) \leq 1 + 2^{-\lambda+1}.$$

[28] From the two standard inequalities, $1 + x \leq e^x$ $(x \geq 0)$ and $e^y \leq 1 + 2y$ $(0 \leq y \leq 1)$ we get that $(1+x)^K \leq e^{Kx} \leq 1 + 2Kx$, for $0 \leq Kx \leq 1$.

[29] It is easy to check that for $0 \leq x \leq 1/2$, $\frac{1}{1-x} \leq 1 + 2x$. For $\lambda \geq 1$ and $\sigma \geq 1$, we have $0 \leq 2^{-\lambda}/\sigma^2 \leq 1/2$.

It follows that, for $\lambda \geq 5$,

$$16\sigma^2/E[X^2] \leq 16 + 2^{-\lambda+5} \leq 17.$$

The claim follows from the fact that if Y is $\mathsf{subE}(\alpha)$, then given $\beta \geq \alpha$, Y is also $\mathsf{subE}(\beta)$. □

Interestingly, the above Lemma also works when X is a linear combination over iid discrete Gaussian deviates.

Lemma 15. *Let $\lambda \geq 5$, $\alpha \in \mathbb{Z}^L$ $(\alpha \neq 0)$, and $X = \sum_{i=0}^{L-1} \alpha_i Y_i$, $\{Y_i \xleftarrow{\$} \chi_\sigma\}_{i=0}^{L-1}$ (iid), with $\sigma \geq 1$ satisfying inequality (19). Then, $\frac{X^2}{E[X^2]} - 1$ is $\mathsf{subE}(17)$.*

Proof. Let $A = \sum_{i=0}^{L-1} \alpha_i^2$, then X is $\mathsf{subG}(\sigma\sqrt{A})$ [Wai19, p. 24]. Then, letting $m_2 = E[Y_0^2] = \ldots = E[Y_{L-1}^2]$, $E[X^2] = \sum_{i=0}^{L-1} \alpha_i^2 E[Y_i^2] = m_2 A$. Since, from Lemma 13, $X^2 - E[X^2]$ is $\mathsf{subE}(16A\sigma^2)$ then $\frac{X^2}{E[X^2]} - 1$ is $\mathsf{subE}(16A\sigma^2/(Am_2))$, i.e. $\mathsf{subE}(16\sigma^2/m_2)$. Following Lemma 8, $m_2 \geq \sigma^2 - 2^{-\lambda}$. The rest of the proof is identical to that of Lemma 14. □

Lemma 16. *Let $\lambda \geq 5$ and $\{X_k \xleftarrow{\$} \chi_\sigma\}_{k=0}^{K-1}$ (iid), with $\sigma \geq 1$ satisfying inequality (19). Then, for any $\alpha \in \mathbb{Z}^L$, $\frac{1}{E[X^2]} \sum_{k=0}^{K-1} X_k^2 - K$ is $\mathsf{subE}(17\sqrt{K}, 17)$.*

Proof. Directly from Lemmas 14 and 9. □

Proposition 8. *Let $\lambda \geq 5$ and $\{X_k \xleftarrow{\$} \chi_\sigma\}_{k=0}^{K-1}$, with $\sigma \geq 1$ satisfying inequality (19). Further denote $S^2 = \frac{1}{K} \sum_{k=0}^{K-1} X_k^2$. Then, for $K > 2 \cdot 17^2 \lambda \log 2$,*

$$P\left(E[X^2] > \frac{KS^2}{K - 17\sqrt{2K\lambda \log 2}}\right) \leq 2^{-\lambda}.$$

Proof. From Lemma 16, $\frac{KS^2}{E[X^2]} - K$ is $\mathsf{subE}(17\sqrt{K}, 17)$. From Lemma 10, we thus get that,

$$P\left(\frac{KS^2}{E[X^2]} - K < -c\right) = P\left(E[X^2] > \frac{KS^2}{K - c}\right) \leq U(c, 17\sqrt{K}, 17),$$

for $K - c > 0$, i.e. $0 < c < K$. Now, Eq. (22),

$$U(c, \nu, \alpha) = \begin{cases} e^{-\frac{c^2}{2 \cdot 17^2 K}}, & 0 \leq c \leq 17K, \\ e^{-\frac{c}{2 \cdot 17}}, & c > 17K. \end{cases}$$

Since $c < K < 17K$, we are in the first of the above two regimes. Then solving $e^{-\frac{c^2}{2 \cdot 17^2 K}} = 2^{-\lambda}$ yields,

$$c = 17\sqrt{2K\lambda \log 2}.$$

It therefore follows (when $c < K$ so when $17\sqrt{2K\lambda \log 2} < K$, i.e. $K > 2 \cdot 17^2 \lambda \log 2$) that

$$P\left(E[X^2] > \frac{KS^2}{K - 17\sqrt{2K\lambda \log 2}}\right) \leq 2^{-\lambda}.$$

□

Corollary 1. *Let $\lambda \geq 5$, $\alpha \in \mathbb{Z}^L$ $(\alpha \neq 0)$, and $X = \sum_{i=0}^{L-1} \alpha_i Y_i$, $\{Y_i \xleftarrow{\$} \chi_\sigma\}_{i=0}^{L-1}$ (iid), with $\sigma \geq 1$ satisfying inequality (19). Further let $\{X_k\}_{k=0}^{K-1}$ be a random iid sampling of X and $S^2 = \frac{1}{K} \sum_{k=0}^{K-1} X_k^2$. Then, for $K > 2 \cdot 17^2 \lambda \log 2$,*

$$P\left(E[X^2] > \frac{KS^2}{K - 17\sqrt{2K\lambda \log 2}}\right) \leq 2^{-\lambda}.$$

Proof. Identical to that of Prop. 8, using Lemma 15 rather than 14. □

Although this approach is stable under linear combinations, it is quite coarse since for a precision of 128 bits, it requires a sample size of $K = 2 \cdot 17^2 \cdot 128 \log 2 \approx 51281$.

7.5 A tighter one-sided confidence interval for the mean of squares of discrete Gaussian deviates

In this section, we derive a one-sided confidence interval for the mean of squares of discrete Gaussian deviates of the form,

$$P(E[X^2] > \phi(S^2, K, \lambda)) \lesssim_\lambda 2^{-\lambda},$$

for some function ϕ and, given $\{X_k \xleftarrow{\$} \chi_\sigma\}_{k=0}^{K-1}$, with $S^2 = \frac{1}{K}\sum_{i=0}^{K-1} X_i^2$. This result is given in Prop. 10. The analysis in this section is explicitly non-asymptotic and uses specific properties of the discrete Gaussian distribution that we establish first.

Interestingly, for a centered real-valued random variable X that follows a *continuous* Gaussian distribution with variance σ^2 we have that $Y = X^2 - E[X^2] \sim \mathsf{subE}(2\sigma^2, 4\sigma^2)$ [Wai19, Example 2.8]. The next proposition shows that, in the variance regime of Lemma 8, a random variable that follows the discrete Gaussian distribution almost yields the same property.

Proposition 9. *Let $\lambda > 0$, $\kappa \geq 2\lambda + 7$ and $X \xleftarrow{\$} \chi_\sigma$, with $\sigma \geq 1$ satisfying inequality (from Lemma 8),*

$$\sigma \geq \sqrt{\frac{\kappa \log 2 + \log(208\pi^2)}{2\pi^2 - 4}} = B(\kappa), \tag{24}$$

Then $Y = X^2 - E[X^2] \sim_\lambda \mathsf{subE}(2\sigma^2, 4\sigma^2)$.

Proof. To prove this we consider,

$$\begin{aligned} E[e^{sY}] &= \frac{1}{Z(\sigma)} \sum_{k\in\mathbb{Z}} e^{s(k^2 - E[X^2])} e^{-\frac{k^2}{2\sigma^2}} \\ &= \frac{e^{-sE[X^2]}}{Z(\sigma)} \sum_{k\in\mathbb{Z}} e^{k^2\left(s - \frac{1}{2\sigma^2}\right)}. \end{aligned} \tag{25}$$

First remark that $E[e^{sY}] < \infty$ if $s < \frac{1}{2\sigma^2}$. In the rest of this proof, we will consider $|s| < \frac{1}{4\sigma^2} \leq \frac{1}{2\sigma^2}$ ($\sigma \geq 1$). Let $-\frac{1}{2\rho^2} = s - \frac{1}{2\sigma^2}$ i.e., $\rho^2 = \frac{1}{2}\left(\frac{1}{2\sigma^2} - s\right)^{-1} = \frac{\sigma^2}{1 - 2\sigma^2 s}$. Therefore,

$$\frac{2}{3}\sigma^2 \leq \rho^2 \leq 2\sigma^2. \tag{26}$$

Recall that, for $\xi > 0$,

$$Z(\xi) = \sum_{k\in\mathbb{Z}} e^{-\frac{k^2}{2\xi^2}} = \sqrt{2\pi}\xi\left(1 + 2\sum_{k\geq 1} e^{-2\pi^2\xi^2 k^2}\right).$$

For $k \geq 1$, $k^2 \geq k$, hence,

$$\begin{aligned} \sum_{k\geq 1} e^{-2\pi^2\xi^2 k^2} &\leq \sum_{k\geq 1} e^{-2\pi^2\xi^2 k} \\ &\leq \frac{e^{-2\pi^2\xi^2}}{1 - e^{-2\pi^2\xi^2}}. \end{aligned} \tag{27}$$

Returning to Eq. (25), we obtain,

$$E[e^{sY}] = e^{-sE[X^2]} \frac{Z(\rho)}{Z(\sigma)}, \tag{28}$$

with, following Eq. (27),

$$\begin{aligned}\frac{Z(\rho)}{Z(\sigma)} &\leq \frac{\sqrt{2\pi}\rho\left(1+2\frac{e^{-2\pi^2\rho^2}}{1-e^{-2\pi^2\rho^2}}\right)}{\sqrt{2\pi}\sigma}\\ &\leq \frac{\rho}{\sigma}(1+\varepsilon(\rho)),\end{aligned}$$

with $\varepsilon(\rho) = 2\frac{e^{-2\pi^2\rho^2}}{1-e^{-2\pi^2\rho^2}}$.
From Eq. (26), we get that,

$$1 - e^{-2\pi^2\rho^2} \geq 1 - e^{-4\pi^2/3} \geq \frac{1}{2},$$

so that,

$$\varepsilon(\rho) \leq 4e^{-2\pi^2\rho^2}.$$

Finally, from Eqs. (27) and (24), we obtain (recall that $\kappa \geq 2\lambda + 7$ so that $4e^{-\kappa/2} \leq 2^{-\lambda-2}$),

$$\begin{aligned}\varepsilon(\rho) &\leq 4e^{-(4/3)\pi^2\sigma^2},\\ &\leq 4e^{-(4/3)\pi^2\frac{\kappa\log 2+\log(208\pi^2)}{2\pi^2-4}},\\ &\leq 4e^{-\frac{4\pi^2\kappa\log 2}{3(2\pi^2-4)}}e^{-\frac{4\pi^2\log(208\pi^2)}{3(2\pi^2-4)}},\\ &\leq 4e^{-\frac{4\pi^2\kappa\log 2}{3(2\pi^2-4)}},\\ &\leq 4e^{-\kappa/2},\\ &\leq 2^{-\lambda-2}.\end{aligned}$$

Returning to Eq. (28), we thus now have,

$$E[e^{sY}] \leq e^{-sE[X^2]}\frac{\rho}{\sigma}(1+2^{-\lambda-2}),$$

i.e.,

$$\log E[e^{sY}] \leq -sE[X^2] + \log\frac{\rho}{\sigma} + \log(1+2^{-\lambda-2}). \tag{29}$$

Recall that $\rho^2 = \frac{\sigma^2}{1-2\sigma^2 s}$ so that $\rho = \frac{\sigma}{\sqrt{1-2\sigma^2 s}}$, $\frac{\rho}{\sigma} = \frac{1}{\sqrt{1-2\sigma^2 s}}$ and $\log\frac{\rho}{\sigma} = -\frac{1}{2}\log(1-2\sigma^2 s)$. We now use the elementary fact that, for $|x| \leq 1/2$, $-\log(1-x) \leq x + x^2$. Letting $x = 2\sigma^2 s$, yields $|x| \leq 1/2$, since $|s| \leq \frac{1}{4\sigma^2}$. Hence,

$$-\frac{1}{2}\log(1-2\sigma^2 s) \leq \frac{1}{2}(2\sigma^2 s + 4\sigma^4 s^2) = \sigma^2 s + 2\sigma^4 s^2.$$

Plugging this into Eq. (29), gives,

$$\begin{aligned}\log E[e^{sY}] &\leq -sE[X^2] + \sigma^2 s + 2\sigma^4 s^2 + \log(1+2^{-\lambda-2}),\\ &\leq s(\sigma^2 - E[X^2]) + 2\sigma^4 s^2 + \log(1+2^{-\lambda-2}).\end{aligned} \tag{30}$$

If $s \leq 0$ in Eq. (30), since $E[X^2] \leq \sigma^2$, we straightaway get that,

$$\log E[e^{sY}] \leq 2\sigma^4 s^2 + \log(1+2^{-\lambda-2}),$$

and, hence, that,

$$\begin{aligned}E[e^{sY}] &\leq e^{2\sigma^4 s^2}(1+2^{-\lambda-2}),\\ &\leq e^{\nu^2 s^2/2}(1+2^{-\lambda-2}),\end{aligned}$$

with $\nu^2 = 4\sigma^4$. Finally, if $s > 0$ in Eq. (30), Lemma 8 gives that

$$s(\sigma^2 - E[X^2]) \le s2^{-\kappa} \le \frac{2^{-\kappa}}{4\sigma^2} \le 2^{-\kappa} \le 2^{-\lambda-2}.$$

In this case, Eq. (30) thus becomes,

$$\log E[e^{sY}] \le 2^{-\lambda-2} + 2\sigma^4 s^2 + \log(1 + 2^{-\lambda-2}),$$

yielding[30],

$$\begin{aligned} E[e^{sY}] &\le e^{2^{-\lambda-2}} e^{2\sigma^4 s^2}(1 + 2^{-\lambda-2}), \\ &\le e^{\nu^2 s^2/2}(1 + 2^{-\lambda-1})(1 + 2^{-\lambda-2}), \\ &\le e^{\nu^2 s^2/2}(1 + 2^{-\lambda}). \end{aligned}$$

also with $\nu^2 = 4\sigma^4$. This completes the proof. □

Lemma 17. *Let $3 \le \lambda \le (\kappa - 7)/2$, for some $\kappa > 0$, and $X \xleftarrow{\$} \chi_\sigma$, with $\sigma \ge 1$ satisfying inequality (24), i.e. $\sigma \ge B(\kappa)$. Then, $\frac{X^2}{E[X^2]} - 1 \sim_\lambda \mathsf{subE}(2 + 2^{-\lambda+2}, 4 + 2^{-\lambda+3})$.*

Proof. It is easy to check the following facts:

1. If $Y \sim_\lambda \mathsf{subE}(\nu, \alpha)$ then $aY \sim_\lambda \mathsf{subE}(|a|\nu, |a|\alpha)$.
2. If $Y \sim_\lambda \mathsf{subE}(\nu, \alpha)$ then $Y \sim_\lambda \mathsf{subE}(\nu', \alpha)$, for $\nu' \ge \nu$.
3. If $Y \sim_\lambda \mathsf{subE}(\nu, \alpha)$ then $Y \sim_\lambda \mathsf{subE}(\nu, \alpha')$, for $\alpha' \ge \alpha$.

Let $Y = X^2 - E[X^2] \sim_\lambda \mathsf{subE}(2\sigma^2, 4\sigma^2)$ (from Prop. 9). From Fact 1, $\frac{X^2}{E[X^2]} - 1 \sim_\lambda \mathsf{subE}(2\sigma^2/E[X^2], 4\sigma^2/E[X^2])$.

Following Lemma 8, $E[X^2] \ge \sigma^2 - 2^{-\lambda}$. Hence[31],

$$\frac{\sigma^2}{E[X^2]} \le \frac{1}{1 - 2^{-\lambda}/\sigma^2} \le 1 + 2 \cdot (2^{-\lambda}/\sigma^2) \le 1 + 2^{-\lambda+1}.$$

It follows that, for $\lambda \ge 3$,

$$2\sigma^2/E[X^2] \le 2 + 2^{-\lambda+2} \text{ and } 4\sigma^2/E[X^2] \le 4 + 2^{-\lambda+3}.$$

Hence, from Facts 2 and 3 above, $\frac{X^2}{E[X^2]} - 1 \sim_\lambda \mathsf{subE}(2 + 2^{-\lambda+2}, 4 + 2^{-\lambda+3})$. □

Lemma 18. *Let $\lambda \ge 3$ and K polynomially bounded in λ. Further let $\lambda + 1 + \log_2 K \le \lambda' \le (\kappa - 7)/2$, for some $\kappa > 0$, let $\{X \xleftarrow{\$} \chi_\sigma\}$ and $\{X_k \xleftarrow{\$} \chi_\sigma\}_{k=0}^{K-1}$ an iid sample of length K, with $\sigma \ge 1$ satisfying inequality (24), i.e. $\sigma \ge B(\kappa)$. Then, $\frac{1}{E[X^2]} \sum_{k=0}^{K-1} X_k^2 - K \sim_\lambda \mathsf{subE}((2 + 2^{-\lambda+2})\sqrt{K}, 4 + 2^{-\lambda+3})$.*

Proof. From Lemma 17 applied with parameter λ', each $Y_k = \frac{X_k^2}{E[X^2]} - 1$ satisfies,

$$Y_k \sim_{\lambda'} \mathsf{subE}(2 + 2^{-\lambda'+2}, 4 + 2^{-\lambda'+3}).$$

[30] For $0 \le x \le 1$, we have that $e^x \le 1 + 2x$. Hence, $e^{2^{-\lambda-2}}(1 + 2^{-\lambda-2}) \le (1 + 2^{-\lambda-1})(1 + 2^{-\lambda-2}) = 1 + 2^{-\lambda-1} + 2^{-\lambda-2} + 2^{-2\lambda-3} \le 1 + 2^{-\lambda-1} + 2^{-\lambda-2} + 2^{-\lambda-3} = 1 + 2^{-\lambda}(2^{-1} + 2^{-2} + 2^{-3}) \le 1 + 2^{-\lambda}$.

[31] It is easy to check that for $0 \le x \le 1/2$, $\frac{1}{1-x} \le 1 + 2x$. For $\lambda \ge 1$ and $\sigma \ge 1$, we have $0 \le 2^{-\lambda}/\sigma^2 \le 1/2$.

Since $\lambda' \geq \lambda + 1 + \log_2 K$, Lemma 12 yields,

$$\sum_{k=0}^{K-1} Y_k \sim_\lambda \mathsf{subE}((2+2^{-\lambda'+2})\sqrt{K}, 4+2^{-\lambda'+3}).$$

Finally, as $\lambda' \geq \lambda$, we have $2+2^{-\lambda'+2} \leq 2+2^{-\lambda+2}$ and $4+2^{-\lambda'+3} \leq 4+2^{-\lambda+3}$, which gives the claim. □

Proposition 10. *Let $\lambda \geq 3$ and K polynomially bounded in λ. Further let $\lambda + 1 + \log_2 K \leq \lambda' \leq (\kappa - 7)/2$, for some $\kappa > 0$, let $\{X \xleftarrow{\$} \chi_\sigma\}$ and $\{X_k \xleftarrow{\$} \chi_\sigma\}_{k=0}^{K-1}$ an iid sample of length K with $\sigma \geq 1$ satisfying inequality (24), i.e. $\sigma \geq B(\kappa)$. Further denote $S^2 = \frac{1}{K}\sum_{k=0}^{K-1} X_k^2$. Then, for $K > 2(2+2^{-\lambda+2})^2\lambda\log 2$,*

$$P\left(E[X^2] > \frac{KS^2}{K-(2+2^{-\lambda+2})\sqrt{2K\lambda\log 2}}\right) \lesssim_\lambda 2^{-\lambda}.$$

Proof. From Lemma 18, $\frac{KS^2}{E[X^2]} - K \sim_\lambda \mathsf{subE}((2+2^{-\lambda+2})\sqrt{K}, 4+2^{-\lambda+3})$. From Lemma 11, we thus get that,

$$P\left(\frac{KS^2}{E[X^2]} - K < -c\right) = P\left(E[X^2] > \frac{KS^2}{K-c}\right) \lesssim_\lambda U(c, (2+2^{-\lambda+2})\sqrt{K}, 4+2^{-\lambda+3}),$$

for $K - c > 0$, i.e. $0 < c < K$. Now, Eq. (22),

$$U(c,\nu,\alpha) = \begin{cases} e^{-\frac{c^2}{2(2+2^{-\lambda+2})^2 K}}, & 0 \leq c \leq (2+2^{-\lambda+2})K/2, \\ e^{-\frac{c}{2(4+2^{-\lambda+3})}}, & c > (2+2^{-\lambda+2})K/2. \end{cases}$$

Since $c < K < (2+2^{-\lambda+2})K/2$, we are in the first of the above two regimes. Then solving $e^{-\frac{c^2}{2(2+2^{-\lambda+2})^2 K}} = 2^{-\lambda}$ yields,

$$c = (2+2^{-\lambda+2})\sqrt{2K\lambda\log 2}.$$

It therefore follows (when $c < K$ so when $(2+2^{-\lambda+2})\sqrt{2K\lambda\log 2} < K$, i.e. $K > 2(2+2^{-\lambda+2})^2\lambda\log 2$) that

$$P\left(E[X^2] > \frac{KS^2}{K-(2+2^{-\lambda+2})\sqrt{2K\lambda\log 2}}\right) \lesssim_\lambda 2^{-\lambda}.$$

□

The previous proposition shows that a precise estimator of the variance can be computed over a random sample of size K of a discrete Gaussian variable. *This approach yields reasonable sample sizes since $8 \times 128\log 2 \approx 710$ to reach a precision of 128 bits.* However, unlike the result in the previous section, it is not stable under linear combinations and therefore will not directly apply to evaluated ciphertext noises in our constructions. We retain it as a result of independent interest and as a useful benchmark.

7.6 Comparison with a heuristic confidence interval from the χ^2 distribution

For comparison's sake with the previous sections, if X follows a centered *continuous* Gaussian distribution with variance σ^2, then,

$$Y = \frac{1}{\sigma^2}\sum_{i=0}^{K-1} X_i^2,$$

follows a χ^2_K distribution. Also $E[Y] = K$.

Then we have the following two bounds.

Theorem 1 ([Gho21], Theorem 2)**.** *Let Y follow χ^2_K, then for $0 < c < K$,*

$$P(Y < K - c) \leq e^{-\frac{c^2}{4K}}.$$

Lemma 19. *Let $K \geq 4\lambda \log 2 + 1$, given a set $X_0, ..., X_{K-1}$ of iid centered Gaussian deviates with variance σ^2, then,*

$$P\left(\sigma^2 \leq \frac{KS^2}{K - 2\sqrt{K\lambda \log 2}}\right) \geq 1 - 2^{-\lambda}.$$

where $S^2 = \frac{1}{K}\sum_{i=0}^{K-1} X_i^2$.

Proof. Under the statement assumption, $\frac{KS^2}{\sigma^2}$ follows χ^2_K. Theorem 1 then tells us that,

$$P\left(\frac{KS^2}{\sigma^2} < K - c\right) = P\left(\sigma^2 > \frac{KS^2}{K - c}\right) \leq e^{-\frac{c^2}{4K}}.$$

Thus, letting $e^{-\frac{c^2}{4K}} = 2^{-\lambda}$ yields $c = 2\sqrt{K\lambda \log 2}$. It therefore follows (when $c < K$ so when $2\sqrt{K\lambda \log 2} < K$, i.e. $K \geq 4\lambda \log 2 + 1$) that,

$$P\left(\sigma^2 > \frac{KS^2}{K - 2\sqrt{K\lambda \log 2}}\right) \leq 2^{-\lambda}.$$

□

The above bound is heuristic, as it implicitly relies on the CLT, but yields sample sizes of 355 (for $\lambda = 128$). This number should be compared to the value 710 obtained in Sect. 7.5.

8 LWE-based constructions

We now give up the comfort of working under the correctness assumption. In this "jungle", we have decided to straightaway focus on concrete candidate LWE-based constructions. Still, a natural question is whether black-box results, such as Prop. 4 and 5, can be obtained in the case of approximate or somewhat correct schemes? I.e. may any CPA^D secure linearly homomorphic scheme with the LOH property be vCCA^D secure? A natural proof strategy for obtaining such results, for example in the private-key setting, is to perform a reduction towards a CPA^D challenger. For such a reduction to work, it then has to handle decryption requests over well-formed evaluated ciphertexts from the vCCA^D adversary by means of its CPA^D challenger's one. To do so, the reduction then has to populate the internal state of its CPA^D challenger with the exact same ciphertext provided by the vCCA^D adversary, by means of evaluation requests parameterized by the LOH extractor output and the set of fresh well-formed ciphertexts output by the encryption oracle. We discuss this further in Sect. 9. However, as we unveil in this section, building approximate or somewhat correct (LWE-based) schemes credibly satisfying even weak variants of the LOH property is particularly delicate. So it seems to us that this kind of black-box results have a limited relevance.

Now, the Knowledge-of-Exponent pattern is also natural to apply to LWE-based schemes: start from the multi-secret variant of Regev, put the message in the first slot and multiples of that message in the subsequent ones, for a large-enough set of secret multipliers. However, in the LWE setting, we have to deal with CPA^D security[32]. There are two ways to

[32]At the very least, a simple adaptation of the attack path in [CSBB24] allows to retrieve the L_∞ norm of the noise vector in such a ciphertext.

do so. On the one hand, we can put additional restrictions on the adversary abilities, via the cryptosystem specification which a CPA^D adversary is bounded to follow, so as to achieve correctness [ABMP24]. However, in the CCA adversary regime, compliance with these constraints (e.g. a bound on the L_2 norm of the linear combinations that the adversary can evaluate over fresh ciphertexts) has to be enforced and this appears difficult to achieve without advanced proof-of-knowledge techniques. On the other hand, if we wish to avoid such additional constraints, we have to embed *within the cryptosystem* some mechanism (usually some form of smudging) that eventually allows to handle CPA^D decryption requests in a reduction towards a CPA challenger. This is the path we follow in this section, by designing a line of schemes consistently following the Knowledge-of-Exponent pattern and achieving some degree of CPA^D security by means of smudging.

8.1 The basic Regev Scheme (FS_0)

We start from the usual Regev scheme. We consider the symmetric variant, which is parameterized by a security parameter λ, a dimension n, an integer q and a (discrete Gaussian) probability distribution χ_{σ_0} on $\mathbb{Z}_q$ with standard deviation σ_0. Plaintexts are elements of $\mathbb{Z}_t$ and ciphertexts are elements of $\mathbb{Z}_q^n \times \mathbb{Z}_q$. *Unless otherwise stated, we work under the assumption that t divides q.* The scheme $\mathsf{FS}_0(\lambda, n, q, t, \sigma_0)$ is then defined as follows:

- FS_0.KeyGen: pick a secret key $\mathsf{sk} \in \mathbb{Z}_q^n$ uniformly at random.
- FS_0.Enc: given a plaintext $m \in \mathbb{Z}_t$ and sk, pick $a \in \mathbb{Z}_q^n$ uniformly at random, pick e in $\mathbb{Z}_q$ according to χ_{σ_0}, and return (a, b) with $b = \langle a, \mathsf{sk}\rangle + \Delta m + e$ and $\Delta = q/t$.
- FS_0.Dec: given a ciphertext $c = (a, b)$ and sk, return $\left[\left\lceil \frac{1}{\Delta}(b - \langle a, \mathsf{sk}\rangle)\right\rfloor\right]_t$.

This scheme is trivially linearly homomorphic, offering homomorphic addition, mult-by-const ($\mathtt{mul}_\alpha$, $\alpha \in \mathbb{Z}_t$) and add-by-const ($\mathtt{add}_\alpha$, $\alpha \in \mathbb{Z}_t$) operators.

Regarding the security notions of interest to us in this paper, it is well-known that FS_0 is CPA secure under the LWE assumption [Reg05]. However, this scheme is *not correct*[33] since, as soon as $|e| \geq \frac{\Delta}{2}$, the noise is not entirely removed by the rounding in the decryption function. Also, keep in mind that the noise may be arbitrarily amplified through legit homomorphic operations, even when limited to homomorphic additions, so choosing parameters such that fresh ciphertexts decrypt correctly with overwhelming probability is necessary but not sufficient at all for correctness. Additionally, FS_0 is CPA^D insecure [CSBB24, CCP+24] and also trivially CCA1 insecure. For example, the ill-formed ciphertext $(-\Delta\mathbf{1}_i, 0)$ decrypts to $[\mathsf{sk}_i]_t$, where $\mathbf{1}_i$ is the i-th standard basis vector, which is enough to recover the secret key with n decryption requests when its components are in $\mathbb{Z}_t$. When, as above, sk is picked uniformly in $\mathbb{Z}_q$ it is also easy for a CCA1 adversary to retrieve sk by means of this kind of decryption requests (we describe such an attack in Sect. 8.2.1 below for completeness).

8.2 Folklore CCA1 attacks against Regev

In this subsection, we briefly present two lines of folklore CCA1 attacks against the Regev scheme (FS_0).

8.2.1 A non-adaptive ill-formed ciphertext-based attack

As discussed in Sect. 8.1, let us remark that the the ill-formed ciphertext $(-\Delta\mathbf{1}_i, 0)$ decrypts to $[\mathsf{sk}_i]_t$, where $\mathbf{1}_i$ is the i-th standard basis vector. As a slight simplification,

[33] As Regev is the baseline blueprint for BFV/BGV and TFHE, this is also the case for these schemes.

let us assume that $\Delta = \frac{q}{t} = \frac{2^k}{2^l} = 2^{k-l}$ and asks for the decryption of ill-formed ciphertext $c_l = \left(-\frac{\Delta}{t}\mathbf{1}_i, 0\right) = (-2^{k-2l}\mathbf{1}_i, 0)$ which gives $\left[\left\lceil \frac{\mathsf{sk}_i}{t} \right\rfloor\right]_t$. Since $\frac{\mathsf{sk}_i}{t} = \left\lfloor \frac{\mathsf{sk}_i}{t} \right\rfloor + \frac{[\mathsf{sk}_i]_t}{t}$, two cases can then occur,

- When $\frac{[\mathsf{sk}_i]_t}{t} < \frac{1}{2}$ (recall that we know $[\mathsf{sk}_i]_t$ from above), $\mathsf{FS}_0.\mathsf{Dec}(\mathsf{FS}_0.\mathsf{sk}, c_l) = \left[\left\lceil \frac{\mathsf{sk}_i}{t} \right\rfloor\right]_t = \left[\left\lfloor \frac{\mathsf{sk}_i}{t} \right\rfloor\right]_t$ and the decryption of c_l straightaway gives us bits $l, l+1, ..., 2l-1$ of sk_i.
- When $\frac{[\mathsf{sk}_i]_t}{t} \geq \frac{1}{2}$, $\mathsf{FS}_0.\mathsf{Dec}(\mathsf{FS}_0.\mathsf{sk}, c_l) = \left[\left\lceil \frac{\mathsf{sk}_i}{t} \right\rfloor\right]_t = \left[\left\lfloor \frac{\mathsf{sk}_i}{t} \right\rfloor + 1\right]_t$. So either $\mathsf{FS}_0.\mathsf{Dec}(\mathsf{FS}_0.\mathsf{sk}, c_l) = 0$ and thus $\left[\left\lfloor \frac{\mathsf{sk}_i}{t} \right\rfloor\right]_t = t-1$ or, $\left[\left\lfloor \frac{\mathsf{sk}_i}{t} \right\rfloor\right]_t = \mathsf{FS}_0.\mathsf{Dec}(\mathsf{FS}_0.\mathsf{sk}, c_l) - 1$ otherwise. So $\mathsf{FS}_0.\mathsf{Dec}(\mathsf{FS}_0.\mathsf{sk}, c_l) - 1 \mod t$ gives away bits $l, l+1, ..., 2l-1$ of sk_i.

A CCA1 adversary may then further proceeds similarly by exploiting ill-formed ciphertext $c_{2l} = \left(-\frac{\Delta}{t^2}\mathbf{1}_i, 0\right) = (-2^{k-3l}\mathbf{1}_i, 0)$ and its knowledge of $[\mathsf{sk}_i]_{t^2}$ (from above) to retrieve bits $2l, 2l+1, ..., 3l-1$ of sk_i, and so on and so forth. Let us emphasize that this attack is non-adaptive.

8.2.2 An adaptive noise recovery attack

Another line of folklore attacks follows a dichotomic search pattern (which has been used numerous times, e.g. [LMSV11, CSBB24]) where the adversary starts from a well-formed encryption of 0, $c_0 = (a, b)$, with $b = \langle a, \mathsf{sk}\rangle + e$, to find the critical value α^* such that $\mathsf{Dec}(\mathsf{sk}, a, b + \alpha^*) \neq 0$, i.e. such that $e + \alpha^* = \frac{\Delta}{2}$. The adversary may then conclude that the noise in c_0 is equal to $\frac{\Delta}{2} - \alpha^*$ thereby getting one linear equation in sk. By repeating this (adaptive) process over n encryptions of 0, the adversary can eventually retrieve sk by means of linear algebra techniques.

Remark that this attack does not use the legit add-by-const operator, which only allows to add multiples of Δ to the b-term of an LWE pair.

8.3 Achieving (non-adaptive) CPA^D security (FS_1)

8.3.1 Preliminaries.

The intuition behind our first construction is as follows: we append a clean noise sample of size K to each ciphertext and we use it as a (linearly-updatable) variance monitoring mechanism. To do so, we consider a multi-secret variant of FS_0 with $K+1$ slots, with the first slot carrying a message and the K others being vanilla LWE samples carrying only noise. Each slot is under key material *independent* of that of the others with *the same a vector* used for all slots. Upon decryption, we then use the noises that we recover in the K last slots to put a confidence interval with $1 - 2^{-\lambda}$ probability on the noise variance (we are able to do that with the results in Sect. 7) and use this safe bound to generate an appropriate smudging noise at decryption.

Following Corr. 1 and in the regime where $\lambda \geq 128$, we can use

$$\bar{\sigma}_K^2 = \frac{K\hat{\sigma}^2}{K - 17\sqrt{2K\lambda\log 2}} \tag{31}$$

as a safe upper bound for σ^2 given the sample $E_0, ..., E_{K-1}$ (for $K \geq 2 \cdot 17^2\lambda\log 2 + 1$).

8.3.2 Scheme FS_1.

Let $K \geq 2 \cdot 17^2\lambda\log 2 + 1$, the scheme $\mathsf{FS}_1(\lambda, n, q, t, \sigma_0, K)$ is then defined as follows:

- $\mathsf{FS}_1.\mathsf{KeyGen}$: for $k \in [\![0, K]\!]$, uniformly pick $\mathsf{sk}^{(k)} \in \mathbb{Z}_q^n$.

- $\mathsf{FS}_1.\mathsf{Enc}$: given a plaintext $m \in \mathbb{Z}_t$ and $\mathsf{sk}^{(k)}$, $k \in [\![0, K]\!]$, uniformly pick a *single* $a \in \mathbb{Z}_q^n$ as well as vector $E \in \mathbb{Z}_q^{K+1}$ with each component drawn independently from χ_{σ_0}. Then, return $(a, B) \in \mathbb{Z}_q^n \times \mathbb{Z}_q^{K+1}$ such that

$$B_0 = \langle a, \mathsf{sk}^{(0)} \rangle + \Delta m + E_0.$$

and, for $k \in [\![1, K]\!]$,

$$B_k = \langle a, \mathsf{sk}^{(k)} \rangle + E_k.$$

Note that we will sometimes refer to B_0 as the *payload slot* and to the B_k's ($k \in [\![1, K]\!]$) as the *noise slots*.

- $\mathsf{FS}_1.\mathsf{Dec}$: given ciphertext $c = (a, B) \in \mathbb{Z}_q^n \times \mathbb{Z}_q^{K+1}$ and $\mathsf{sk}^{(k)}$, $k \in [\![0, K]\!]$, first compute

$$\hat{\sigma}^2 = \frac{1}{K} \sum_{k=1}^{K} \left(B_k - \langle a, \mathsf{sk}^{(k)} \rangle \right)^2.$$

Then (following Eq. 31 and Lemma 3), uniformly pick $v \in [\![-B, B]\!]$ with $B = 2^\lambda \bar{\sigma}_K \sqrt{2(\lambda+1)\log 2}$. If $|v| \geq \frac{\Delta}{2}$, return $\perp$. Else, return $\mathsf{FS}_0.\mathsf{Dec}(\mathsf{sk}^{(0)}, a, B_0)$.

It is well-known that such multi-secret variants of the Regev scheme are CPA secure. The linearly homomorphic operators of FS_1 (homomorphic addition and mult-by-const) are trivially defined from those of FS_0.

8.3.3 CPA_0^D security of FS_1.

To prove the CPA^D security of FS_1, we proceed via a reduction to the CPA security of FS_0. In a nutshell, the reduction operates the K noise slots, while the CPA challenger against FS_0 operates the first slot with the message payload. For processing decryption requests (over state indices in CPA^D), the reduction uses the noises it recovers from the noise slots to generate a smudging noise of large-enough variance (with overwhelming probability) to provide outputs that are indistinguishable from those of a true CPA^D decryption oracle. *However, the reduction is valid only in the* non-adaptive *setting (i.e. CPA_0^D, as defined towards the end of Sect. 4.3.1) where the adversary specifies all its queries in advance, straight after the key generation step* [LM21]. As we shall see in the proof of Proposition 11, this restriction is necessary to maintain the independence of the noises which are retrieved from the noise slots[34], an assumption which we require for proper smudging (essentially in order to be able to apply Corr. 1 to obtain a safe bound on the ciphertext noise variance).

Let us first consider the following lemma which essentially states that FS_1's decryption function guarantees correct decryption when it does not return $\perp$.

Lemma 20. *Let $(a, b) \in \mathbb{Z}_q^n \times \mathbb{Z}_q$ be such that $b = \langle a, \mathsf{sk} \rangle + \Delta m + e$ where e is a subgaussian deviate of variance proxy σ^2. Let v be drawn uniformly in $[\![-B, B]\!]$, independently of e, with $B \geq 2^\lambda \bar{\sigma} \sqrt{2(\lambda+1)\log 2}$ and $\bar{\sigma}^2 \geq \sigma^2$. Then, if $|v| < \frac{\Delta}{2}$, with overwhelming probability,*

$$\left\lceil \frac{1}{\Delta} (b - \langle a, \mathsf{sk} \rangle) \right\rfloor = m.$$

Proof. In the conditions of the statement, Lemma 3 applies and $e + v \overset{i}{=} v$. Hence, when $|v| < \frac{\Delta}{2}$, $|e| << \frac{\Delta}{2}$ (with overwhelming probability). The claim follows. □

[34] For example, as soon as an adversary homomorphically computes ciphertext-dependent functions, this iid assumption is jeopardized. This observation will later serve as a basis for an attack against the LOH property of our next scheme (Prop. 13).

Then the following lemma captures the independence of the noises (in both the payload and noise slots) within an evaluated FS_1 ciphertext, as long as the coefficients of the linear combination are independent of the ciphertexts on which it has been applied to produce the said ciphertext.

Lemma 21. *Let $\pi \in \mathbb{Z}_t^L$, let $c^{(0)}, ..., c^{(L-1)}$ be any set of L fresh well-formed FS_1 ciphertexts. Further let*

$$c = (a, B) = \mathsf{FS}_1.\mathsf{Eval}(\texttt{lincomb}_\pi, c^{(0)}, ..., c^{(L-1)}),$$

then the $E_k = B_k - \langle a, \mathsf{sk}^{(k)} \rangle$'s ($k \in [\![0, K]\!]$) are independent and subgaussian.

Proof. The claim follows trivially from the facts that fresh well-formed ciphertexts under FS_1 have this property, that FS_1's homomorphic operators proceed component-wise as well as that π is a priori chosen, *independently* of the $c^{(i)}$'s. □

Remark that E_0 also follows a subgaussian distribution but centered on Δm.

Following this, we now establish the CPA_0^D security of FS_1, i.e. its CPA^D security in the non-adaptive setting.

Proposition 11. *Let $K \geq 2 \cdot 17^2 \lambda \log 2 + 1$, if there exists an adversary $\mathcal{A}$ against the CPA_0^D security of $\mathsf{FS}_1(\lambda, n, q, t, \sigma_0, K)$, then there exists an adversary $\mathcal{B}$ against the (LOR-)CPA[35] security of $\mathsf{FS}_0(\lambda, n, q, t, \sigma_0)$ that achieves the same advantage.*

Proof. Recall that the CPA^D game allows multiple challenge requests. So $\mathcal{B}$ starts by initializing an initially empty state $S := [\,]$ which will contain message-message-ciphertext triplets and by uniformly picking $\mathsf{sk}^{(k)} \in \mathbb{Z}_q^n$, for $k \in [\![1, K]\!]$ (i.e. $\mathsf{sk}^{(0)}$ is in the CPA challenger against FS_0 and the other keys are in the reduction). Then, *since we are in the non-adaptive setting*, $\mathcal{A}$ sends all its requests to the reduction which processes them as follows:

- When processing an *encryption request* for message m, $\mathcal{B}$ first transfers it as is to the CPA challenger getting $(a, b = \langle a, \mathsf{sk}^{(0)} \rangle + \Delta m + e)$, for unknown $\mathsf{sk}^{(0)}$ and e. It then picks $E \in \mathbb{Z}_q^K$ following χ_{σ_0} and constructs (a, B) such that $B_0 = b$ and, for $k \in [\![1, K]\!]$,
$$B_k = \langle a, \mathsf{sk}^{(k)} \rangle + E_{k-1}.$$
After adding it to its internal state by doing $S := [S; (m, m, (a, B))]$, $\mathcal{B}$ returns the ciphertext (a, B) to $\mathcal{A}$.

- When processing a *challenge request* for messages $m_0 \neq m_1$, $\mathcal{B}$ also first transfers it as is to the CPA challenger getting $(a, b = \langle a, \mathsf{sk}^{(0)} \rangle + \Delta m_\gamma + e)$, for unknown $\mathsf{sk}^{(0)}$, challenge bit γ and noise e. It then proceeds as for encryption requests above, however updating its internal state as $S := [S; (m_0, m_1, (a, B))]$.

- When processing an *evaluation request* (wlog broken-down in unitary $\texttt{sum}$ and $\texttt{mul}_\alpha$ homomorphic operations),
  - $(\texttt{eval}, \texttt{sum}, i, j)$: $\mathcal{B}$ creates a new *evaluated* ciphertext
  $$c = \mathsf{FS}_1.\mathsf{Eval}(\texttt{sum}, S[i].c, S[j].c),$$
  and returns c to $\mathcal{A}$ after updating its internal state as
  $$S := [S; (S[i].m_0 + S[j].m_0, S[i].m_1 + S[j].m_1, c)].$$

[35] Where (LOR-)CPA is the multiple challenge variant of (IND-)CPA, or (FTG-)CPA in the terminology of [BDJR97, BDPR98] which show the equivalence between the two notions.

– $(\texttt{eval}, \texttt{mul}_\alpha, i)$: $\mathcal{B}$ similarly creates a new *evaluated* ciphertext,

$$c = \mathsf{FS}_1.\mathsf{Eval}(\texttt{mul}_\alpha, S[i].c),$$

and returns c to $\mathcal{A}$ after updating its internal state as

$$S \coloneqq [S; (\alpha S[i].m_0, \alpha S[i].m_1, c)].$$

- Lastly, for processing a decryption request with state index i, $\mathcal{B}$ returns $\perp$ if $S[i].m_0 \neq S[i].m_1$. Otherwise, i.e. when $S[i].m_0 = S[i].m_1$, it first computes

$$\hat{\sigma}^2 = \frac{1}{K}\sum_{k=1}^{K}\left(B_k - \langle a, \mathsf{sk}^{(k)}\rangle\right)^2. \tag{32}$$

Then, it uniformly picks $v \in [\![-B, B]\!]$ with $B = 2^\lambda \bar{\sigma}_K \sqrt{2(\lambda+1)\log 2}$. Finally, it returns $\perp$ if $|v| \geq \frac{\Delta}{2}$, and $S[i].m_0$ (or, equivalently, $S[i].m_1$) otherwise.

The key point which makes the reduction works is that $\mathcal{B}$'s replies to $\mathcal{A}$'s decryption requests are indistinguishable from a true FS_1 decryption oracle. Let e be the (unknown to $\mathcal{B}$) noise in the first payload slot of $S[i].c$ and σ^2 denotes its variance. Since we are in the non-adaptive setting, Lemma 21 applies (i.e. the $B_k - \langle a, \mathsf{sk}^{(k)}\rangle$'s in Eq. 32 are iid). Therefore we can use Corr. 1 (and Eq. 31) to claim that $\bar{\sigma}_K^2 \geq \sigma^2$ (with overwhelming probability). Then, since $\bar{\sigma}_K^2$ is independent from e, Lemma 20 applies. It thus follows that, in the reduction's processing of decryption requests, $\mathsf{FS}_0.\mathsf{Dec}(\mathsf{FS}_0.\mathsf{sk}, S[i].c.a, S[i].c.B_0) = S[i].m_0$ (or, equivalently, $S[i].m_1$) whenever $|v| < \frac{\Delta}{2}$ (with overwhelming probability). □

Remark that, stricto sensu, the noises in the noise slots of an FS_1 ciphertext could even be given in cleartext form since, *in the non-adaptive private-key setting* under which the above proof operates, the adversary eventually specifies all of her requests before seeing any ciphertexts. However, in our next construction, built on FS_1, we will also use the noise slots to store additional private information. Hence, it will not be desirable to keep the content of the noise slots in cleartext.

Because it is using a ciphertext-dependent variance estimation procedure, FS_1 falls in a category of schemes, referred to as Dynamic Error Estimation-based schemes, introduced and studied in Sect. 5 of [LMSS22] in the context of CKKS. In a nutshell, that paper presents an approach (which the authors of that paper attribute to Y. Polyakov) leveraging on a special message encoding which fixes many of the coordinates of CKKS message space to be 0 and to use these to obtain, at decryption, an estimation of the noise variance for the ciphertext and to use this estimation to set the variance of their noise flooding mechanism (which essentially is equivalent to the noise smudging mechanism initially introduced in [AJL+12] for threshold FHE). Then, still in the context of CKKS, the authors of [LMSS22] present CPA^D attacks on a "natural" class of such schemes and leave open the following problem: "While our results on "dynamic" error estimation are negative, we have not ruled out achieving some weaker security notion with these techniques (for natural schemes)." In essence, the previous proposition is a positive answer to this open problem, with the weaker security notion being non-adaptive CPA^D security as defined in [LM21]. Let us also emphasize that FS_1 departs slightly from the blueprint depicted in [LMSS22], in that we are using the dynamic variance estimation to obtain a safe-bound on the true variance which leads to a correct variance for the smudging noise (with overwhelming probability). Let us also emphasize that the attacks in [LMSS22] do not apply to FS_1 as they leverage on the noise/message dependencies which naturally arise in CKKS (and other schemes such as BGV and BFV) when performing homomorphic multiplications. In contrast, the noises in an FS_1 ciphertext are message-independent (and remain so under the linear homomorphic operators).

8.4 Achieving "non-adaptive" LOH (FS'_1)

Let us emphasize that the schemes considered in this section are not new, to the exception of the dynamic estimation based smudging technique that we use in the decryption function of the FS'_1 scheme below.

8.4.1 $\mathsf{FS}_0^{(K)}$ and FS'_1.

We now consider the usual multi-secret variant of FS_0, where K messages are encrypted by ciphertexts under K instances of FS_0 with *independent* key material but *using the same a vector*. Then $\mathsf{FS}_0^{(K)}(\lambda, n, q, t, \sigma_0)$ is defined as follows:

- $\mathsf{FS}_0^{(K)}.\mathsf{KeyGen}$: for $k \in [\![0, K-1]\!]$, uniformly pick $\mathsf{sk}^{(k)} \in \mathbb{Z}_q^n$.
- $\mathsf{FS}_0^{(K)}.\mathsf{Enc}$: given plaintext $M \in \mathbb{Z}_t^K$ and $\mathsf{sk}^{(k)}$, $k \in [\![0, K-1]\!]$, uniformly pick a *single* $a \in \mathbb{Z}_q^n$ as well as vector $E \in \mathbb{Z}_q^K$ with each component drawn independently from χ_{σ_0}. Then, return $(a, B) \in \mathbb{Z}_q^n \times \mathbb{Z}_q^K$ such that for $k \in [\![0, K-1]\!]$
$$B_k = \langle a, \mathsf{sk}^{(k)} \rangle + \Delta M_k + E_k.$$
- $\mathsf{FS}_0^{(K)}.\mathsf{Dec}$: given ciphertext $(a, B) \in \mathbb{Z}_q^{n+K}$ and $\mathsf{sk}^{(k)}$, $k \in [\![0, K-1]\!]$, let
$$\mu_k = \mathsf{FS}_0.\mathsf{Dec}(\mathsf{sk}^{(k)}, a, B_k),$$
for $k \in [\![0, K-1]\!]$. Then return $(\mu_0, ..., \mu_{K-1})$.

We now define a variant of $\mathsf{FS}_0^{(K+1)}$, FS'_1, with built-in additional verifications following the Knowledge-of-Exponent pattern as well as smudging in the decryption function.

Recall Corr. 1 and let

$$K \geq \max\left\{\left\lceil \frac{\lambda}{\log_2 t} \right\rceil, 2 \cdot 17^2 \lambda \log 2 + 1\right\}. \tag{33}$$

We then build FS'_1 from $\mathsf{FS}_0^{(K+1)}$ as follows:

- $\mathsf{FS}'_1.\mathsf{KeyGen}$: run $\mathsf{FS}_0^{(K+1)}.\mathsf{KeyGen}$ and let $\xi_1, ..., \xi_K$ be *secret multipliers* uniformly picked in $\mathbb{Z}_t^*$.
- $\mathsf{FS}'_1.\mathsf{Enc}$: given a plaintext $m \in \mathbb{Z}_t$ and $\mathsf{sk}^{(k)}$, $k \in [\![0, K]\!]$, return
$$(a, B) = \mathsf{FS}_0^{(K+1)}.\mathsf{Enc}(\{\mathsf{sk}^{(k)}\}_{k \in [\![0,K]\!]}, m, [\xi_1 m]_t, ..., [\xi_K m]_t).$$
- $\mathsf{FS}'_1.\mathsf{ImVer}$: given ciphertext $c = (a, B) \in \mathbb{Z}_q^n \times \mathbb{Z}_q^{K+1}$ and $\mathsf{sk}^{(k)}$, $k \in [\![0, K]\!]$, let
$$(\mu_0, \mu_1, ..., \mu_K) = \mathsf{FS}_0^{(K+1)}.\mathsf{Dec}(\{\mathsf{sk}^{(k)}\}_{k \in [\![0,K]\!]}, c).$$
When[36] $a = \vec{0}$ or if
$$\exists k \in [\![1, K]\!], [\xi_k \mu_0]_t \neq \mu_k, \tag{34}$$
then return False. Otherwise (i.e. when $a \neq \vec{0}$ and $\forall k \in [\![1, K]\!], [\xi_k \mu_0]_t = \mu_k$), let
$$\varepsilon_k = B_k - \langle a, \mathsf{sk}^{(k)} \rangle - \Delta \mu_k,$$

[36] This is to eliminate a corner case in a later security proof.

and compute $\hat{\sigma}^2 = \frac{1}{K}\sum_{i=1}^{K}\varepsilon_i^2$ *(remark that* $\hat{\sigma}^2$ *is computed over the* K *last slots in order to preserve independence between* $\hat{\sigma}^2$ *and* ε_0*, when all the* ε_k*'s,* $k \in [\![0, K]\!]$*, are independent).* Then, pick $v \in \mathbb{Z}_q$ following a centered discrete Gaussian distribution of variance (following Eq. 31 and Lemma 6).

$$\sigma^2_{\mathrm{smg}} = \frac{2^{2\lambda}\bar{\sigma}_K^2(\lambda+1)\log 2}{\pi}. \tag{35}$$

Finally, return False when $|v| \geq \frac{\Delta}{2}$ and True otherwise.

- $\mathsf{FS}'_1.\mathsf{Dec}$: given ciphertext $c = (a, B) \in \mathbb{Z}_q^{K+1}$ and $\mathsf{sk}^{(k)}$, $k \in [\![0, K]\!]$, if $\mathsf{FS}'_1.\mathsf{ImVer}(\{\mathsf{sk}^{(k)}\}_{k\in[\![0,K]\!]}, c) = \mathsf{False}$, then return $\perp$. Otherwise, let $(\mu_0, \mu_1, ..., \mu_K) = \mathsf{FS}_0^{(K+1)}.\mathsf{Dec}(\{\mathsf{sk}^{(k)}\}_{k\in[\![0,K]\!]}, c)$ and return μ_0.

We emphasize that FS'_1 has no direct add-by-const operator, since the multipliers are private.

8.4.2 CPA_0^D security of FS'_1.

To prove the CPA_0^D security of FS'_1 we now relate it to the FS_1 scheme that we have studied in the previous section. Indeed, in FS'_1 the K last slots serve the double purpose of enforcing ciphertext verification as well as of estimating the noise variance for smudging the first (payload) slot. Lemma 22 essentially shows that the two schemes are equivalent over well-formed ciphertexts. This will then allow us to prove the CPA_0^D security of FS'_1 based on the CPA_0^D security of FS_1 which we have previously established.

The following lemma implies that FS'_1's decryption function admits an alternate version which "knows the noise".

Lemma 22. *Let* $\mathsf{ct} = (a, B) \in \mathbb{Z}_q^n \times \mathbb{Z}_q^{K+1}$ *denote a well-formed* FS'_1 *ciphertext, i.e. such that for* $k \in [\![0, K]\!]$ *(*$\xi_0 = 1$*),*

$$B_k = \langle a, \mathsf{sk}^{(k)}\rangle + \Delta\xi_k m + e_k, \tag{36}$$

and the e_k*'s are iid and subgaussian. Define,*

$$\widehat{\mathsf{FS}'_1.\mathsf{Dec}}(\{\mathsf{sk}^{(k)}\}_{k\in[\![0,K]\!]}, a, B) = \mathsf{FS}_1.\mathsf{Dec}(\{\mathsf{sk}^{(k)}\}_{k\in[\![0,K]\!]}, a, (B_0, B_1 - \Delta\xi_1 m, ..., B_K - \Delta\xi_K m)).$$

Then the outputs of $\widehat{\mathsf{FS}'_1.\mathsf{Dec}}$ *are indistinguishable from those of* $\mathsf{FS}'_1.\mathsf{Dec}$*.*

Proof. Let $V[e_k] = \sigma^2_{\mathrm{ct}}$.

Case 1 (correct $\mathsf{FS}_0^{(K+1)}$ *decryption).* In this case, condition (34) is satisfied and, therefore, FS'_1's decryption function proceeds, following Eq. (35), with smudging based on $\bar{\sigma}_K^2 \geq \sigma^2_{\mathrm{ct}}$ (with overwhelming probability, from Corr. 1 and Eq. (31)). Hence, FS'_1's decryption function proceeds as $\widehat{\mathsf{FS}'_1.\mathsf{Dec}}$.

Case 2 (incorrect $\mathsf{FS}_0^{(K+1)}$ *decryption #1).* We now consider the case where a decryption error occurs *only* in the first slot, i.e. $\mathsf{FS}_0.\mathsf{Dec}(\mathsf{sk}^{(0)}, a, B_0) \neq m$. On the one hand, condition (34) is not satisfied and $\mathsf{FS}'_1.\mathsf{Dec}(\mathsf{FS}'_1.\mathsf{sk}, c) = \perp$. On the other hand, $\widehat{\mathsf{FS}'_1.\mathsf{Dec}}$ proceeds following Eq. (35), with smudging based on $\bar{\sigma}_K^2 \geq \sigma^2_{\mathrm{ct}}$ (with overwhelming probability, from Corr. 1 and Eq. 31). As a consequence, it picks a smudging noise ν such that $e_0 + v \overset{i}{=} v$ i.e. such that $|v| >> |e_0| \geq \frac{\Delta}{2}$ (with overwhelming probability). Hence, $\widehat{\mathsf{FS}'_1.\mathsf{Dec}}$ also returns $\perp$ (with overwhelming probability).

Case 3 (incorrect $\mathsf{FS}_0^{(K+1)}$ *decryption #2).* Lastly we consider the case where a decryption error occurs in at least one of the K last slots (and possibly also in the first one)

i.e. $\exists k \in [\![1, K]\!] : \mathsf{FS}_0.\mathsf{Dec}(\mathsf{sk}^{(k)}, a, B_k) \neq \xi_k m$. On the one hand, condition (34) is again not satisfied and $\mathsf{FS}'_1.\mathsf{Dec}(\mathsf{FS}'_1.\mathsf{sk}, c) = \perp$. However, $\widehat{\mathsf{FS}'_1.\mathsf{Dec}}$ (which eventually knows the e_k's) proceeds by smudging as in the previous cases. Since an error occurs in at least one of the K last slots, we have $\|(e_1 \ldots e_K)\|_\infty \geq \frac{\Delta}{2}$, hence $\hat{\sigma}^2 \geq \frac{\Delta^2}{4K}$, and (Eq. 31),

$$\bar{\sigma}_K^2 = \frac{K\hat{\sigma}^2}{K - 17\sqrt{2K\lambda\log 2}} \geq \frac{\Delta^2}{4(K - 17\sqrt{2K\lambda\log 2})}.$$

Thus, the discrete Gaussian smudging in Lemma 6, leads to smudge with a discrete Gaussian noise having variance

$$\sigma^2_{\mathrm{smg}} = \frac{(\lambda+1)2^{2\lambda}\bar{\sigma}_K^2 \log 2}{\pi} \geq \frac{(\lambda+1)2^{2\lambda}\Delta^2 \log 2}{4\pi(K - 17\sqrt{2K\lambda\log 2})}. \tag{37}$$

Consider the following bound for the discrete Gaussian: $P(|X| \leq a) \leq \frac{2a}{Z} \leq \frac{2a}{\sqrt{2\pi}\sigma}$ (since $Z > \sqrt{2\pi}\sigma$). Then, plugging (37) in that bound yields,

$$\begin{aligned} P\left(|v| \leq \frac{\Delta}{2}\right) &\leq \frac{\Delta}{\sqrt{2\pi}}\sqrt{\frac{4\pi(K - 17\sqrt{2K\lambda\log 2})}{(\lambda+1)2^{2\lambda}\Delta^2\log 2}}, \\ &\leq \frac{\sqrt{K - 17\sqrt{2K\lambda\log 2}}}{2^{\lambda-1}\sqrt{2(\lambda+1)\log 2}} \\ &\leq \mathrm{neg}(\lambda), \end{aligned}$$

as long as $K \leq O(\mathrm{poly}(\lambda))$. Therefore, $\widehat{\mathsf{FS}'_1.\mathsf{Dec}}$ also replies $\perp$ with overwhelming probability. □

Remark that Lemma 22 above tells us that FS'_1's smudging is equivalent to that of FS_1 (with the discrete Gaussian smudging of Lemma 6 replacing the uniform smudging of Lemma 3), as $\widehat{\mathsf{FS}'_1.\mathsf{Dec}}$ is exactly $\mathsf{FS}_1.\mathsf{Dec}$. As the next proposition establishes, the (non-adaptive) CPA^D security of FS'_1 then follows from that of FS_1 (Prop. 11).

Proposition 12. *Let $K \geq 2 \cdot 17^2\lambda\log 2 + 1$, FS'_1 is CPA_0^D-secure.*

Proof. Let G_0 denote the CPA_0^D game against FS'_1 and G_1 the CPA_0^D game against FS_1. The claim follows from the fact that the two games cannot be distinguished since, for both schemes, well-formed ciphertexts are indistinguishable from uniform over $\mathbb{Z}_q^{n+K+1}$ (from the LWE assumption) and from Lemma 22 which tells that over well-formed ciphertexts, $\mathsf{FS}'_1.\mathsf{Dec}$'s behavior in G_0 is indistinguishable from that of $\mathsf{FS}_1.\mathsf{Dec}$ in G_1. □

8.4.3 FS'_1 and LOH.

Several works since [BCI+13], e.g. [BISW17, GMNO18, Nit19], have consistently made the assumption that schemes like FS'_1 satisfy a *weaker* variant of the LOH property with the constraint that the L^2-norm of the linear combinations that the adversary may apply is bounded[37] by $\frac{q\sqrt{\pi}}{2t\sigma_0\sqrt{(\lambda+1)\log 2}}$, a condition under which the scheme achieves (statistical) correctness.

Without such kind of restrictions, FS'_1 does not satisfy the LOH property as the next proposition establishes.

[37] This follows from the Banaszczyk bound ([LP11, Lemma 2.2] and [Ban95]) which states that given a vector x of iid discrete Gaussian deviates with variance σ_0^2, any $T \in \mathbb{R}^+$ and any $a \in \mathbb{R}^L$ we have that $P(|\langle x, a\rangle| \geq T\sigma_0\|a\|) \leq 2e^{-\pi T^2}$. Equating $2e^{-\pi T^2} = 2^{-\lambda}$ then yields $T = \sqrt{\frac{(\lambda+1)\log 2}{\pi}}$. A sufficient condition for correct decryption of a ciphertext of the form $c = \mathsf{Eval}(\mathtt{lincomb}_a, c_0, \ldots, c_{L-1})$, where $c_0, \ldots, c_{L-1}$ denotes L fresh well-formed ciphertexts LWE ciphertexts, is thus that $T\sigma_0\|a\| < \frac{q}{2t}$, i.e. $q > 2tT\sigma_0\|a\|$.

Proposition 13. *FS'_1 does not have the LOH property.*

Proof. Recall the notations in Def. 7 (p. 15) and consider that the set of messages $(a_0, ..., a_{m-1})$ in that definition is $(1, 0, ..., 0)$. Hence c_0 is an encryption of 1, when $c_1, ..., c_{m-1}$ are all encryptions of 0. However, as soon as $m \geq n + K + 2$ is sufficiently large that $c_1, ..., c_{m-1}$ forms a generating set of vectors for $\mathbb{Z}_q^{n+K+1}$ with non-negligible probability, the adversary can find a linear combination of the $c_1, ..., c_{m-1}$ with coefficients in $\mathbb{Z}_q$ *thus depending on the c_j's* such that,

$$c'_0 = \sum_{j=1}^{m-1} \alpha_j(c_0, ..., c_{m-1})c_j = c_0,$$

Then, since c_0 is a fresh well-formed encryption of 1 under FS'_1, we have that $\mathsf{FS}'_1.\mathsf{ImVer}(\mathsf{FS}'_1.\mathsf{sk}, c'_0) = \mathsf{FS}'_1.\mathsf{ImVer}(\mathsf{FS}'_1.\mathsf{sk}, c_0) = \mathsf{True}$ as well as $\mathsf{FS}'_1.\mathsf{Dec}(\mathsf{FS}'_1.\mathsf{sk}, c'_0) = \mathsf{FS}'_1.\mathsf{Dec}(\mathsf{FS}'_1.\mathsf{sk}, c_0) = 1$ (with high probability). However, since c'_0 has been obtained by linearly combining encryptions of 0 we have $a'_0 = \Pi \cdot (a_0 \ ... \ a_{m-1})^T = 0 \neq \mathsf{FS}'_1.\mathsf{Dec}(\mathsf{FS}'_1.\mathsf{sk}, c'_0)$, in violation of the LOH property. □

Interestingly, this attack strategy also yields a new CPA^D attack path on the vanilla Regev scheme, its RLWE variant and the other mainstream FHE schemes which are based on them. We describe such an attack in Sect. 8.4.4 below.

Now, recall that we have done our CPA^D security proofs in the non-adaptive private-key setting in which the adversary specifies all its requests under the drastic restriction that it has not yet seen any ciphertext. In this setting, the previous attack is not applicable and only a degenerate variant of the LOH property with $m = 0$ is relevant. We refer to this (rather weak) variant as LOH$_0$[38] and now assume that FS'_1 satisfies it.

Assumption 1. *Let $K \geq \left\lceil \frac{\lambda}{\log_2 t} \right\rceil$, FS'_1 satisfies the LOH$_0$ property.*

As discussed in Sect. 4.5, the LOH property requires a deterministic decryption function and FS'_1's is not. However, this does not disrupt LOH$_0$. Indeed, in the private-key setting of FS'_1, remark that the only valid ciphertexts that a LOH$_0$ adversary could a priori create on its own would be "trivial" encryptions of 0 of the form $(\vec{0}, E)$, where E is a $K + 1$ dimensional vector such that $\|E\|_\infty < \frac{\Delta}{2}$ (as, indeed, such a ciphertext decrypts to $\vec{0}$ under $\mathsf{FS}_0^{(K+1)}$ and, as $\vec{0}$ satisfies condition 34, could decrypt to 0 under FS'_1 depending on smudging). However, this corner case is not an issue, as in the specification of our scheme, $\mathsf{FS}'_1.\mathsf{ImVer}$ returns False whenever $a = \vec{0}$. It follows that, for FS'_1, the LOH$_0$ property tells that an adversary can ex nihilo create on its own a ciphertext c such that $\mathsf{FS}'_1.\mathsf{ImVer}(\mathsf{FS}'_1.\mathsf{sk}, c) = \mathsf{True}$ only with negligible probability.

We now show that FS'_1 achieves CCA0 security, as defined in Sect. 4.3.2.

Proposition 14. *FS'_1 is CCA0 secure under the assumption that it has the LOH$_0$ property.*

Proof. As just discussed, for FS'_1, the LOH$_0$ property tells that an adversary can create on its own a ciphertext c such that $\mathsf{FS}'_1.\mathsf{ImVer}(\mathsf{FS}'_1.\mathsf{sk}, c) = \mathsf{True}$ only with negligible probability. In the non-adaptive setting of CCA0 where the adversary specifies all its decryption requests before seeing any ciphertexts, the claim thus follows by a straightforward reduction to the CPA security of FS'_1. The reduction simply forwards encryption and challenge requests as is to its CPA challenger. Lastly, the reduction handles a decryption request over a ciphertext c by just returning $\perp$ unconditionally. □

[38] In the *public-key* case, it turns out that LOH$_0$ coincides with a "non-adaptive" variant of PA1 [BP04d] in which the ciphertext creator has to specify all its (decryption) oracle requests straight after seeing the public material of the scheme. Then, recall Eq. (7), the (non-adaptive) PA1 plaintext extractor may be obtained from the LOH extractor by doing $(\pi, \beta) = \mathsf{extract}(c, \mathsf{aux})$ and returning β (as $\pi = 0$ in the case of LOH$_0$).

So, eventually, the LOH_0 property earned us a little CCA security increment. At least sufficient to thwart (non-adaptive) CCA1 attacks such as the one we discussed at the end of Sect. 8.1 (see also Sect. 8.2.1).

As a last remark, note that FS'_1 can essentially be abstracted out of the proof of the previous proposition which thus establishes that, *in the private-key setting*, CPA+LOH_0 implies CCA0[39] (hence, with FS'_1 as a heuristic construction of an LWE-based LHE scheme that plausibly satisfies LOH_0). However, in the public-key setting, we already saw that LOH_0 corresponds to a non-adaptive variant of PA1 [BP04d] and the reduction in the proof of the above proposition does not work as is: since, in that latter setting, the adversary can build valid ciphertexts on its own, replying $\perp$ to every decryption request is not appropriate. We leave it as an open question whether the previous implication also holds in the public-key setting.

8.4.4 Yet another CPA^D attack path on "exact" FHE

Interestingly, in the wake of [CSBB24, CCP+24], the attack in Proposition 13 gives us yet another attack path on the vanilla Regev scheme, its RLWE variant and the other mainstream FHE schemes which are based on them.

For simplicity sake, we state the attack against vanilla RLWE encryption with a binary secret key $\mathsf{sk} \in \{0,1\}^n$. In such a condition, remark that the ill-formed ciphertext $(-\Delta \cdot 1, 0)$ decrypts to sk (similarly to the attack in Sect. 8.2.1). Let $c^{\triangle}$ denotes this ciphertext.

The CPA^D attack then goes as follows:

- The adversary, say $\mathcal{A}$, requests encryptions of 0, until it gets a generating set of vectors for[40] $\mathbb{Z}_q^{2n}$. Let $c_0, ..., c_{N-1}$ denotes these ciphertexts.

- Using elementary linear algebra, the adversary then finds a linear combination with coefficients in $\mathbb{Z}_q$ such that,
$$c^{\triangle} = \sum_{i=0}^{N-1} \pi_i c_i,$$
with $\pi \in \mathbb{Z}_q^N$.

- For $i \in [\![0, N-1]\!]$, the adversary then builds ciphertext $c_i^{\triangle}$ with the following legit homomorphic operations, which it performs by means of CPA^D game evaluation oracle calls over the appropriate indices. Assuming $t > 2$, let $k = \lceil \log_{t-1} \pi_i \rceil$ and let the $\pi_{i,j}$'s denote the k digits in the base $(t-1)$ decomposition of π_i, i.e.
$$\pi_i = \sum_{j=0}^{k-1} \pi_{i,j} (t-1)^j.$$
Then $\mathcal{A}$ performs,
$$c_{i,j}^{\triangle} = \pi_{i,j} \otimes \underbrace{(t-1) \otimes ... \otimes (t-1)}_{j \text{ times}} \otimes c_i,$$
where $\alpha \otimes c$ denotes $\mathsf{Eval}(\mathrm{mul}_\alpha, c)$, $\alpha \in \mathbb{Z}_t$, and,
$$c_i^{\triangle} = \mathsf{Eval}(\mathrm{sum}, c_{i,0}^{\triangle}, ..., c_{i,k-1}^{\triangle}).$$

[39] As a consequence, CCA0 does not imply CPA_0^D. For the same reasons that CPA^D is independent from CCA1 [BCF+25], CPA_0^D is independent of CCA0 although it may be desirable that a scheme achieves both security notions. For example, FS'_1 without smudging in its decryption function would achieve CCA0 but not CPA_0^D, when full-blown FS'_1 achieves both notions.

[40] When $q \geq O(2^\lambda)$ is prime, $2n$ encryptions of 0 are enough (with overwhelming probability).

- Following this, by an evaluation request of the form $\mathsf{Eval}(\mathrm{sum}, c_0^{\triangle}, ..., c_{N-1}^{\triangle})$, the adversary eventually puts $c^{\triangle}$ in the internal state of its CPA^D challenger in association to 0 left and right cleartext evaluations (since the c_i's are all encryptions of 0).
- Finally, a single decryption request on state index $|S| - 1$, where S is the internal state of the CPA^D challenger, gives sk to $\mathcal{A}$.

Remark that the above attack also shows that, unless the number of ciphertexts it has access to is suitably restricted, a CPA^D adversary against the vanilla RLWE scheme can get *any* ciphertext into the internal state of a CPA^D challenger and, as such, has as much power as a CCA adversary (in the special case of these LWE-based schemes).

This attack is of independent interest and has a number of far reaching consequences that we address in a follow up work [CCRS26].

8.5 Achieving vCCA^D security, under the conjectured (adaptive) CPA^D security of FS'_1 (and FS_1)

We are now left with the question asking how far can we go in the adaptive setting with our LWE-based line of schemes? If we conjecture that FS'_1 is (adaptive) CPA^D secure, then it still does not satisfy the LOH assumption as the attack in the proof of Prop. 13 still works against it. However, that attack works under the assumption that $m \geq n + K + 1$ (recall Def. 7 notations) and similarly so for the adaptive CPA^D attack in Sect. 8.4.4. It turns out that the (adaptive) vCCA^D security of FS'_1 can be established under the assumption that the adversary has access to only $m < n + K + 1$ ciphertexts, so under the conjecture that it is (adaptive) CPA^D secure and the assumption that it has the LOH property (in this same restrictive setting for both). To do so, we however have to associate a linearly homomorphic hash to FS'_1 ciphertexts in order to deal with yet another corner case involving trivial encryptions of 0. This path is the one pursued in this section.

8.5.1 The ℓ-LOH property

Even if we are willing to conjecture that FS'_1 is (adaptive) CPA^D secure, then it still does not satisfy the LOH assumption as the attack in the proof of Proposition 13 still works against it. However, that attack works under the assumption that $m \geq n + K + 1$ and similarly so for the adaptive CPA^D attack in Sect. 8.4.4. In this section, we thus investigate the restricted case where the adversary only has access to $m < n + K + 1$ ciphertexts. Indeed, let $N = n + K + 1$ and assume that the adversary has access to only $0 \leq \ell < N$ randomly chosen ciphertexts. Then the probability for an arbitrary ciphertext to be in the span of these ℓ ciphertexts is upper bounded by[41] $\frac{q^\ell}{q^N} = q^{\ell - N}$. Equating $q^{\ell-N} = 2^{-\lambda}$ then yields $\ell = N - \lambda \frac{\log 2}{\log q}$. *We refer to the LOH property in this restricted setting as ℓ-LOH.* We thus now attempt to achieve (adaptive) ℓ-vCCA^D security in the private-key setting (i.e. vCCA^D security where the adversary can cumulate no more than $\ell \leq N - \lambda \frac{\log 2}{\log q}$ encryption or challenge oracle calls). Further remark that since $\log_2 q$ generally is in $O(\lambda)$, we have $\ell \leq N - O(1)$.

Following this, there remain two issues to deal with.

The first is that FS'_1's decryption function is not deterministic (as discussed in Sect. 4.5, the LOH property requires deterministic decryption). This however can be dealt with by considering a variant of FS'_1 (or FS_1 in the rest of this Sect.) with *deterministic* decryption, e.g. by generating the smudging noise by means of a keyed PRF seeded on $H(c)$ for some hash function H). When this is so, our CPA$^D_0$ security proofs (Prop. 11 and 12 as well as Lemma 22) remain valid "only" at the cost of modelling H as a Random Oracle.

[41] The bound is tight when $\mathbb{Z}_q$ is a field [Bro92].

The last issue is as follows. Let us consider a trivial encryption of 0 under $\mathsf{FS}_0^{(K+1)}$ of the form $(\vec{0}_n, E)$ where E is a $K+1$ dimensional vector such that $0 < \|E\|_\infty < \frac{\Delta}{2}$. Recall that $\mathsf{FS}'_1.\mathsf{ImVer}(\mathsf{FS}'_1.\mathsf{sk}, (\vec{0}_n, E)) = \mathsf{False}$ (as $\mathsf{FS}'_1.\mathsf{ImVer}(\mathsf{FS}'_1.\mathsf{sk}, (a, B))$ returns False whenever $a = 0$). Because of this, a ciphertext of the form

$$c'_0 = \mathsf{FS}'_1.\mathsf{Eval}(\mathtt{lincomb}_\pi, c_0, ..., c_{\ell-1}) + (\vec{0}_n, E), \tag{38}$$

where $c_0, ..., c_{\ell-1}$ are the fresh well-formed ciphertexts available to the ℓ-LOH adversary, should yield $\mathsf{extract}(c'_0) = \varnothing$ and $a_0 = \perp$ (in Def. 7's notations, p. 15). However, when $\|E\|_\infty << \frac{\Delta}{2}$, $\mathsf{FS}'_1.\mathsf{ImVer}(\mathsf{FS}'_1.\mathsf{sk}, c'_0) = \mathsf{True}$ and $\mathsf{FS}'_1.\mathsf{Dec}(\mathsf{FS}'_1.\mathsf{sk}, c'_0) \neq \perp$, with high probability. We finally deal with this last issue by associating a tag to FS'_1 ciphertexts obtained by means of a collision-resistant (keyed) hash function, thus preventing a ℓ-LOH adversary to build ciphertexts of the form (38) on its own (as, as just argued, it can get a valid hash for vector $(\vec{0}_n, E)$ only with negligible probability when $\ell \leq N - \lambda\frac{\log 2}{\log q}$). Interestingly, this also makes our next scheme immune to the folklore (adaptive) CCA1 attack in Sect. 8.2.2 (under the same restriction).

Lastly, we emphasize that the conjecture that FS'_1 is (adaptive) ℓ-CPA^D secure (i.e. when only $\ell < n + K + 1$ ciphertexts are available to the adversary) is not so far disproved by any attacks, indeed, to the best of our knowledge:

- The attacks in [LMSS22] do not apply to FS'_1 as they leverage on the noise/message dependencies which naturally arise in CKKS, and other schemes such as BGV and BFV, when performing homomorphic multiplications. In contrast, the noises in a FS'_1 ciphertext are message independent.

- Among the CPA^D attacks in [CCP+24], only the *non-adaptive* attack in Sect. 4 of that paper would be applicable to FS'_1. However, since it is non-adaptive, the CPA$^D_0$ security of FS'_1 implies that it is not subject to it. The others attacks in [CCP+24] either need multiplications or bootstrappings.

- The adaptive LWE noise recovery dichotomic attack in [CSBB24] manipulates only ciphertexts of the form $\alpha \cdot c_0$ (with c_0 a well-formed fresh encryption of 0) and is heuristically thwarted by FS'_1's smudging (under the assumption of the independence of the E_k's, in the notations of Lemma 21, which is not stricto sensu true since the α coefficient choice during the dichotomy depends on outputs of the CPA^D decryption oracle which depend on the E_k's. However we conjecture that this dependence is benign when the adversary is restricted to evaluate single-coefficient linear combinations as in the latter attack).

- Lastly, FS'_1 is *provably* immune against the attack we present in Sect. 8.4.4 as soon as the adversary is restricted to see less than $N - \lambda\frac{\log 2}{\log q}$ ciphertexts (an assumption that we explicitly make in the rest of this Sect.).

8.5.2 Linearly homomorphic hash functions

The last tool we need is thus a *collision-resistant* keyed linearly homomorphic one-way hash function

$$\hat{H} = (\mathsf{KeyGen}, \mathsf{Digest}, \mathsf{Add}, \mathsf{Mulc}),$$

with $\hat{H}.\mathsf{Digest}(\hat{H}.\mathsf{sk}, .) : \mathbb{Z}_q^n \times \mathbb{Z}_q^{K+1} \longrightarrow \mathcal{H}$, $\hat{H}.\mathsf{Add} : \mathcal{H}^2 \longrightarrow \mathcal{H}$ and $\hat{H}.\mathsf{Mulc} : \mathcal{H} \times \mathbb{Z}_q \longrightarrow \mathcal{H}$ such that, with the convention that $\oplus \equiv \hat{H}.\mathsf{Add}$ and $\odot \equiv \hat{H}.\mathsf{Mulc}$, the following property holds,

$$\hat{H}.\mathsf{Digest}\left(\hat{H}.\mathsf{sk}, \sum_{l=0}^{L-1} \pi_l c_l\right) = \bigoplus_{l=0}^{L-1} \pi_l \odot \hat{H}.\mathsf{Digest}(\hat{H}.\mathsf{sk}, c_l).$$

Although the construction can be instantiated from any linearly homomorphic hash function (with domain $\mathbb{Z}_q^N$), there are only a very limited number of candidate constructions in the state-of-the-art. As a concrete example, we can use the Fiore-Gennaro-Pastro hash function introduced in [FGP14a] ([FGP14b], Sect. 4.2) in order to build several VC schemes on top of the BV scheme [BV11]. We give the hash function details in Sect. A for completion.

8.5.3 Scheme FS_2'

Following the discussion just above, we now consider an additional scheme, FS_2', which we build from a variant of FS_1' with deterministic decryption (Sect. 8.5.1), by further onboarding hash function $\hat{H}$ in the following Encrypt-then-Hash fashion:

- $\mathsf{FS}_2'.\mathsf{KeyGen}$: run $\mathsf{FS}_1'.\mathsf{KeyGen}$ to get $\mathsf{FS}_1'.\mathsf{sk}$, as well as $\hat{H}.\mathsf{KeyGen}$, to get $\hat{H}.\mathsf{sk}$. We note $\mathsf{FS}_2'.\mathsf{sk} = (\mathsf{FS}_1'.\mathsf{sk}, \hat{H}.\mathsf{sk})$.
- $\mathsf{FS}_2'.\mathsf{Enc}$: given $m \in \mathbb{Z}_t$, $\mathsf{FS}_1'.\mathsf{sk} = \{\mathsf{sk}^{(k)},\ k \in [\![0, K]\!]\}$, and $\hat{H}.\mathsf{sk}$, return
$$\mathsf{ct} = (c, \tau) = (\mathsf{FS}_1'.\mathsf{Enc}(\mathsf{FS}_1'.\mathsf{sk}, m), \hat{H}.\mathsf{Digest}(\hat{H}.\mathsf{sk}, c)).$$
- $\mathsf{FS}_2'.\mathsf{ImVer}$: given $\mathsf{ct} \in \mathbb{Z}_q^n \times \mathbb{Z}_q^{K+1} \times \mathcal{H}$, and $\mathsf{FS}_2'.\mathsf{sk} = (\mathsf{FS}_1'.\mathsf{sk}, \hat{H}.\mathsf{sk})$, return False if either $\mathsf{FS}_1'.\mathsf{ImVer}(\mathsf{FS}_1'.\mathsf{sk}, \mathsf{ct}.c) = \mathsf{False}$ or
$$\hat{H}.\mathsf{Digest}(\hat{H}.\mathsf{sk}, \mathsf{ct}.c) \neq \mathsf{ct}.\tau, \tag{39}$$
and True otherwise.
- $\mathsf{FS}_2'.\mathsf{Dec}$: given $\mathsf{ct} \in \mathbb{Z}_q^n \times \mathbb{Z}_q^{K+1} \times \mathcal{H}$, $\mathsf{FS}_2'.\mathsf{sk} = (\mathsf{FS}_1'.\mathsf{sk}, \hat{H}.\mathsf{sk})$, if $\mathsf{FS}_2'.\mathsf{ImVer}(\mathsf{FS}_2'.\mathsf{sk}, \mathsf{ct}) = \mathsf{False}$, then $\perp$ is returned. Otherwise return $\mathsf{FS}_1'.\mathsf{Dec}(\mathsf{FS}_1'.\mathsf{sk}, \mathsf{ct}.c)$.

Consistently with Sect. 8.5.1, we will now work under the following assumption.

Assumption 2. *Let $\ell \leq n + K + 1 - \lambda\frac{\log 2}{\log q}$, FS_2' has the ℓ-LOH property.*

Additionally, for typical FS_2' parameters, the number of ciphertexts, ℓ, that the adversary may observe would be several orders of magnitude below that required for the quantum attack in [DAFS24a, Sect. 6.3] to apply (recall also the discussion at the end of Sect. 4.5 on p. 17). However, it should be emphasized that the latter argument is subject to caution since the approaches in [DAFS24a] admit more optimized variants that are not public [Ste25].

8.5.4 ℓ-vCCA^D security of FS_2'

For the proof in this section, we assume a slightly modified mult-by-const operator and associated extractor. Indeed, we assume that the mult-by-const operator, $\mathtt{mul}_\alpha$, accepts $\alpha \in \mathbb{Z}_q$ rather than $\mathbb{Z}_t$ with the effect of producing an encryption of $[[\alpha]_t m]_t$ (under our running assumption that t divides q, Sect. 8.1). This operator naturally occurs whenever usual mult-by-const operators are chained during an homomorphic evaluation. Consistently, given a FS_2' ciphertext ct, we also assume that (π, β), as returned by $\mathsf{extract}(\mathsf{ct}, \mathsf{aux})$, is in $Z_q^m \times Z_q$ rather than $Z_t^m \times Z_t$ (in the notations of Def. 7, p. 15) and that the following property holds,
$$(\pi, \beta) = \mathsf{extract}(\mathsf{ct}, \mathsf{aux}) \Leftrightarrow \mathsf{ct}.c = c_e, \tag{40}$$
with $\pi \in \mathbb{Z}_q^{|S_\mathcal{F}|}$, $\beta = 0$ (in the private-key setting) and,
$$c_e = \mathsf{FS}_1'.\mathsf{Eval}(\mathtt{lincomb}_\pi, S_\mathcal{F}[0].c, ..., S_\mathcal{F}[|S_\mathcal{F}| - 1].c),$$

where $S_{\mathcal{F}}$ denotes a common list of well-formed ciphertexts on which an adversary and its challenger agree. This modification is a mild assumption as the usual extractor output is naturally a reduction modulo t of the modified one's output (again, when t divides q). This slight modification is important as it will allow us to rebuild the ciphertexts given by the adversary when we need to do so in the proof of the next proposition.

Proposition 15. *FS'_2 is (adaptive) ℓ-vCCA^D secure, under the assumptions that it satisfies the ℓ-LOH property and that FS_1 is (adaptive) CPA^D secure.*

Proof. The proof works under the natural assumption that $\mathcal{A}$ and the challenger share a common numbering for the ciphertexts output by the encryption oracle (i.e., *fresh well-formed ciphertexts*).

We then start by one step of game hopping.

First game hop. Let G_0 be the ℓ-vCCA^D game against FS'_2 and G_1 be the same game as G_0 where we modify the challenger as follows. First (consistently with the above common numbering assumption), the challenger $\mathcal{C}_1$ stores the fresh ciphertexts it generates in an internal state $S_{\mathcal{F}}$ containing message-message-ciphertext triplets (and $\mathcal{A}$ is assumed to maintain a similar array). Then, when handling a decryption request on ciphertext ct, it first invokes the LOH extractor to verify that

$$\mathsf{extract}(\mathsf{ct}, \mathsf{aux}) \neq \varnothing,$$

rather than checking conditions (34) and (39) in $\mathsf{FS}'_2.\mathsf{ImVer}(\mathsf{FS}'_2.\mathsf{sk}, \mathsf{ct})$, and returns $\perp$ when this is not the case. Otherwise, $\mathcal{C}_1$ proceeds as follows. Let $(\pi, \beta) = \mathsf{extract}(\mathsf{ct}, \mathsf{aux})$[42],

$$\mu_0 = \sum_{i=0}^{|S_{\mathcal{F}}|-1} \pi_i S_{\mathcal{F}}[i].m_0 \text{ and } \mu_1 = \sum_{i=0}^{|S_{\mathcal{F}}|-1} \pi_i S_{\mathcal{F}}[i].m_1. \tag{41}$$

Then, *whenever* $\mu_0 = \mu_1$ (otherwise it returns $\perp$), *rather than returning* $\mathsf{FS}'_1.\mathsf{Dec}(\mathsf{FS}'_1.\mathsf{sk}, (\mathsf{ct}.c.a, \mathsf{ct}.c.B))$, $\mathcal{C}_1$ returns $\mathsf{FS}_1.\mathsf{Dec}(\mathsf{FS}'_1.\mathsf{sk}, (\mathsf{ct}.c.a, B'))$ with

$$B'_0 = B_0 \tag{42}$$

and, for $k \in [\![1, K]\!]$,

$$B'_k = B_k - \Delta\xi_k\mu_0. \tag{43}$$

We now argue that G_0 and G_1 cannot be distinguished. Indeed, in the case where $\mathsf{extract}(\mathsf{ct}) = \varnothing$, the indistinguishability of the two games follows from the ℓ-LOH property as, in this case, we have that $\mathsf{FS}'_2.\mathsf{ImVer}(\mathsf{FS}'_2.\mathsf{sk}, \mathsf{ct}) = \mathsf{False}$ with overwhelming probability. When, $\mathsf{extract}(\mathsf{ct}) \neq \varnothing$, then (by Eq. 40) $\mathsf{ct}.c$ is a well-formed FS'_1 ciphertext, i.e. is such that for $k \in [\![0, K]\!]$ ($\xi_0 = 1$),

$$\mathsf{ct}.c.B_k = \langle \mathsf{ct}.c.a, \mathsf{sk}^{(k)} \rangle + \Delta\xi_k\mu + e_k,$$

and the e_k's are iid and subgaussian, which may decrypt to $\perp$ depending on the e_k's (remark that μ may not be $\mathbb{Z}_t$). Now (recall the properties of the extractor discussed at the beginning of this Sect.), when $\mu_0 = \mu_1$ in Eq. (41), we exactly have that $\mu = \mu_0$. Hence, Eq. (43) turns $(\mathsf{ct}.c.a, \mathsf{ct}.c.B)$ into a well-formed ciphertext $(\mathsf{ct}.c.a, B')$ under FS_1 with

$$B'_0 = \langle \mathsf{ct}.c.a, \mathsf{sk}^{(0)} \rangle + \Delta\mu + e_0,$$

and, for $k \in [\![1, K]\!]$,

$$B'_k = \langle \mathsf{ct}.c.a, \mathsf{sk}^{(k)} \rangle + e_k.$$

[42] Recall, following Def. 7 (p. 14), that the LOH assumption enforces linear (rather than affine) combinations in the private-key setting. Hence, in the context of this proof, when the LOH extractor does not return $\varnothing$, we get $(\pi, \beta) = \mathsf{extract}(\mathsf{ct}, \mathsf{aux})$ with $\beta = 0$.

The indistinguishability of the two games, in the case where $\mathsf{extract}(\mathsf{ct}) \neq \varnothing$, thus follows from the indistinguishability between FS_1's and FS'_1's decryption functions which we have established in Lemma 22.

Second game hop. We now consider game G_2 where we further modify the challenger $\mathcal{C}_1$ such that (recall $\mathsf{FS}'_1.\mathsf{Enc}$ definition on p. 44) the new challenger $\mathcal{C}_2$ replies to a challenge request on $m_0 \neq m_1 \in \mathbb{Z}_t^2$ with ciphertext

$$\mathsf{ct}' = (c', \tau') = (\mathsf{FS}_0^{(K+1)}.\mathsf{Enc}(\mathsf{FS}_0^{(K+1)}.\mathsf{sk}, m_\gamma, \xi_1 m_0, ..., \xi_K m_0), \hat{H}.\mathsf{Digest}(\hat{H}.\mathsf{sk}, c')),$$

rather than, in G_1,

$$\mathsf{ct}^* = (c^*, \tau) = (\mathsf{FS}_0^{(K+1)}.\mathsf{Enc}(\mathsf{FS}_0^{(K+1)}.\mathsf{sk}, m_\gamma, \xi_1 m_\gamma, ..., \xi_K m_\gamma), \hat{H}.\mathsf{Digest}(\hat{H}.\mathsf{sk}, c^*)).$$

These two games cannot be distinguished for the following reasons:

- Both c' and c^* are indistinguishable from uniform (by the LWE assumption).
- When handling a decryption request over a ciphertext ct such that $\varnothing \neq \mathsf{extract}(\mathsf{ct}, \mathsf{aux}) = (\pi, \beta)$ (again, with $\beta = 0$ in the private-key setting), and $\mu_0 = \mu_1$ in Eq. (41), in which case we also have $\mu_0 = \mu_\gamma$, both challengers reply with $\mathsf{FS}_1.\mathsf{Dec}(\mathsf{FS}'_1.\mathsf{sk}, (\mathsf{ct}.c.a, B'))$ (with B' obtained, in both cases, following Eqs. 42 and 43 above). When this is not the case, i.e. when either $\mathsf{extract}(\mathsf{ct}, \mathsf{aux}) = \varnothing$ or $\mu_0 \neq \mu_1$ (in Eq. 41), then both challengers consistently reply $\perp$.

Final reduction. We next prove that an adversary $\mathcal{B}$ against the CPA^D security of FS_1 (consistently with $\mathcal{C}_2$ specification) can be built using an adversary $\mathcal{A}$ against G_2 as a subroutine such that it achieves the same advantage as the latter. The reduction thus maintains an initially empty state $S_{\mathcal{F}}$ containing message-message-ciphertext (under FS'_2) triplets (and $\mathcal{A}$ is assumed to maintain a similar state). The reduction also maintains a conversion table between ciphertext indices in $S_{\mathcal{F}}$ and S, where S is the internal state of the CPA^D challenger against FS_1, $\mathfrak{I} : [\![0, |S_{\mathcal{F}}| - 1]\!] \longrightarrow [\![0, |S| - 1]\!]$ such that $S_{\mathcal{F}}[l].c = S[\mathfrak{I}(l)].c$ (it does so by means of a counter ctr initially set to 0). *Note that to ease understanding, we also show the internal state of the CPA^D challenger in the following.* The reduction subsequently picks the multipliers $\xi_1, ..., \xi_K$ uniformly in $\mathbb{Z}_t^*$ and runs $\hat{H}.\mathsf{KeyGen}$. It then handles $\mathcal{A}$'s requests as follows:

- When receiving an encryption request over message $m \in \mathbb{Z}_t$, $\mathcal{B}$ transfers it as is to the CPA^D challenger to get ciphertext $c = (a, B)$. It then computes ciphertext $c' = (a, B')$, with $B'_0 = B_0$ and $B'_k = B_k + \Delta \xi_k m$ (for $k \in [\![1, K]\!]$) and returns $\mathsf{ct} = (c', \hat{H}.\mathsf{Digest}(\hat{H}.\mathsf{sk}, c'))$ to $\mathcal{A}$ after updating its internal state as $S_{\mathcal{F}} := [S_{\mathcal{F}}; (m, m, \mathsf{ct})]$ and its conversion table as $\mathfrak{I}(|S_{\mathcal{F}}| - 1) := \mathsf{ctr}$ (then doing $\mathsf{ctr} := \mathsf{ctr} + 1$). *This has the side effect of updating the CPA^D challenger's internal state as $S := [S; (m, m, c)]$ (with the invariant that $\mathsf{ctr} = |S|$).*
- When receiving a challenge request over messages $m_0 \neq m_1 \in \mathbb{Z}_t^2$, then $\mathcal{B}$ also transfers it as is to the CPA^D challenger to get ciphertext $c^* = (a^*, B^*)$. It then computes ciphertext $c'^* = (a^*, B'^*)$, with $B'^*_0 = B^*_0$ and $B'^*_k = B^*_k + \Delta \xi_k m_0$ (for $k \in [\![1, K]\!]$), yielding $c'^* = \mathsf{FS}_0^{(K+1)}.\mathsf{Enc}(\mathsf{FS}_0^{(K+1)}.\mathsf{sk}, m_\gamma, \xi_1 m_0, ..., \xi_K m_0)$ consistently with $\mathcal{C}_2$'s specification, and returns $\mathsf{ct} = (c'^*, \hat{H}.\mathsf{Digest}(\hat{H}.\mathsf{sk}, c'^*))$ to $\mathcal{A}$ after updating its internal state and conversion table as in the above case. *As in the previous case, this has the side effect of updating the CPA^D challenger's internal state as $S := [S; (m_0, m_1, c^*)]$.*
- When receiving a decryption request over ciphertext $\mathsf{ct} \in \mathbb{Z}_q^n \times \mathbb{Z}_q^{K+1} \times \mathcal{H}$, $\mathcal{B}$ proceeds as follows:

– *Fresh ciphertext.* If $\exists l : S_{\mathcal{F}}[l].c = \mathsf{ct}$, it issues a decryption request with index $\mathfrak{I}(l)$ towards its CPA^D challenger and return the result to $\mathcal{A}$. Remark that the adversary expects the decryption of FS'_1 ciphertext $\mathsf{ct}.c = S_{\mathcal{F}}[l].c$ while the reduction returns in fact the decryption of FS_1 ciphertext $S[\mathfrak{I}(l)].c$ however with the guarantee that $\mathsf{FS}_1.\mathsf{Dec}(\mathsf{FS}'_1.\mathsf{sk}, S[\mathfrak{I}(l)].c)$ is indistinguishable $\mathsf{FS}'_1.\mathsf{Dec}(\mathsf{FS}'_1.\mathsf{sk}, S_{\mathcal{F}}[l].c)$ (by Lemma 22).

– *Evaluated ciphertext.* Otherwise, it invokes $\mathsf{extract}(\mathsf{ct})$ getting either $\varnothing$, in which case $\perp$ is returned to $\mathcal{A}$ (following $\mathcal{C}_1$'s specification in the first game hop), or a vector $\pi \in \mathbb{Z}_q^{|S_{\mathcal{F}}|}$. Let $l_0, ..., l_{L-1}$ denote the indices such that $\pi_{l_j} \neq 0$, then $\mathcal{B}$ issues an evaluation request over indices $\mathfrak{I}(l_0), ..., \mathfrak{I}(l_{L-1})$ towards its CPA^D challenger to get ciphertext

$$c_e \quad = \quad \mathsf{FS}_1.\mathsf{Eval}(\mathtt{lincomb}_{\pi_{l_0},...,\pi_{l_{L-1}}}, S[\mathfrak{I}(l_0)].c, ..., S[\mathfrak{I}(l_{L-1})].c).$$

Remark that, when $\mu_0 = \mu_1$ in Eq. (41), we have $\mathsf{ct}.c.a = c_e.a$, $\mathsf{ct}.c.B_0 = c_e.B_0$ as well as $\mathsf{ct}.c.B_k = c_e.B_k + \Delta\xi_k\mu_0$ ($k \in [\![1, K]\!]$). *This has the side effect of augmenting the internal state of the CPA^D challenger with the triplet*

$$\left(\sum_{i=0}^{L-1} \pi_{l_i} S[\mathfrak{I}(l_i)].m_0, \sum_{i=0}^{L-1} \pi_{l_i} S[\mathfrak{I}(l_i)].m_1, c_e\right). \tag{44}$$

Additionally $\mathcal{B}$ increments ctr to keep $\mathfrak{I}$ consistent with the internal state of its challenger (i.e. maintain the invariant that $\mathsf{ctr} = |S|$). Finally, $\mathcal{B}$ issues a decryption request with index $\mathsf{ctr} - 1$ towards its CPA^D challenger to get a decryption of c_e (which may be $\perp$) which it sends back to $\mathcal{A}$. Remark that because of the above relation between $\mathsf{ct}.c$ and c_e we have that $\mathsf{FS}'_1.\mathsf{Dec}(\mathsf{FS}'_1.\mathsf{sk}, \mathsf{ct}.c)$ is indistinguishable from $\mathsf{FS}_1.\mathsf{Dec}(\mathsf{FS}'_1.\mathsf{sk}, c_e)$ (by Lemma 22). Recall that, when handling a decryption request on state index l, the CPA^D challenger's decryption oracle verifies that $S[l].m_0 = S[l].m_1$ and returns the decryption of $S[l].c$ only when this is the case (and $\perp$ otherwise) so the reduction does not even have to check this.

The claim then follows from the (adaptive) CPA^{D}security of FS_1 which we conjecture in this section. □

As a last remark, it may be worth noting that applying the Encrypt-then-Hash blueprint that yielded FS'_2 from FS'_1 (Sect. 8.5.3) *directly to FS_1* may be sufficient to obtain a scheme that also credibly satisfies the ℓ-LOH property, i.e. under the not too far-fetched assumption that the (linearly homomorphic) hash verification is sufficient to get this property. Although this depart from the more conventional Knowledge-of-Exponent blueprint, the resulting scheme (say FS_2) would be simpler and would also admit a less cumbersome proof of ℓ-vCCA^D security, in particular with a more direct final reduction towards a CPA^D challenger against FS_1 which is left to the reader.

9 Discussion on general results for approximate schemes?

As discussed at the beginning of Sect. 8, a natural question is whether black-box results, such as Prop. 4 and 5, can be obtained in the case of approximate schemes? E.g. if any CPA^D secure linearly homomorphic scheme with the LOH property is vCCA^D secure? A natural proof strategy for obtaining such a result (say, first, in the private-key setting) is to perform a reduction towards a CPA^D challenger. For such a reduction to work, it then has to handle decryption requests over well-formed evaluated ciphertexts from the vCCA^D adversary by means of its CPA^D challenger's one. To do so, the reduction then has to

populate the internal state of its CPA^D challenger with the exact same ciphertext provided by the vCCA^D adversary, by means of evaluation requests parameterized by the LOH extractor output and the set of fresh well-formed ciphertexts output by the encryption oracle. Then, in the notations of Def. 7 (p. 15) and following Eq. (6), the reduction works by relying on the property that[43],

$$(\pi, \beta) = \mathsf{extract}(c', \mathsf{aux}) \Leftrightarrow c' = \mathcal{E}_H.\mathsf{Eval}(\mathtt{lincomb}_\pi, c_0, ..., c_{m-1}),$$

(recall that $\beta = 0$ in the private-key setting). Still, as illustrated in Sect. 8 and 8.5, building approximate or somewhat correct (LWE-based) schemes satisfying even weak variants of the LOH property is particularly delicate. So it seems to us that the kind of black-box results sketched above has a limited practical relevance.

We also emphasize that, in the public-key setting, in order for a reduction to properly rely on a CPA^D challenger's decryption oracle to handle decryption requests from a vCCA^D adversary, it further has to populate the challenger's internal state with the well-formed fresh ciphertexts generated on its own by the adversary (which, in that case, controls the encryption randomness). To do so, the reduction hence necessarily has to operate in the adversarially-chosen encryption randomness setup which is accounted for by the notion of Strong CPA^D security introduced in [BJSW24][44]. We think the only way whereby this reliance on Strong CPA^D could be avoided, would be by proceeding via a reduction which does not have to rely on a challenger with a decryption oracle (e.g a reduction to the CPA rather than "CPA^D" security of the homomorphic scheme). This however appears difficult to achieve without introducing non-black box assumptions on $\mathcal{E}_H$ as the reduction would then have to handle the adversary's decryption requests without relying on any decryption oracle.

10 Concluding remarks

In this paper, our goal has been to investigate whether the LOH property and the associated Knowledge-of-Exponent design blueprint could help obtaining "beyond CCA1" secure linearly homomorphic schemes without relying on advanced SNARK machinery.

When working under the correctness assumption, the short answer is yes. Indeed, this paper has unveiled a fruitful connection between the LOH property and vCCA security, eventually yielding simple constructions achieving this strong CCA security notion at twice the cost of achieving only CPA security.

Relaxing the correctness assumption has, as is usually the case, revealed a more complicated picture. Although it can be expected that a CPA^D secure scheme with the LOH property achieves vCCA^D security (Sect. 9), the second part of this paper shows that it is quite delicate to build schemes having these former properties by starting from an LWE-based scheme and extending it by naturally following the KEA design blueprint. Eventually, this strategy works only in a non-adaptive setting in which vCCA^D collapses onto a weak relaxation of CCA1. If we conjecture that FS'_1 is (adaptive) CPA^D secure, then it still does not satisfy the LOH assumption as the attack in the proof of Prop. 13 still works against it. However, that attack works under the assumption that $m \geq n + K + 1$ and similarly so for the adaptive CPA^D attack in Sect. 8.4.4. It turns out that the (adaptive) vCCA^D security of FS'_1 can be established under the assumption that the adversary has access to only $m < n + K + 1$ ciphertexts, so under the conjecture that it is (adaptive) CPA^D secure and the assumption that it has the LOH property (in this same restrictive setting for both). To do so, we however have to associate a linearly homomorphic hash to

[43] Although, stricto sensu, Eq. (6) only gives us the right to left implication (see the discussion at the beginning of Sect. 8.5.4 leading to Eq. 40).

[44] Similarly to the reductions in the proof of Prop. 17 and 18 in [BCF+25].

FS'_1 ciphertexts in order to deal with yet another corner case involving trivial encryptions of 0.

Generalization to the public-key setting however appears more problematic, even in the non-adaptive setting. Indeed, it is tempting to apply the well-known Regev's trick for turning FS_0 into a public-key scheme [Reg05, Reg09]: define the public key as a large enough set of encryptions of 0 under FS_0 along with an additional public encryption of 1 for injecting messages in ciphertexts via the mult-by-const operator[45]. Then, all well-formed ciphertexts would end up being linear combinations of the ciphertexts forming the public key, paving the way for a reduction able to feed all well-formed ciphertexts built by a vCCA^D adversary into the internal state of a (private-key) CPA^D challenger by means of evaluation requests parameterized on the LOH extractor output. However, Claim 5.3 in [Reg09][46] implies that we have to use $N \geq 4\lambda + (n + K + 1)\log_2 q$ ciphertexts to form a public-key. Unfortunately, this number of ciphertexts (which are observed, in the public-key setting, by a non-adaptive adversary before it specifies its requests) is much larger than the limit under which we can credibly claim adaptive vCCA^D security. Another trail, could be to adapt our schemes from the multi-secret to the RLWE setting [LPR10] leading to a public key formed by a single encryption of 0. However, notwithstanding the other issues that may crop up when doing so, a reduction towards a CPA^D challenger would have to operate in the adversarially-chosen encryption randomness setup[47] (contrary to the above "Regev-style" approach where the LOH extractor would be able to retrieve the encryption randomness). This setup then would require the baseline scheme to achieve Strong CPA^D rather than only "standard" CPA^D security [BJSW24].

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[45] With the subtlety that an appropriate large noise would further have to be added to the encryptions of 0 forming the public key so as to smudge out the resulting message-noise dependency.

[46] Claim 5.3 in [Reg09] establishes that, given a finite Abelian group G and any N elements g_0, ..., g_{N-1} in G, the probability that the statistical distance between the uniform distribution on G and the distribution given by the sum of a random subset of g_0, ..., g_{N-1} exceeds $\sqrt[4]{|G|/2^N}$ is bounded by $\sqrt[4]{|G|/2^N}$.

[47] Similarly to the reductions in the proof of Prop. 17 and 18 in [BCF+25].

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A The $\widehat{\mathsf{FGP}}$ linearly homomorphic hash function [FGP14a]

For illustrative purpose, we briefly describe the Fiore-Gennaro-Pastro hash function introduced in [FGP14a] ([FGP14b], Sect. 4.2) in order to build several VC schemes on top of the BV scheme [BV11]. Although FS'_2 can be instantiated from any linearly homomorphic hash function (with domain $\mathbb{Z}_q^N$), this is an example of a concrete candidate.

The hash function consists in interpreting a FS_1 ciphertext as a polynomial in $\mathbb{Z}_q[X][Y]$ with degree $\max(K, n-1)$ in X and degree 1 in Y, i.e. a ciphertext $\mathsf{ct} = (a, B) \in \mathbb{Z}_q^n \times \mathbb{Z}_q^{K+1}$ is associated to polynomial

$$p_{\mathsf{ct}}(X, Y) = \sum_{i=0}^{n-1} a_i X^i + Y \sum_{i=0}^{K} B_i X^i.$$

Given α and β, two *secret* values uniformly picked in $\mathbb{Z}_q^2$, the hash function is then simply defined as

$$\mathsf{FGP.Digest}(\alpha, \beta, \mathsf{ct}) = p_{\mathsf{ct}}(\alpha, \beta) = \sum_{i=0}^{n-1} a_i \alpha^i + \sum_{i=0}^{K} B_i \beta \alpha^i.$$

This function is trivially linearly homomorphic (and *not* affine homomorphic). It is further shown in [FGP14b] (Theorem 2) that it is universal one-way for $q > 2^{\lambda}$, although not collision resistant. To achieve collision resistance, [FGP14a, FGP14b] then extends the hash function to work in the exponent of a multiplicative group $\mathbb{G}$ for which the discrete logarithm is hard, e.g. with a prime modulus $M = q + 1$, with $q = tp$, where p is a suitably large prime (and t is the plaintext modulus[48]),

$$\widehat{\mathsf{FGP}}.\mathsf{Digest}(\alpha, \beta, \mathsf{ct}) = g^{p_{\mathsf{ct}}(\alpha,\beta)} \mod M.$$

where g in a generator of $\mathbb{G}$. Note that [FGP14a, FGP14b] uses bilinear groups rather than cyclic groups in order for the homomorphic property to hold for degree-2 functions which we do not need as the present work explicitly focuses on linear-only homomorphic schemes. In summary, the specification of $\widehat{\mathsf{FGP}}$ is as follows:

- $\widehat{\mathsf{FGP}}.\mathsf{KeyGen}$: uniformly pick $\alpha, \beta \in \mathbb{Z}_q^2$, choose a prime M as above and a generator g of $\mathbb{Z}/M\mathbb{Z}$.
- $\widehat{\mathsf{FGP}}.\mathsf{Digest}$: given $\mathsf{ct} = (a, B) \in \mathbb{Z}_q^n \times \mathbb{Z}_q^{K+1}$ and α, β, return

$$g^{\sum_{i=0}^{n-1} a_i \alpha^i + \sum_{i=0}^{K} B_i \beta \alpha^i} \mod M.$$

- $\widehat{\mathsf{FGP}}.\mathsf{Add}$: given $h, h' \in (\mathbb{Z}/p\mathbb{Z})^2$, return $hh' \mod M$.
- $\widehat{\mathsf{FGP}}.\mathsf{Mulc}$: given $h \in \mathbb{Z}/p\mathbb{Z}$ and $\alpha \in \mathbb{Z}_q$, return $h^{\alpha} \mod M$.

Still, in terms of practical implications, the fact that the LWE ciphertext modulus connects to the discrete log modulus requires using unusually large LWE moduli. As already investigated in [FGP14a], there are protocols in which function FGP, rather than $\widehat{\mathsf{FGP}}$, can be used (essentially when only one FHE calculation is performed). In such cases, the aforementioned difficulty disappear.

To the best of our knowledge, there are only a very limited number of candidate constructions in the state-of-the-art. Other approaches may consider using the Boneh-Freeman lattice-based linearly homomorphic signature scheme [BF11] (modified to authenticate vectors with coefficients over extension fields $\mathbb{F}_{2^k}$, as explained in that paper) or the SWIFFT hash function [LMPR08] (although a variant working over non-binary inputs remains to be defined and studied). Both approaches would lead to constructions consistently only based on lattices.

[48] It is desirable that t divides q for a number of reasons (including but not limited to the discussion in Sect. 8.5.4).

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