On Maximum Size Simultaneous Linear Approximations in Ascon and Keccak and Related Translation and Differential Properties

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Abstract. In this paper we study the S-box known as Chi or χ initially proposed by Daemen in 1995 and very widely used ever since in Keccak, Ascon, and many other. This type of ciphers is typically analyzed [in recent research] in terms of subspace trail attacks [TeDi19] and vector space invariants. An interesting question is then, when different spaces are mapped to each other by translations with a constant. In this paper we relax this fundamental question and we consider arbitrary sets of points and their translations. We generalize previous S-box partial linearization results on Keccak and Ascon from Eurocrypt 2017. We basically introduce new ways to linearize the Ascon S-box to the maximum possible extent. On this basis we show further remarkable properties and some surprising connections between [simultaneous] linear and [prominent] differential properties. We exhibit a new family of maximum size and optimal approximation properties with 11 points, beyond the maximum size of any set in the DDT table. We prove a theorem which guarantees that this type of properties are stable by arbitrary input-side translations which holds for all quadratic permutations. All this will be placed in the context of previous work on classification of 5-bit quadratic permutations.

Key Words. Ascon, Keccak, Chi S-box construction, Simultaneous Linear Approximations, LAT, DDT, Mutual Information, rotation invariants, affine space trails, connectors, connectivity tables, invariant attacks, cryptanalysis.

1 Introduction

Linear Cryptanalysis [LC] and Differential Cryptanalysis [DC] are classical attacks in symmetric cryptography [block ciphers, stream ciphers and hash functions]. These methods are typically claimed to be invented either at IBM and NSA in the 1970s, or by academic researchers in late 1980s. In fact it is easy to see that the frequently cited research articles from late 1980 / early 1990s did not invent anything new. It is possible to see that the study of differential and linear properties of ciphers is in many ways much older, directly and indirectly. For example in Section 7 of [CoGr22] we learn that Enigma rotor 3 from 1930s has a very strong linear approximation modulo 26 true with probability 10/26, and that all early German and Swiss Enigma rotors in 1930s had an exceptionally low number of invariant impossible differentials which number has substantially increased in later 1940s Enigma rotors, cf. [CoGr22]. Likewise, several very specific design criteria which amount to resistance against a variety

of linear, differential, higher-order differential and partial collision properties in the block cipher T-310 were already studied in Eastern Germany in 1970s, cf. [CoPoSc08,T310S].

In their very simple or "simplistic" versions, with isolated "single" linear and differential properties, these basic attacks simply do not work very well on modern cipher such as AES, Ascon or Keccak. Or so it seems: see [CoQi20]. Many advanced attacks published in recent years, are all about various ways to combine multiple linear properties [BCQ04,SmAes07,CoDe08] and a variety of differential properties, [TeAs16,HuCu24,CoQi20] or both combined, with "Connectivity Tables" cf. [QSMG17,HaDeEi24]. When everything fails, the same properties can also be used in some algebraic attacks cf. [LiIsMeYa21,CoDe08].

This paper studies the question of existence of [large size] **simultaneous** linear approximations for small size S-boxes and a number of closely related facts and concepts. At Eurocrypt 2017 researchers work on constructing collisions on up to 6 rounds of Keccak with so called "connectors", cf. Section 4.2. in cf. [QSMG17]. Their "connectors" are based on partial linearization properties with 4 points known as LAS, which points in fact form an affine space of dim 2 on both I/O sides of the S-box. with special focus on LSA properties related to DDT() sets. In this paper we drop the requirement of points forming an affine space of Dim=2,3,4, and we study more general types of linearization properties with arbitrary sets of points. Most of the work inside this paper is applicable to Ascon, Keccak, XooDoo, Icepole, Rasta, and other ciphers using the χ S-box.

An important family of cryptanalytic attacks are various subspace trail attacks [TeDi19] which attempt to predict the state of a cipher in terms of sequences of affine spaces. A closely related concept is the one of invariant attacks, such as in [CARG19,GJNQSS16] and many other such as [older] truncated differentials [TeAs16]. In all these attacks we can simply **drop** the requirement of points forming a vector space. Then the classical question of reuse of linear spaces can be re-defined to consider when two arbitrary pairs of sets or points are related by a translation by a bitwise XOR with a constant.

In this paper we study various self-similarity properties of Ascon and other similar S-boxes. We mostly work with the Ascon S-box and we will discover some interesting rather non-trivial properties. This will lead to some new definitions, attempting to propose some relatively robust measures of vulnerability w.r.t. space/set reuse and simultaneous linear approximations [BCQ04,CoDe08]. At the end we will compare Ascon to other similar S-boxes and also study their ASIC implementation cost (and few other important parameters) in order to avoid unfair comparisons. At this moment we do not attempt to design an actual invariant or subspace attack on several rounds of Ascon or Keccak (or not in this paper). The universe of potential attacks involving various forms of affine space trails and/or simultaneous linear and differential approximations [or equations or relations] is extremely large and diverse, to cite only few selected papers: [BCQ04,SmAes07,CoDe08,CARG19,CoQi20,GJNQSS16,LiIsMeYa21,HaDeEi24].

2 Notation and Basic Definitions

We summarize here our notations.

F_2	Finite field $F_2 = \{0, 1\}$
$\oplus$	Bitwise XOR or bitwise addition modulo 2 in F_2^k
S-box	In this paper all S-boxes are assumed to be bijective
$F[], S[]$	Some cryptographic S-box $F_2^k \to F_2^k$
k	In this paper $k = 5$ or 8 most of the time
HW	Hamming Weight
$y = F[x]$	Application of F to x, implies $x, y \in F_2^k$
x_i, y_i	We number bits of x and y from 0 to $k-1$
$co - X$	Complement of set X, $co - X = F_2^5 \setminus X$
BIBO	Bad Input Bad Output property cf. [LMCFW23]
I/O	Input and Output sides
LIO-5	5 linear relations involving terms from both I/O sides x_i/y_j
MC	Multiplicative Complexity cf. [BoFiPe13,BoPePo00]
LAT	Linear Approximation Table
$\alpha{\cdot}x{+}\beta{\cdot}y{=}0$	One LIO equation from LAT where $\cdot$ is the dot product
NL,$NL_1, \mathcal{L}$	NonLinearity $2^{k-1} - \lvert max_{\alpha,\beta\neq 0}(LAT(\alpha,\beta))\rvert$
WZ,WN	set of Walsh Zeroes $LAT(\alpha,\beta) = 0$ and its complement
A|2	Example of LIO encoding for $y_1 = x_1 \oplus x_3$
DDT	Differential Distribution Table
Di-size	Size of Dillon's set $\{F[x] \oplus F[y] \oplus F[z] \oplus F[x \oplus y \oplus z]\}$
δ_{in}, δ_{ou}	Two 5-bit I/O differences
MI	Mutual Information
DMI	$\mathrm{MI}(\delta_{in}; \delta_{ou})$
DMI_S	$\mathrm{MI}(\delta_{in}, \delta_{ou} \lvert x, x' \in S)$
LAS-2	Linearizable Affine Subspaces of dim 2
LSS-11	A Linearizable Sub Set of points of size 11
VecLin, Max(LSS)	max size of an LSS set
NL_v	Vectorial Nonlinearity cf. [Nagy22], $NL_v = 2^k - Max(LSS)$
$\sum_{in}(V)$	XORs of all I/O for all $x \in V \subset F_2^5$
$\sum_{ou}(V)$	XORs of all outputs $S[x]$ for all $x \in V \subset F_2^5$
UDB	Undisturbed Bits property $F[x]_j \oplus F[x \oplus \delta_{in}]_j = \mathrm{Cst}$ for several j
S_{in}	Linear function s.t. $S_{in}[x]$ = the input of embedded Keccak S-box
T_{ou}	Affine function s.t. $T_{ou}^{-1}[y]$ = the output of embedded Keccak S-box
$Rot5$	Circular left bit rotation on 5 bits; e.g. $Rot5[16] = 1$
SI	Shift Invariant $S[Rot5[x]] = Rot5[S[x]]$ for all x, cf. [DaPhD95]
ρ_{in}	Input-side rotation in Ascon, $\rho_{in}[x] = S_{in}^{-1} \circ Rot5 \circ S_{in}$
ρ_{ou}	Output-side rotation in Ascon, $\rho_{ou}[x] = T_{ou} \circ Rot5 \circ T_{ou}^{-1}$
kX	Values of 5 internal products; equal to $S_{in}[x] \oplus T_{ou}[S[x]]$ in Ascon

2.1 Single Linear Approximations - LAT and LIO

Definition 1 (LAT Table). *Let S be an S-box on k bits. We define the Linear Approximation Table (LAT) as follows:*

$$LAT(\alpha, \beta) = \sum_{x \in F_2^k} (-1)^{\alpha.x+\beta.S[x]}$$

where . is the dot product. We call NL=$\mathcal{L}_S = |max_{\alpha,\beta\neq 0}(LAT(\alpha, \beta))|$. This is the same as NL_1 in [Nagy22].

Definition 2 (WZ). *We call Walsh Zeros (WZ) all pairs s.t. $LAT(\alpha, \beta) = 0$ with $\alpha \neq 0, \beta \neq 0$.*

Definition 3 (LIO relation). *For any subset of points $V \subset F_2^k$ we call an LIO relation for an S-box F any affine or linear combination $\alpha.x + \beta.y + c$ mixing input and output bits $x_i \in F_2$ and $y_i \in F_2$ which is true with probability 1, i.e. it holds for every pair x, y where $x \in V$ and $y = F[x]$.*

LIO equations are a special case of "implicit equations" in mathematics, a.k.a. "I/O equations" cf. [CoDe08] a.k.a. "indirect polynomials" in sagemath. We restrict to consider only implicit equations which are of degree at most 1 in both variables on both sides, i.e. products of type $x_i y_i$ are not allowed. Our LIO and general I/O equations form a linear space stable by addition, and therefore the number of non-zero LIO relations is always of the form $2^m - 1$.

Definition 4 (LIO-k). *We say that a set of points $V \subset F_2^k$ is LIO-k when it has exactly $2^k - 1$ LIO (i.e. linear I/O relations).*

Example 1a - Ascon. We consider the set $V = \{0, 3, 8, 9, 10, 11, 22, 23, 29, 31\}$. For these 10 points by linear algebra we get exactly $3 = 2^2 - 1$ LIO relations which are in hex: $C|B$ $14|0$ $18|B$ which form a linear space of dimension 2. The last one $18|B$ could be transcribed as: $y_0 \oplus y_1 \oplus y_3 = x_4 \oplus x_3$ where $\oplus$ denotes the addition modulo 2.

Example 1b - Ascon. One can do better than size 10. It is possible to see that there exist seven sets of 20 points out of 32, such that exactly 3 distinct linear approximations (dim=2) exist and hold in all 20 pairs x, y and these sets, with corresponding linear I/O masks in hex, are:

- $1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 14, 15, 17, 18, 21, 22, 25, 26, 29, 30$ $\quad 1|6$ $E|4$ $F|2$
- $0, 3, 4, 7, 8, 11, 12, 15, 16, 17, 19, 20, 21, 23, 24, 25, 27, 28, 29, 31$ $\quad F|4$ $13|6$ $1C|2$

It is important to notice that these two sets are equivalent by a XOR with a constant x, this for any $x \in \{18, 22, 26, 30\}$, and this sort of **translation similarity** situation happens extremely frequently in this paper.

Example 2 - Probabilistic Single LIO. Keccak S-box has an undesirable property: it has 5 probabilistic LIO where $y_i = x_i$ with probability of 24/32 or 0.75 each, see Property 2 in [LiIsMeYa21]. In this paper we study only deterministic LIO true with probability 1, when restricted to sets of say 11 values out of 32. This amounts to study of **simultaneous** linear I/O equations and the size of our set can be used to effectively **lower-bound** the probabilities for various sets of linear equations to hold jointly.

2.2 Multiple Linear Approximations - LAS and New Definitions LSS

An important notion already studied for Ascon and Keccak S-boxes is the notion of LAS introduced at Eurocrypt 2017 cf. [QSMG17].

Definition 5 (LAS = Linearizable Affine Subspace). *Let S be an S-box on k bits. We call LAS or Linearizable Affine Subspace a set of points V which forms an affine subspace such that*

$$S[x] = A \cdot x + c \quad \forall x \in V$$

where A is $k \times k$ matrix and c is a constant vector in F_2^k.

In this paper we need to relax this notion slightly:

Definition 6 (LSS = Linearizable Sub Set). *Let S be an S-box on k bits. We call LSS or Linearizable Sub Set any set of points V such that*

$$S[x] = A \cdot x + c \quad \forall x \in V.$$

There is some asymmetry in the study of LAS and LSS notions: we look at sets at input side mostly. The situation at the input side is typically more complex. This is due to the fact the in many cryptographic S-boxes, the inverse S-box is more complex (e.g. in Ascon and Keccak it is cubic vs. quadratic).

2.3 Related Concepts - Vectorial Nonlinearity

The set of LSS properties is closely related to the concept of **vectorial nonlinearity $\mathbf{NL_v}$** which is defined as a distance towards the set of all affine functions $A \cdot x + c$. It is easy to see that the best or the largest LSS property in existence has size $2^k - NL_v$ where k is the input size.

Following [Nagy22] we have

$$NL \leq NL_v = 2^k - Max(LSS) < 2^k - k - 1.$$

Another name for Max(LSS) is Vectorial Linearity or VecLin cf. Definition 6 in [BiTuUd25] also studied in [BiTuUd26].

There exist more precise lower and upper bounds for $Max(LSS)$, see Carlet/Nagy in Appendix C.1 and LMCC conjecture in Appendix C.5. These bounds are not quite satisfactory, for example for AES. For example we know for sure that $18 \leq Max(LSS) \leq 23$, cf. Lemma 13 in [Nagy25], We have for few years conjectured that $Max(LSS) = 18$ exactly for AES. A complee mathematical proof of this fact can be found in a companion paper to be published soon. A computational verification of this fact was also done several times by the authors. Likewise, the same result was also claimed during Eurocrypt 2026 cf. [BiTuUd26]. A new software library for solving this problem in a systematic way is now available cf. [BiTuUd2G].

2.4 On Space Covering with Disjoint LSS Approximations

We also look at a further type of global property, which is about approximating a function with several approximations where the matrices used live in small space.

Definition 7 (LSS Affine Dimension). *We call the LSS Affine Dimension of the S-box F, the smallest integer D such that there exist matrices $A_1, \dots A_D$ and a constant translation matrix A_0, such that the set of all possible 2^D linear combinations of the form, $A_0 \oplus \sum_i A_i$ corresponds to an LSS property for some set V and union of these sets V is the whole space.*

Observations: It is easy to see that the LSS dimension of any S-box on k bits is at most k. The LSS dimension is stable w.r.t extended (EA) affine equivalence.

LSS Dim with Quadratic S-boxes: The case of a quadratic S-box is particularly interesting, see Section 8. For example in Ascon, as will see later, there exist just 5 matrices A_i on 25 bits with corresponding LSS-11 properties which covers the whole space 11 times see Section 7, and another set of 5 matrices with LSS-10 properties which covers the space 10 times, see Section 8.5. In general it is NOT necessary to cover the whole space 11 times, one time (with slight overlap if any) would be enough, hence we expect that the minimum dimension is frequently smaller than k.

LSS Dim is Smaller than Expected: In fact the LSS dimension of Ascon S-box is **only** 2 and for all known S-boxes not only quadratic the LSS dimension is at most 3, see Table 1 and Section 4 in [CoNi25], and it is also very low and only 4 in AES.

2.5 On Rotation Properties

Inside Ascon S-box there is anther S-box embedded: the Keccak S-box. This has some very serious consequences as we will see in this paper. Keccak S-box has many symmetries and in particular it is stable w.r.t. a rotation of five wires like $i \mapsto i+1 \mod 5$. With an affine variable change on each side, the internal circular bit rotations of Keccak can be translated to Ascon as follows. These functions are of order 5 and orbits for these functions are six sets of size 5 and two singletons (size 1). These 6+2 sets play an important role in this paper and will be later called by letters a, b, q, r, s, t, y, z, cf. Table 15 in page 38. We recall that $\rho_{in}[x] = S_{in}^{-1} \circ Rot5 \circ S_{in}$ and $\rho_{ou}[x] = T_{ou} \circ Rot5 \circ T_{ou}^{-1}$. It goes without saying that orbits for ρ_{in} are mapped to orbits for ρ_{ou} by the Ascon S-box.

f	specification
ρ_{in}	0,2,23,21,12,14,27,25,28,30,11,9,16,18,7,5,17,19,6,4,29,31,10,8,13,15,26,24,1,3,22,20
ρ_{in}^{2}	0,23,8,31,16,7,24,15,1,22,9,30,17,6,25,14,19,4,27,12,3,20,11,28,18,5,26,13,2,21,10,29
ρ_{in}^{-2}	0,8,28,20,17,25,13,5,2,10,30,22,19,27,15,7,4,12,24,16,21,29,9,1,6,14,26,18,23,31,11,3
ρ_{in}^{-1}	0,28,1,29,19,15,18,14,23,11,22,10,4,24,5,25,12,16,13,17,31,3,30,2,27,7,26,6,8,20,9,21
ρ_{in}^{5}	0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31
ρ_{ou}	10,1,12,7,4,15,2,9,18,25,20,31,28,23,26,17,3,8,5,14,13,6,11,0,27,16,29,22,21,30,19,24
ρ_{ou}^{2}	20,1,28,9,4,17,12,25,5,16,13,24,21,0,29,8,7,18,15,26,23,2,31,10,22,3,30,11,6,19,14,27
ρ_{ou}^{-2}	13,1,21,25,4,8,28,16,15,3,23,27,6,10,30,18,9,5,17,29,0,12,24,20,11,7,19,31,2,14,26,22
ρ_{ou}^{-1}	23,1,6,16,4,18,21,3,17,7,0,22,2,20,19,5,25,15,8,30,10,28,27,13,31,9,14,24,12,26,29,11
ρ_{ou}^{5}	0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31

Table 1. Two linear internal isomorphisms or rotation functions relevant to Ascon.

With rotations, we can introduce a new type of property:

Definition 8 (LRS = Linearizable Rotation Stable Sub-Set). *We call LRS any set of points V, such that for every rotation of type ρ_{in}^{a} for any integer a including $a = 0$, there exist a matrix $A[a]$ and a vector $c[a]$ such that*

$$S[x] = A[a] \cdot x + c[a] \quad \forall x \in \rho_{in}^{a}[V]$$

2.6 On Maximal Size Properties

We also define:

Definition 9 (Maximum Size LAS/LSS/LRS). *Let S be an S-box on k bits and V be a set of points $V \in F_2^k$. We say that V is maximal size LAS / LSS / LRS respectively if*

$$S[x] = A \cdot x + c \quad \forall x \in V.$$

and satisfies one of the previous definitions respectively AND the equality above does no longer hold if we add any additional points to V.

Definition 10 (Maximum Size LIO-k). *We say that V is maximal size LIO-k if at least one of $2^k - 1$ LIO relations is no longer correct for all $x \in V$, if we add any additional points to V.*

2.7 On Mutual Information and DMI

A very nice measure of quality of the S-box is to measure the Mutual Information between the input difference δ_{in} and the output difference δ_{ou} across all possible pairs of values. We call this quantity DMI and a simple formula to compute DMI is given below. The idea that ciphers with high DMI might be potentially problematic was first proposed by Nicolas Courtois at a NIST workshop about Ascon in June 2023: cf. [CoNi23,CoNi25].

The notion of DMI is motivated by and closely related to Differential Cryptanalysis and the DDT() sets we will study below in Section 3. In this paper we will show that it is also very strongly related and correlated to LC and more advanced forms of LC, and even with technical questions such Multiplicative Complexity (MC) and ASIC implementation cost. We use a standard formula for MI mixing joint and marginal probabilities found in wikipedia [wikiMI]. We define:

Definition 11 (DMI and Restricted DMI$_S$). *We define DMI as follows:*

$DMI = MI(\delta_{in}; \delta_{ou})$, done for all possible pairs of inputs, more precisely:

$$\sum_{\forall \delta_{in}, \delta_{ou} \in F_2^k} Pr_x(S[x] \oplus S[x \oplus \delta_{in}] = \delta_{ou}) \cdot log_2\left(\frac{Pr_x(S[x] \oplus S[x \oplus \delta_{in}] = \delta_{ou})}{2^{-k} \cdot Pr_{x,x'}(S[x] \oplus S[x'] = \delta_{ou})} \right)$$

In this paper we will sometimes restrict this definition to a subset $S \subseteq F_2^5$ as follows, and consider all pairs within a certain subset. The main point is that our MI can and will increase for some well-chosen subsets. We define: $DMI_S = MI(\delta_{in}; \delta_{ou} | x, x' \in S)$, i.e.

$$DMI_S = \sum_{\forall \delta_{in}, \delta_{ou} \in F_2^5} Pr_{x,x' \in S}(x \oplus x' = \delta_{in} \wedge S[x] \oplus S[x'] = \delta_{ou}) \cdot log_2\left(\frac{Pr_{x,x' \in S}(x \oplus x' = \delta_{in} \wedge S[x] \oplus S[x'] = \delta_{ou})}{Pr_{x,x' \in S}(x - x' = \delta_{in}) \cdot Pr_{x,x' \in S}(S[x] \oplus S[x'] = \delta_{ou})} \right)$$

2.8 On 5-bit S-boxes and their DMI

In this table we recall some of the most prominent 5-bit S-boxes used in applied and lightweight cryptography.

sbox	specification
Ascon	4,11,31,20,26,21,9,2,27,5,8,18,29,3,6,28,30,19,7,14,0,13,17,24,16,12,1,25,22,10,15,23
Keccak	0,9,18,11,5,12,22,15,10,3,24,1,13,4,30,7,20,21,6,23,17,16,2,19,26,27,8,25,29,28,14,31
Icepole	31,9,18,11,5,12,22,15,10,3,24,1,13,4,30,7,20,21,6,23,17,16,2,19,26,27,8,25,29,28,14,0
Thakor	10,3,11,22,17,4,1,8,12,28,23,18,26,6,31,20,15,24,29,13,14,19,30,5,25,27,7,0,16,21,2,9
Fides-5	1,0,25,26,17,29,21,27,20,5,4,23,14,18,2,28,15,8,6,3,13,7,24,16,30,9,31,10,22,12,11,19
SC2000-5	16,14,13,2,11,17,21,30,7,24,18,28,26,1,12,6,31,25,0,23,20,22,8,27,4,3,19,5,9,10,29,15
Shamash	16,14,13,2,11,17,21,30,7,24,18,28,26,1,12,6,31,25,0,23,20,22,8,27,4,3,19,5,9,10,29,15
SYCON	8,19,30,7,6,25,16,13,22,15,3,24,17,12,4,27,11,0,29,20,1,14,23,26,28,21,9,2,31,18,10,5

Table 2. Some well-known cryptographic S-boxes on 5 bits.

2.9 On Prediction of LAS-2 Properties

A well known folklore result is such that zero of LAS-2 properties exist for APN S-boxes. This also holds for few other classes of cryptographically strong near-optimal S-boxes such as based on power, inverse and exponential functions.

sbox	#0s in DDT	#8s in DDT	DMI	MC	LAS-2
SYCON	707	20	1.906	?	80
Ascon	707	20	1.906	5	80
Keccak	707	20	1.906	5	80
Shamash	707	20	1.906	?	80
Icepole	687	10	1.819	6	70
Thakor	637	4	1.589	?	45
Inv-5	527	0	1.125	8	0
Fides-5	527	0	1.125	8	0
SC2000-5	527	0	1.125	?	0

Table 3. Basic parameters of S-boxes on 5 bits studied in this paper.

We claim that the number of LAS-2 properties can be predicted with a decent level of precision from DMI and from DMI alone.

To show this we have done extensive computer simulations with S-boxes created essentially at random or by mutations from other known S-boxes, to show that there is a strong near-affine relationship between the number of distinct LAS-2 properties and the DMI value, see Table 1.

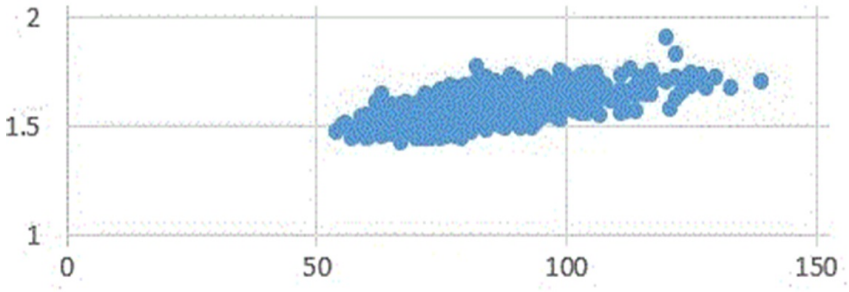

Fig. 1. A near-affine relationship which shows that DMI and LAS-2 properties are related and that the number of LAS-2 mappings drops to zero below $DMI \approx 1.1$.

It is important to see that this is an empirical law based on observation of large numbers of candidates for cryptographic S-boxes with low DMI. It suggests that LAS-2 is "small" when $DMI \approx 1.1$ but it does NOT guarantee it will be exactly zero (APN case). For example for AES we get 85 with $DMI = 1.05$ which were until recently the smallest values ever observed for an 8-bit bijective S-box, which questions we will revisit below.

In general the approximation curve is not and cannot be a simple affine curve. Instead, the number of undesirable properties is expected to grow faster for S-boxes with very poor parameters (close to linear), i.e. when DMI is very large. To show this we look at all 75 representative classes of quadratic permutations on 5 bits based in Table 1 in [BoBiSa17]. The results are shown in Fig. 2 below.

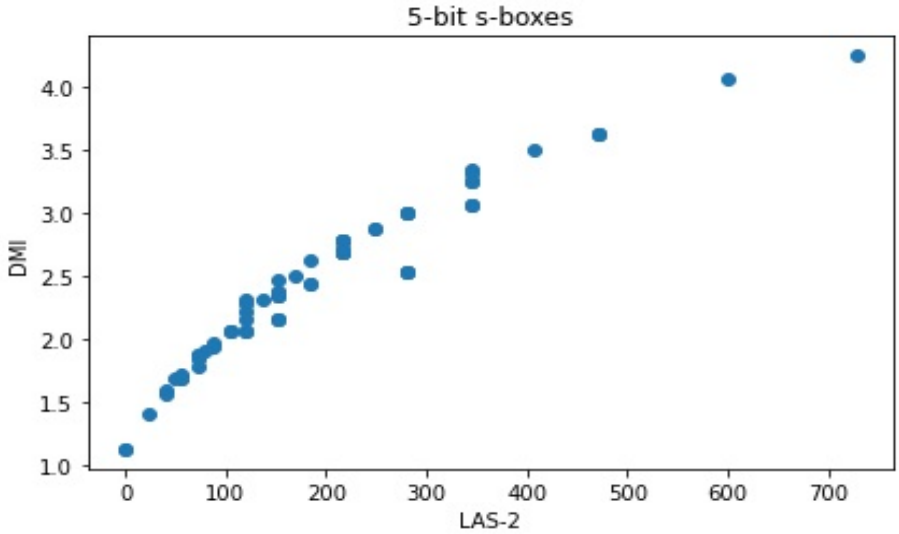

Fig. 2. DMI and LAS-2 count for all 75 affine classes of permutations on 5 bits following Table 1 in [BoBiSa17].

Some Observations on Fig. 2. Some classes of S-boxes such as 8,9,10,11 out of 75 and with the numbering of [BoBiSa17] are outliers lying below our curve. These 4 classes 8,9,10,11 are very similar and probably isomorphic in some sense. They appear clearly weaker than expected here in our data points: we obtain LAS-2=280, more than could be in general predicted from their DMI being 2.53. Some results for 8-bit S-boxes are shown on Fig. 3.

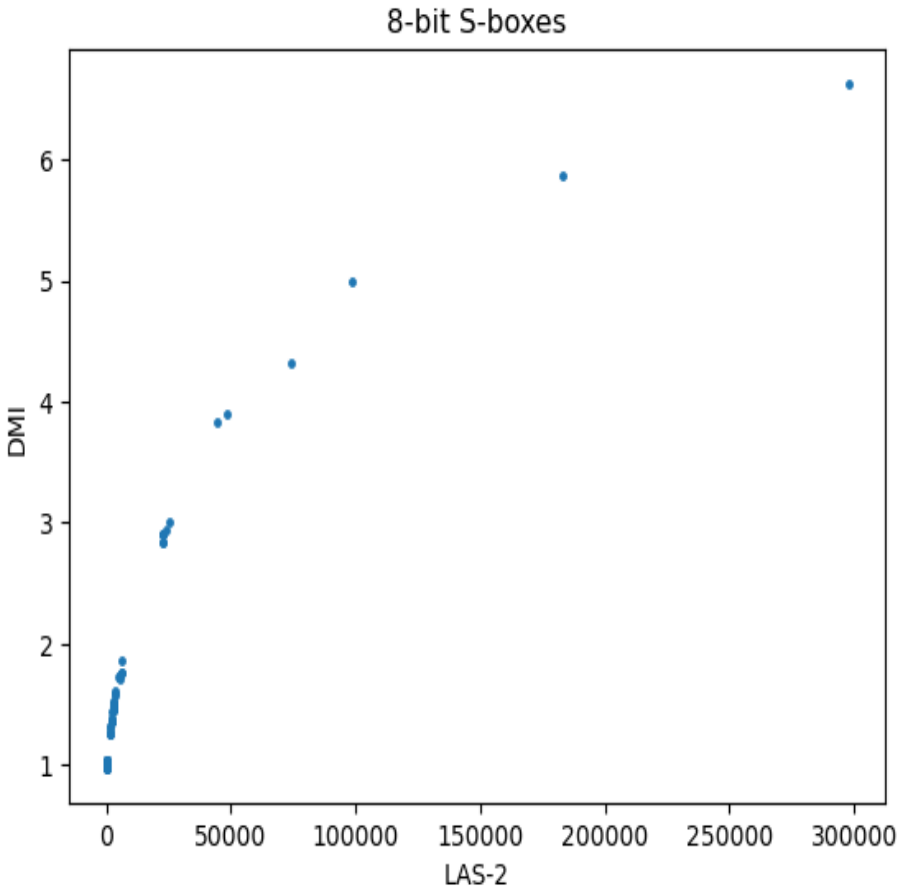

Fig. 3. DMI and LAS-2 count for a selection of 8-bit S-boxes. See also Fig. 17 page 68.

Here what we show is very incomplete: there exist vast quantities of 8-bit S-boxes and most such S-boxes have never been studied.

2.10 On Enumeration of LAS-2 Properties

A summary of all LSS properties of all sizes in existence for Ascon and Keccak can be found in Table 11 page 29. For AES it is in Table 13 page 30.

3 The DDT Connection

The idea of how DDT (Differential Distribution Table) reveals some affine spaces and their transformations is not new, see Section 5.2 in [GJNQSS16]. This paper explores further connections between LAS, LSS and DDT. Following [QSMG17] we define:

Definition 12 (DDT Sets). *Let S be an S-box on k bits. We define the following set:*

$$DDT(\delta_{in}, \delta_{ou}) = \{x \in F_2^k | S[x] \oplus S[x \oplus \delta_{in}] = \delta_{ou}\}$$

In Observation 2 in [QSMG17] the authors already show that the sets of points of type $DDT(\delta_{in}, \delta_{ou})$ are sometimes LAS of size 4 and are affine spaces, sometimes they are of size 8 and not affine spaces, and contain LAS as subsets. This question is also studied in Proposition 1/2 in [Sun24] In addition we also observed that in many cases DDT sets of size 8 are perfect disjoint unions of two LAS of size 4.

3.1 On Translations of Sets

It is possible to see that in S-boxes generated at random, the sets of type $DDT(\delta_{in}, \delta_{ou})$ behave like random sets which are stable by a translation by δ_{in} and these sets are **not** affine spaces typically (cf. Section 5.2.3. in [GJNQSS16]). However with Ascon or Keccak S-box we get some very special situations. For example in Section 3.4 we will discover that all sets of the form $DDT(0x10, \delta_{ou})$ are disjoint cosets and translations of each other by a constant, and they also fix some bits on both sides in a regular and systematic way.

3.2 On Disjoint Sets of δ_{ou}

We observe that in Ascon and in Keccak there are exactly 20 sets of type $DDT(\delta_{in}, \delta_{ou})$ which are of size 8 and are "behind" each 8 which appears in the DDT table of Ascon. The set of δ_{in} for which this happens contains 5 elements, and there are 5 lines in the DDT table which contain four 8's each. All the output differences are also disjoint across all the four entries in each of 5 lines: the set of 20 δ_{ou} which are used is a set of 20 distinct elements of F_2^5 which is a bit surprising. All δ_{in} for which we obtain have the "undisturbed bits" UDB property extensively studied by Tezcan in multiple papers for Ascon [TeAs16,TeDi19] and other ciphers [MaTe14].

3.3 On Dillon's Sums and Simplified Dillon's Sums

In this paper we denote by Di-size the size of the Dillon's set:

$$\text{Di-size} \stackrel{def}{=} |\{F[x] \oplus F[y] \oplus F[z] \oplus F[x \oplus y \oplus z]\}|,$$

cf. [AbCaVi23]. All very high quality S-boxes have Di-size=32 including APNs and many more, cf. [AbCaVi23]. Conversely any S-box with Di-size being less than 32 should be considered as weak and rather unsuitable for cryptographic use. Dillon sets are sums over affine spaces of dimension 2. As the main point in this paper is that we do NOT need to work with affine spaces, it is also useful to define:

$$\text{Ci-size} \stackrel{def}{=} |\{F[x] \oplus F[y] \oplus F[x \oplus y]\}|,$$

which size is frequently the same as Di-size defined above and both are at or beyond the saturation point (equal to 2^k) for all "good" bijective cryptographic S-boxes (e.g. Ascon, Keccak, AES, etc). These quantities are related to the so called polar form cf. [FeHAPa21,PaGo98] and the study of 2nd order differentials.

3.4 On DDT and LSS Sets Which Fix Some Bits

Definition 13 (ioab). *We say that a set of points $V \subset F_2^k$ is ioab for two integers a, b if for every x, y with $y = S[x]$ there are a bits out of 5 fixed on input side and there are b bits out of 5 fixed on the output side.*

Example 1: We consider one of the LAS sets of size 4 already studied in [QSMG17] which is 13,14,29,30↦3,6,10,15 and which has a property which can be summarized as ?11??=>0??1? in binary with 4 bits being fixed. This set is of type io22: two input bits are fixed at input side, and two output bits are fixed at output side, this simultaneously for all 4 points and their transformations $y = S[x]$. In the same way the set 9,10,25,26↦5,8,12,1 has property ?10??=>0??0? and is of type io22 also.

Example 2: Finally the union of the two sets listed above, which is exactly: $\{9, 10, 13, 14, 25, 26, 29, 30\}$ is of type io11. Moreover this set of 8 values is always in the form (in binary):

$$C_8 = \{9, 10, 13, 14, 25, 26, 29, 30\} \quad ?1x?? => 0??x?$$

where x is the same value, x =0, or 1 on both sides. Moreover all possible translations of this set $C_8 \oplus b$ are also always of type io11, and they are always of the form $DDT(0x10, \delta_{ou})$. Moreover these sets form 4 disjoint cosets with 4 distinct $\delta_{ou} \in \{0x9, 0xB, 0x1A, 0x18\}$ covering the whole space of 32 points. This type of translation uniformity is quite surprising.

Example 3: By exhaustive enumeration we found that there exists exactly 8 sets of type LSS-7 which are io11. For example the set $\{0, 5, 9, 13, 20, 24, 28\}$ which is of type ???0? =>?0??? or set $\{1, 4, 8, 12, 21, 25, 29\}$ which is of type ???0? =>?1???.

Example 4: In Keccak S-box and for χ functions of any size there is a plethora of situations where the χ function fixes some bits and these bits remain

invariant for any number of iterations χ^k and tend to be the same bits on both sides, and will even take the same exact values on both sides (!). This is a typical situation with χ functions and this happens essentially all the time, see Section 5 in [ScDa24], and we see a glimpse of this later in Fig. 5 in page 28.

Applications in Cryptanalysis: It is easy to see that if all Ascon S-boxes fix some bit say $y_i = C$ at output of one round, then [avoiding the action of round constants] at least 56 S-boxes in the next round will have $x_i = C$ at the input of next round, for any constant C. This is due to the fact that 0^{64} and 1^{64} are invariants for all five linear diffusion layer permutations used in Ascon. If in addition, our set of say 4 or 7 points is stable by XOR with 4=0x4, then the action of round constants is neutral and has no effect.

On this basis the attacker can try to construct probabilistic invariant properties on full Ascon, which is very difficult task with a lot of coding and processing of large datasets for favorable events, falling outside the scope of this paper. Numerous examples of how attacker can combine and extend similar partial linearization properties for several rounds can be found in the cryprographic literature cf. [HaFu10] and many more.

3.5 Examples of DDT

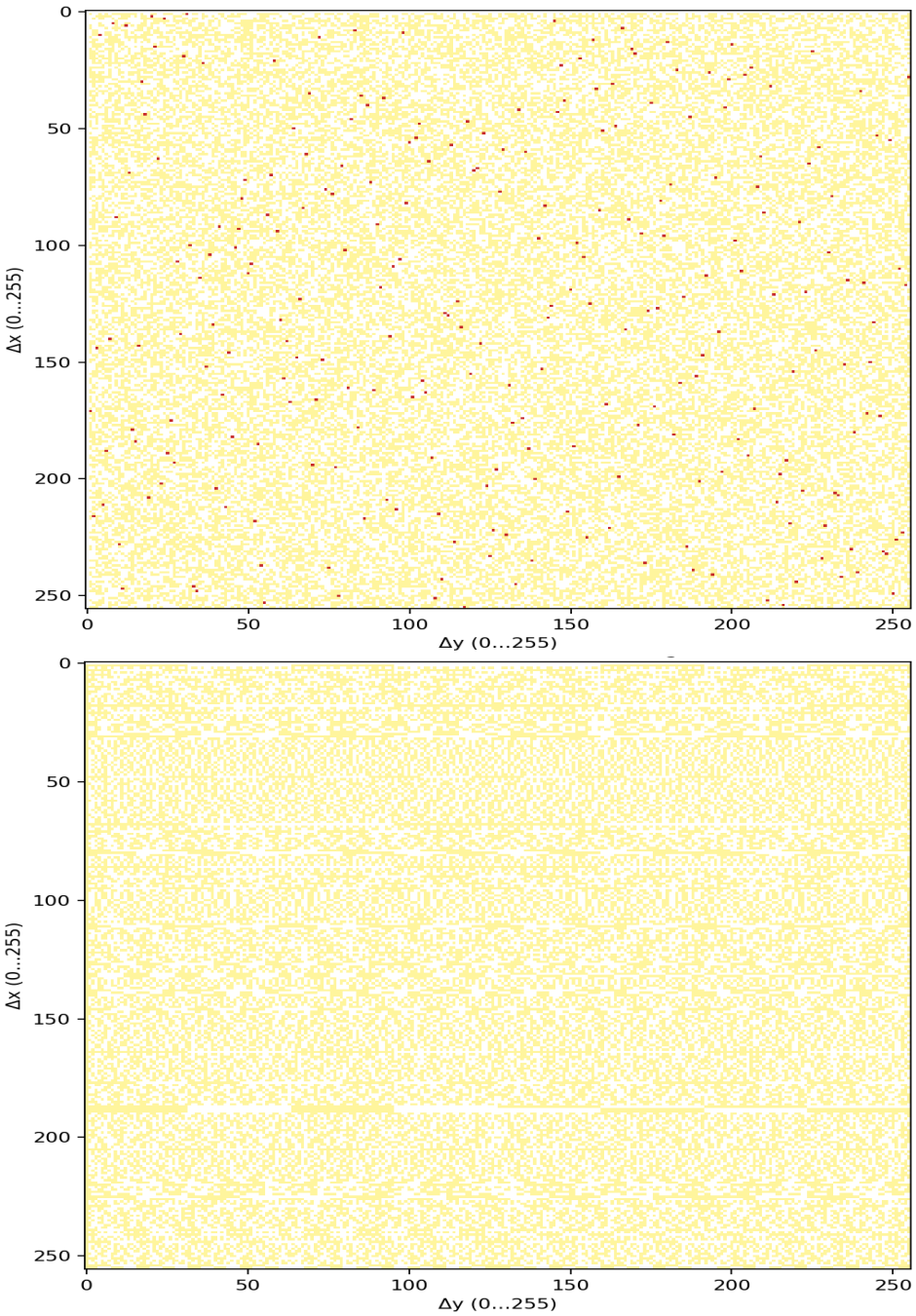

Fig. 4. DDT of the AES S-box vs. x^9 Gold exponent, with 2s in yellow and 4s in red.

4 Pairs of Hyperplanes in LAT and Related Sets

We would like to discover different ways how the Ascon can be partly linear in the spirit of and generalizing our notion of LAS. Before we study arbitrary sets of points, we will first discover many interesting sets by an indirect method. We work in the spirit of LAT (Linear Approximation Table), and we will consider all possible pairs of hyperplanes: (maximum dimension affine spaces). It was already noted in [QSMG17] that we cannot hope that Ascon S-box maps affine spaces of dimension 3 to a space of dim 3. By extension we can hardly hope that this will happen with hyperplanes or at maximal dimension 4, and nothing like this was ever observed for any major cryptographic S-box. Therefore we must consider a question of more probabilistic or approximative nature:

Key Question: Is it possible that an S-box sends **a "large" subset** of some (maximum size) affine space of dim 4 at the input side, to a "large" subset of another affine space of dim 4 at the output side?

To see this we consider an arbitrary linear space L^i with 16 elements on the input side; and an arbitrary linear space L^o with 16 elements which is intended to be used on the output side. This is related to the well known concepts of LAT tables and Walsh coefficients. In particular we have Walsh Zeros (WZ) which are simply zeroes in LAT tables (sometimes called correlation tables). We define:

$$L^i = \{x \mid \alpha \cdot x = 0\} \quad \textit{and} \quad L^o = \{y \mid \beta \cdot y = 0\}$$

where $\cdot$ is the dot product. There are $961 = 31^2$ possible pairs of linear spaces L^i, L^o. This corresponds to both α, β being non-zero, in other terms we discard the first line and the first column of any LAT table as being degenerated.

In what follows let S^{-1} be the inverse of the Ascon S-box. We will denote the complement by the following notation: $co - X = F_2^5 \setminus X$.

Definition 14 (LAT Hyperplane Pair Approximation Test).
For any **bijective** *S-box $S : F_2^k \to F_2^k$ we consider the following natural partitioning of 2^k elements into four sets:*

$$\begin{array}{cccc} S^{-1}[L^o] \cap L^i & \cup & S^{-1}[L^o] \cap co - L^i & \cup \\ co - S^{-1}[L^o] \cap L^i & \cup & co - S^{-1}[L^o] \cap co - L^i & \end{array}$$

and we generate these for sets for all possible pairs of hyperplanes $L^i = \{x \mid \alpha \cdot x = 0\}$ and $L^o = \{y \mid \beta \cdot y = 0\}$ with $\alpha \neq 0$ and $\beta \neq 0$.

In this paper the S-boxes studied are always bijective and $k = 5$. In this case these 4 sets are then a **disjoint** partitioning of 32 elements into 4 sets (which is not true in general). It is then easy to see that the sum of sizes in each line is $2^{k-1} = 16$ and the same is true for each column. Therefore the cardinals of each two sets lying opposite on each diagonal are identical [again because S is bijective].

Let $s_{00}, 16 - s_{00}$ be the sizes of sets defined above in the 1st line, cf. Def. 14. Then in the second line the are swapped: we have $16 - s_{00}, s_{00}$ elements. Then

the number of solutions to the LIO equation $\alpha \cdot x + \beta \cdot y = 0$ is $2s_{00}$ by the union of $S^{-1}[L^o] \cap L^i$ and the other corner with both sets complemented. Likewise the number of solutions to a negated LIO equation namely $\alpha \cdot x + \beta \cdot y = 1$ will be in general $2^k - 2s_{00}$ and both equations are represented typically by the same unique entry in LAT table. We recall that most authors define LAT in terms of relative numbers or "biases" and our numbers are shifted by 2^{k-1} accordingly and therefore we have:

$$LAT(\alpha, \beta) = 2s_{00} - 2^{k-1}$$

Example with Ascon. For example with Ascon S-box we have 336 entries out of 961 with a partitioning of type 6+10+6+10 or vice versa. This means that in LAT we will see 336 entries of type $16 \pm 4 = 2 \cdot 6$ or $2 \cdot 10$ respectively.

Note. We rediscover a well-known fact that for a bijective S-box, all entries in the LAT table are even. In this paper we emphasize the fact that our pairs of sets of the same size are frequently (but not always) more deeply related: by a translation with a constant.

Observations. It is possible to see that 6+10 situations or $LAT(\alpha, \beta) = \pm 4$ are the most common case in all cryptographic S-boxes at size 5, and other situations are less frequent in any cryptographic S-box. We cannot really hope to have a split of type 0+16 or we would have an extremely weak cipher. We also observe very high frequencies of type 8+8 which cases are perfectly balanced (these pairs of masks α, β are sometimes called Walsh Zeros WZ). They correspond to zeros inside an LAT table, and they do not lead to any linear bias whatsoever. The situations of type 4+12 are less frequent and correspond to stronger biases with $LAT(\alpha, \beta) = \pm 8$ and happen 40 times out of 961 with Ascon S-box. There are also other possibilities such as 5+11 which do not happen in Ascon/Keccak or other quadratic S-boxes studied, yet they do happen for other S-boxes on 5 bits. In the Table 4 we report how all possible 961 pairs of affine spaces of Dim 4 lead to a variety of situations in Ascon and Keccak.

4.1 Unexpected Properties and Translations

In our LAT-based process defined in Definition 14, we always generate pairs of sets of the same size. For example we consider $S^{-1}[L^o] \cap L^i$ and $co - S^{-1}[L^o] \cap co - L^i$. Can these two (disjoint) sets be further more deeply related to each other, for example by a translation (XOR) with a constant?

For example in one case out of 961 with $\alpha = 20$ and $\beta = 11$ we have:

$$\begin{array}{rcl}
S^{-1}[L^o] \cap L^i & = & 0,3,20,21,29,31 \mapsto 0,4,10,13,20,23 \\
S^{-1}[L^o] \cap co - L^i & = & 4,6,12,13,16,17,18,19,24,27 \mapsto 3,7,9,14,16,19,25,26,29,30 \\
co - S^{-1}[L^o] \cap L^i & = & 1,2,8,9,10,11,22,23,28,30 \mapsto 5,8,11,15,17,18,22,24,27,31 \\
co - S^{-1}[L^o] \cap co - L^i & = & 5,7,14,15,25,26 \mapsto 1,2,6,12,21,28
\end{array}$$

It important to see that if these two sets with here $s_{00} = 6$ are related by translation, then also the two other complementary sets of sizes $16 - s_{00} = 10$ are also and always related by the same translation. This is due to the fact that we complement (in several ways) inside affine spaces on two sides, and affine

spaces are more likely to be stable by certain (but not all) translations. Can this be guaranteed to work? Yes, and we have the following result:

Theorem 1. *We assume that if we translate the 1st set $S^{-1}[L^o] \cap L^i$ by a constant C we get the last set $co - S^{-1}[L^o] \cap co - L^i$, then the two remaining sets on the other diagonal are also related by translation with the same constant C.*

Proof. Our translation C is by definition a XOR of one element of L^i with parity=0 and one from $co - L^i$ with parity=1, therefore we have parity=1 and $C \in co - L^i$. Likewise we also have $C \in co - S^{-1}[L^o]$. Therefore our constant belongs to the last set: $C \in co - S^{-1}[L^o] \cap co - L^i$. We define S by adding C to all elements from the second set $S^{-1}[L^o] \cap co - L^i$, in S we always swap the parity on both sides shifting to the other coset of only two, so $S \subseteq co - S^{-1}[L^o] \cap L^i$. Now we know that the sets are of the same cardinality and so $S = co - S^{-1}[L^o] \cap L^i$. This ends the proof. □

4.2 On Sets Related to LAT in Ascon and Other S-boxes

It is easy to see in this process we obtain $4 \cdot 585 + 2 \cdot (336 + 40) = 3092$ distinct sets of points of sizes 4,6,8,10,12. We start by two essential observations.

Fact 4.1. For all quadratic S-boxes on 5 and 6 bits, and for all sizes, all inputs in all four LAT-related sets V add to zero at input side, i.e. $\sum_{in}(V) = 0$.

Counter-Examples. This is not true in AES and for 5-bit S-boxes which are not quadratic, e.g. Icepole [IceP14].

Can Our Sets Be LSS? We conjecture that for S-boxes which are not quadratic no set of any size generated as above are LSS, not even at size 4 (the easiest). It might seem that many sets of 4 will form an LSS property, and the four I/O pairs can be interpolated by linear algebra in several ways (pairs, space of dimension 5). However this is not true, the system of equations could be contradictory and they systematically are with cryptographic S-boxes. More precisely, all our sets of size 4 for quadratic S-boxes are such that the sum (XOR) of 4 inputs $\sum_{in}(V)$ is zero, and the sum of 4 outputs $\sum_{ou}(V) = 0$ is never zero. This is true in Ascon/Keccak and for all quadratic S-boxes of size 5 where the inverse is not quadratic, following Table 2 in [BoBiSa17]. Then it is impossible to obtain an LSS, because an affine space should be sent to an affine space by our affine transformation with matrix+vector.

In some cases our set of 4 points **can** be LSS, for example when the S-box is quadratic and its inverse is quadratic. This happens for example with S-box class 17 from Table 1 in [BoBiSa17] all of which are quite weak. In this case all 96 sets of sizes 4,6 generated are LSS, and 496 out of 2204 sets of size 8 generated are LSS. Then at size 10 no sets are LSS even though plenty of LSS-10 properties exist for this S-box, which can be easily interpolated from our later Table 17. For Ascon, at size 12 we cannot hope to find an LSS-12 property for this S-box, they simply do not exist, cf. Thm. 2 page 24. So far we never observed an example where a LAT-related sets of size 4, 6 or 8 would be LSS for any "strong" or cryptographically significant S-box.

Corollary and Observation. There is no intersection between two remarkable sets of 80 properties on 80 points: 1) the 80 sets of 4 which are LAS-2 properties studied at Eurocrypt 2017 cf. [QSMG17], and which are also LSS-4, and the 80 sets of size 4 found for Ascon related to LAT and which are all possible sets of the form $S^{-1}[L^o] \cap L^i$ or similar of size 4. All these 80+80 properties with 4 points are disjoint: $\sum_{ou}(V) = 0$ for the firtst 80 and never zero for the other 80.

Cases where $\sum_{ou}(V) = 0$. We observed that for all quadratic S-boxes such that the inverse is not quadratic, outputs for all sets of size 4, 6 almost never add to zero at output side. For example in Ascon at size 6 there are only 2 such examples out of 672 which is again our special unique pair $\alpha = 20 = \rho_{in}[31]$ and $\beta = 11 = \rho_{ou}^{-1}[31]$ where $\sum_{ou}(V) = 0$.

On Entropy of $\sum_{ou}(V)$. We observed that only some specific output differences happen for various sets of size 4 or 6. For example there are 80 sets of size 4 and the output side sum is always $y \in \{2, 6, 8, 17, 24\}$ which are also exactly those 5 where the output of the Keccak S-box has HW=1, i.e. $T_{ou}^{-1}[y] \in \{1, 2, 4, 8, 16\}$, and each of these 5 values is taken 16 times.

On $\sum_{ou}(V)$ with LSS In Section 8.5 we find 160 examples of LSS-10 where $\sum_{ou}(V) = 0$ is never zero. In contrast with our 32 maximal size shifted $A_{11} \oplus x$ properties LSS-11 all values $\sum_{in}(V)$ and $\sum_{ou}(V)$ are equally probable.

Relevance of $\sum_{ou}(V) = 0$ in Cryptanalysis. In general, rare cases where $\sum_{ou}(V) = 0$ are those which will have an interest in cryptanalysis for various zero sum attacks, cube distinguishers and many related concepts, see for example [HuCu24].

all	8+8	6+10	4+12	0+16
961	585	336	40	0

Table 4. Ascon S-box interacting with 961 pairs of spaces of dim 4 related to LAT

4.3 On Rotation Invariance with our Partitions

A distinct questions is, does it ever happen that two sets of the same size are related by rotations such as ρ_{in}, cf. Table 1 or their powers? This never happened for any of 961 cases studied.

Another question is if a rotation of one set can produce another "twin" set of the same size with a different pair of α, β. This happens systematically for all sets generated no matter their size due to the fact that rotations transform hyperplanes into hyperplanes. This mapping has some fixed points. In our example above, we had $S^{-1}[L^o] \cap L^i = 0, 3, 20, 21, 29, 31$ etc, and this is a very special example. In this case, all our four sets of size 6 and 10 are stable by rotations ρ_{in} and are mapped to themselves. By inspection we verified that this type of

internal rotational symmetry happens just once in this exact case known as B in Section 5, where $\alpha = 20 = \rho_{in}[31]$ and $\beta = 11 = \rho_{ou}^{-1}[31]$.

Remark. Many LSS properties are stable by rotations, cf. Section 8.5. In Ascon all proper LSS-10 properties are freely translatable by translations and rotations, while LSS-11 properties are not: In Section 8.5 we read for most which is 30 out of 32 properties of size 11, if we apply an "internal" rotation function of type ρ_{in}, we do NOT get another LSS property of size 11, which are also those 30/32 cases where matrices have rank 4 not 5, cf. Section 7.

4.4 Translation Similarities

We have tried all 961 possibilities and found that in balanced WZ cases 8+8, sometimes we get related sets of 8 points, and in **all** unbalanced cases 6+10 or 4+12 we **always** get related sets without any exception. We call these pairs WN: they are Walsh Not-Zeroes. We will call WNT those which are input-shifted in our table.

all	8+8	6+10	4+12	0+16
961	585	336	40	0
inp-shifted	240	**336**	**40**	0
unrelated	345	0	0	0

Table 5. Pairs spaces of the same size which are equivalent by translation with Ascon.

This is a bizarre situation like winning in a game of heads/tails in 336+40 cases in a row, which are exactly those WN cases which are **unbalanced** like 6+10, and therefore are all potentially cryptographically significant. In fact however this property is not specific to Ascon. Similar things happen to all other quadratic S-boxes, cf. Table 20. In contrast it is important to see that this does not happen **at all** for random S-boxes, not even with a low frequency, in other terms we switch from 100% to 0% for all unbalanced WN pairs, as we will see in Table 7 below in page 21. First we look at what happens in Icepole, when the S-box was altered slightly by adding just two high degree products, cf. [IceP14], which destroys overall roughly about half of our translation properties as shown in Table 6 below.

Icepole	8+8	7+9	6+10	5+11	4+12	0+16
961	435	220	240	36	30	0
inp-shifted	120	10	160	6	20	0
unrelated	315	210	80	30	10	0

Table 6. Pairs of spaces equivalent by translation with Icepole S-box.

Finally we see that no weakness whatsoever is observed for Thakor, which has DMI close to that of a typical random S-box, cf. Fig. 1.

Thakor	8+8	7+9	6+10	5+11	4+12	0+16
961	270	420	196	60	15	0
inp-shifted	**0**	**0**	**0**	**0**	**0**	**0**
unrelated	270	420	196	60	15	0

Table 7. Pairs spaces of the same size equivalent by translation with Thakor S-box.

In spite of random S-boxes being devoid of this type of vulnerability, in Table 20 page 20 we show that most or all cryptographically significant quadratic S-boxes have significant translation properties of this type.

4.5 Output Side Translations

We also note that nothing like this happens in a comparably strong proportion at the output side of the Ascon S-box, cf. Table 8 below.

all	8+8	6+10	4+12	0+16
961	585	336	40	0
out-shifted	60	80	40	0
unrelated	525	256	40	0

Table 8. Equivalence by translation at the output side with Ascon.

Equivalence by translation is a one-sided property. In the same way our 80 LAS-2 properties and other LSS properties cf. Thm. 3 are completely invariant by arbitrary translations at the input side, and almost not at all related to each other at the output side.

In this paper there is a strong asymmetry between a great abundance of input side symmetries and translations, and a severe scarcity of output side properties of any type. This observation holds for essentially all quadratic S-boxes which are not too weak and the inverse is not quadratic, not only for Ascon and Keccak-like S-boxes, cf. Thm. 3, Table 5, Table 20 and Table 22, multiple properties concerning I/O sums in Section 4.1 and Section 8.5, and many many other.

5 On Discovery of New I/O Partitioning Properties

We have examined all the 961 possible pairs of dim 4 spaces, and looked at various sets of sizes 4,6,8,10,12 to see if some of them achieve some particularly large numbers of simultaneous LIO relations.

cases	dim	LIO-2	LIO-3	LIO-4	LIO-5	LIO-6	LIO-7
LIO	#	3	7	15	31	63	127
4 from 4+12	40+40	0	0	0	0	0	80
6 from 6+10	336+336	0	0		670	2	0
8 from 8+8	$4 \cdot 585$	0	700	1260	360	20	0
10 from 6+10	336+336	272	320	80	0	0	0

Table 9. Different LIO dimensions in all 961 pairs of spaces tried.

By brute force enumeration of 961 spaces we discover that there exist exactly 22 pairs of dim 4 linear spaces with as many as 63 simultaneous LIO relations. These 22 selected simultaneous linear approximations of Ascon split into two categories:

A 10 pairs L^i, L^o of balanced WZ type 8+8 already studied in [QSMG17] without realizing that we obtain a large number of 63 LIO relations at size 8.
B 1 unique special (not previously studied) unbalanced WN pair of L^i, L^o of type 6+10 with also 63 LIO albeit at size 6.

Observations on Set A of $8 \cdot 8$ points. For each of 10 pairs L^i, L^o we get two sets of 8 points. These 20 sets of points are **exactly those** which are "contained" in the 20 Ascon DDT table entries equal to 8, and all of which can be decomposed in 6 different ways as a disjoint (fact not noticed or not emphasized before) union of one of 80 possible LAS (see Observation 1 in [QSMG17]). This set of 20 sets of 8 points from DDT() is stable by arbitrary translations by a constant. It is also stable by arbitrary rotations (!).

Observations on Set B of $6+6$ points. On the surface, the latter case is just 1 of 336 of "inp-shifted" cases reported above with $\{0, 3, 20, 21, 29, 31\}$ or the same set XORed with 26 generated from the same pair L^i, L^o. This partitioning is however quite special and will also be relevant in few other places as we will see below. We have $L^i_B = \{0, 1, 2, 3, 8, 9, 10, 11, 20, 21, 22, 23, 28, 29, 30, 31\}$ and $\alpha = 20 = \rho_{in}[31]$ and
$L^o_B = \{0, 3, 4, 7, 9, 10, 13, 14, 16, 19, 20, 23, 25, 26, 29, 30\}$ and $\beta = 11 = \rho^{-1}_{ou}[31]$.

6 Do Stronger Properties Exist?

We are now going to show a surprising result: it is possible to achieve 31 LIO (dimension 5) with 11 points (larger than ever before) and this is the best we can hope for with Ascon. Furthermore we will show that all our LIO-5 properties of size 11 are also LSS-11, and that additional strong translation symmetry properties hold.

We have used a SAT solver coding which allows to enumerate **all** possible solution to this problem and show that no other solutions exist and that solutions of size 12 or better do not exist. We found that exactly 32 solutions exist and that they come in two pairs of 11 disjoint points, forming 16 joint dual LIO-5 approximations of Ascon spanning 22 out of 32 points. In order to study these properties we define:

$$A_{11} = \{0, 3, 4, 12, 16, 17, 19, 20, 21, 29, 31\}$$

This set will be later also called $s^i + t^i + \{0\}$ and it forms an LSS-11 property for Ascon. We further define a special well-chosen dual or translation of A_{11} which will be called $B_{11} = A_{11} \oplus 0x1A$, which is the same as shifting by 26 in decimal. It is easy to see that $B_{11} = \{5, 7, 9, 10, 11, 14, 15, 22, 25, 26, 30\}$ which set will sometimes also be called $r^i + q^i + \{26\}$ later, cf. Table 15 in page 38.

Disjoint Sets. We observe that A_{11} and B_{11} are disjoint, which is not obvious at all, as these sets are not affine spaces and the maximum size of intersection of any of these sets with any affine space of Dim 4 is at most 8 not 16. However at the output side, these sets are contained inside an affine space and its complement, which provides an easy explanation why they are disjoint. Moreover this special L^o is **precisely** from the one special and unique case obtained above out of 961 and called "Set B", with 63 LIO and $\alpha = 20 = \rho_{in}[31]$ and $\beta = 11 = \rho_{ou}^{-1}[31]$. We have $L_B^o = \{0, 3, 4, 7, 9, 10, 13, 14, 16, 19, 20, 23, 25, 26, 29, 30\}$ and $S[A_{11}] = \{0, 4, 10, 13, 14, 19, 20, 23, 26, 29, 30\}$.

New LSS-11 Properties. Now we observe that all our $32 = 16 \cdot 2$ properties of size 11 are also LSS-11, which is not obvious at all here, and is not true in general for other LIO-5 properties. To better visualize these properties, we list only those 5 out of our 31 LIO properties which use exactly one output bit.

- $A_{11} = 0, 3, 4, 12, 16, 17, 19, 20, 21, 29, 31$ $\quad B|1\ 1C|2\ F|4\ 17|8\ 16|10$
- $B_{11} = 5, 7, 9, 10, 11, 14, 15, 22, 25, 26, 30$ $\quad 12|1\ F|2\ E|4\ 1D|8\ B|10$

Translation Invariance. It is important to see that the fact that we found 32 properties is not accidental. In fact our property is invariant by arbitrary input-side translation by a constant (but not on the output side) as we will see later, see Thm. 3 below page 32. Moreover the 32 properties here work in pairs, we get 16 pairs of disjoint 11+11 points obtained by translation from A_{11}, B_{11} which all give pairs of distinct LSS-11 approximations with disjoint supports. These properties span uniformly the whole space, and for different configurations of points there will be typically several way to approximate them using these properties.

To summarize here is our strongest known Ascon/Keccak simultaneous approximation LSS-11 result:

Theorem 2 (Ascon Family of LSS-11 Properties). *There exists exactly 32 sets of size 11 such that the Ascon S-box is fully linearized on each set of 11 points under the form*

$$S[x] = A \cdot x + c \quad \forall x \in V$$

which are all of the form $A_{11} \oplus b$ *where*
$A_{11} = \{0, 3, 4, 12, 16, 17, 19, 20, 21, 29, 31\}$ *and no solution exists with 12 points. All solutions at size 11 are isomorphic by a translation with a constant.*

Proof. We have obtained this result by 2 methods. First by an automated proof with a SAT solver returning UNSAT or enumerating 32 solutions. We have used the CryptoMiniSAT 5.8 solver developed by Mate Soos. We have also obtained the same result with a brute force method coded on GPU. A summary of all LSS properties of all sizes in existence can be found in Table 11 page 29. □

Output Side Translations. Nothing even remotely similar happens for output side translations. All 1024 output side translations of all 32 set of type $A_{11} \oplus s$ are distinct sets of 11 points.

On Cryptographic Quality of LIO Equations. The 5 equations we listed above e.g. $B|1$ $D|4$ etc, could be qualified as BO or Bad Output, and Keccak is in this respect substantially weaker than Ascon, see notions of BIBO for DDT and LUT tables and comparison in Table 8 in [LMCFW23].

Do Better S-boxes Exist? If we replace the Ascon S-box by Fides S-box which is an APN, then the highest possible number of points to achieve the same type of full S-box linearization drops from 11 to 7, see Table 16 page 44.

On Inverse S-boxes. We have verified that the maximum size of LSS property with the inverse S-box for Ascon is LSS-10. This is related to the fact that in general cubic S-boxes are stronger and our LSS-11 properties are one-sided properties.

On LIO and S-boxes with Quadratic Inverses. We have also verified that quadratic S-boxes which have quadratic inverses achieve systematically bigger or worse LSS values, for example LSS-14 or even higher. This is true for all such S-boxes without any exceptions (!). Therefore ONLY quadratic permutations in size 5 which have a cubic inverse qualify for cryptographic applications. All the other are easily **disqualified**: for example they always have differentials with probability at least of at least 0.5=16/32, see Table 1 in [BoBiSa17]. Detailed results on maximum achievable LSS-sizes in different S-boxes will be later shown in Table 16 in page 44.

Applications. All the results in this paper apply also to Keccak S-box by a simple affine variable change. This with a notable exception of Section 3.4, where we look at some individual bits in Ascon.

7 Further Study of LSS-11 Properties Based on A_{11}

We give here two examples of matrices obtained in LSS-11 properties, for A_{11} and $B_{11} = A_{11} \oplus 26$. In the case of $A_{11} = \{0, 3, 4, 12, 16, 17, 19, 20, 21, 29, 31\}$:

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{bmatrix} 1\,0\,1\,1\,0 \\ 1\,0\,1\,1\,1 \\ 0\,1\,1\,1\,1 \\ 1\,1\,1\,0\,0 \\ 0\,1\,0\,1\,1 \end{bmatrix} \cdot \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

For example with input $4 = 00100 = x_0x_1x_2x_3x_4$ we get output $26 = 11010 = y_0y_1y_2y_3y_4$ in binary where x_4/y_4 represent the least significant bit.

In the case of $B_{11} = 11, 13, 15, 21, 22, 23, 26, 27, 29, 30, 31$ we have:

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{bmatrix} 0\,1\,0\,1\,1 \\ 1\,1\,1\,0\,1 \\ 0\,1\,1\,1\,0 \\ 0\,1\,1\,1\,1 \\ 1\,0\,0\,1\,0 \end{bmatrix} \cdot \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

For example with input $22 = 10110 = x_0x_1x_2x_3x_4$ we get output $17 = 10001 = y_0y_1y_2y_3y_4$ in binary where x_4/y_4 represent the least significant bit.

On Rank of LSS Matrices. We have verified that for all affine shifts of our LSS-11 property, which are always of the form $A_{11} \oplus x$, the matrices are invertible in 30/32 cases for all $x \neq 0$ and $x \neq 26$. In 2/32 cases which are exactly A_{11} and B_{11} for which the matrices are shown above, the rank drops to 4. It is a bit surprising to discover that the right kernel space of each matrix is the same and contains exactly one non-zero element which is in both cases the same and equal to 26.

Further Observations. In general the 32 matrices we find here are not exactly random matrices and live in a space of a small dimension, see Section 8.3 below. Constant parts do not behave like random numbers either. In Section 8.4 we will discover that in half or 16/32 of these approximations of type $A_{11} \oplus x$, the constant part on 5 bits is the same and equal to $4 = 00100$.

7.1 Interaction of A_{11} and DDT Sets

The reader might think that the 11 points are related to DDT() sets, and have a deep connection to differential cryptanalysis, while by its origin it was derived from linear cryptanalysis. This is what we are going to show now. Initially, research for some DDT set connection seems to fail very badly. We have checked that the set of 11 and their 32 translations do NOT contain any of the total 2594 sets of 4,6,8 or 10 points studied in Section 5, nor one would be included inside any one set of 12 points obtained in this process. However we discovered that very interesting things happen when we study how these sets interact with themselves (their own affine shifts):

Observations on Self Similarity of A_{11}. We found that:

1. The intersection of A_{11} with another variant $A_{11} \oplus x$ is always not empty except for $B_{11} = A_{11} \oplus 0x1A$, when $x = 26 = 0x1A$.
2. The intersection of A_{11} with another variant $A_{11} \oplus x$ is of maximum size 6 in exactly 5 cases where $x \in \{4, 12, 16, 17, 19\}$ which set will later be called t, which are exactly five of those lines δ_{in} in the DDT of Ascon/Keccak S-box which contain 8, and which are therefore exactly all the input differentials in Ascon of maximum strength 8/32. These 5 δ_{in} also amount to half of 10 "undisturbed bits" UDB properties in [TeAs16,TeDi19].
3. The intersection of A_{11} with another variant $A_{11} \oplus x$ is of size 4 when $x \in \{1, 2, 3, 5, 7, 8, 14, 15, 20, 21, 23, 25, 28, 29, 31\}$ which set will later be called $s \cup r \cup b$ and which 15 values are **exactly** those lines in DDT table which contain any 4 numbers.
4. The intersection of A_{11} with another variant $A_{11} \oplus x$ is of size 2 when $x \in \{6, 9, 10, 11, 13, 18, 22, 24, 27, 30\}$ which set will later be called $q \cup a$ which 10 values are **exactly** those lines in DDT table except for the special case of $x = 26 = 0x1A$ already used in B_{11}.

Remark on A_{11} and Differential Cryptanalysis. Each time when intersection of two A_{11} is of maximum size 6, we obtain a property which concerns 16/32 points and yet captures 6/8 of the points which might interest the attacker the most: those concerned by differentials true with maximum probability of 8/32. This question is further studied in Section 12 and leads to the study of the so called DMI profiles. This type of big interesection is expected to lead new types of differential-linear attacks, guess then determine collision finding attacks, or/and algebraic key recovery attacks, where the attacker makes several assumptions about input differences of several S-boxes, and these properties entail $2 \cdot 31$ additional linear I/O relations between 20 bits.

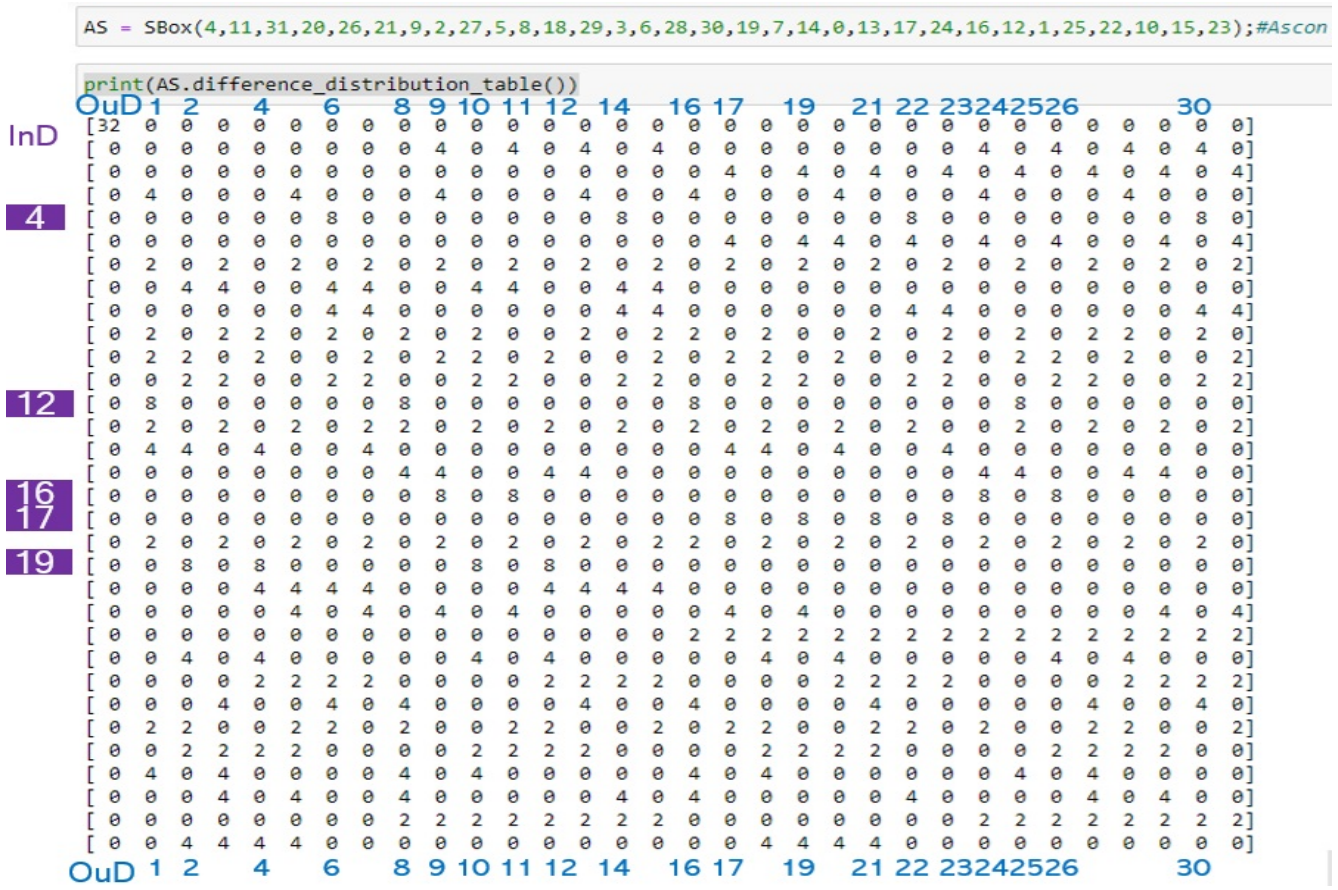

Table 10. The DDT table for Ascon with focus on 5x4 sets DDT() of size 8.

7.2 LSS-11 Properties in Keccak and General χ Mapping Theory

It is interesting to contemplate our property and all their shift in the native Keccak version. This provides further structural (and rotational symmetry) insights into these properties: all matrices become cyclic. In Keccak variant, we obtain the following two sets and matrices for A_{11} and B_{11} respectively. In the case of A_{11} we have: $S_i[A_{11}] = \{0, 1, 2, 4, 5, 8, 9, 10, 16, 18, 20\}$.

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{bmatrix} 1\,0\,1\,0\,0 \\ 0\,1\,0\,1\,0 \\ 0\,0\,1\,0\,1 \\ 1\,0\,0\,1\,0 \\ 0\,1\,0\,0\,1 \end{bmatrix} \cdot \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

For example with input $1 = 00001 = x0x1x2x3x4$ we get output $5 = 00101 = y0y1y2y3y4$ where $x4/y4$ represent the least significant bit.

In the case of B_{11} we have: $S_i[B_{11}] = \{11, 13, 15, 21, 22, 23, 26, 27, 29, 30, 31\}$.

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{bmatrix} 1\,1\,0\,0\,0 \\ 0\,1\,1\,0\,0 \\ 0\,0\,1\,1\,0 \\ 0\,0\,0\,1\,1 \\ 1\,0\,0\,0\,1 \end{bmatrix} \cdot \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

For example with input $23 = 10111 = x0x1x2x3x4$ we get output $7 = 00111 = y0y1y2y3y4$ in binary where $x4/y4$ represent the least significant bit.
Note: Constant part is always either $0^5, 31 = 1^5$ or a power of 2 with HW=1.

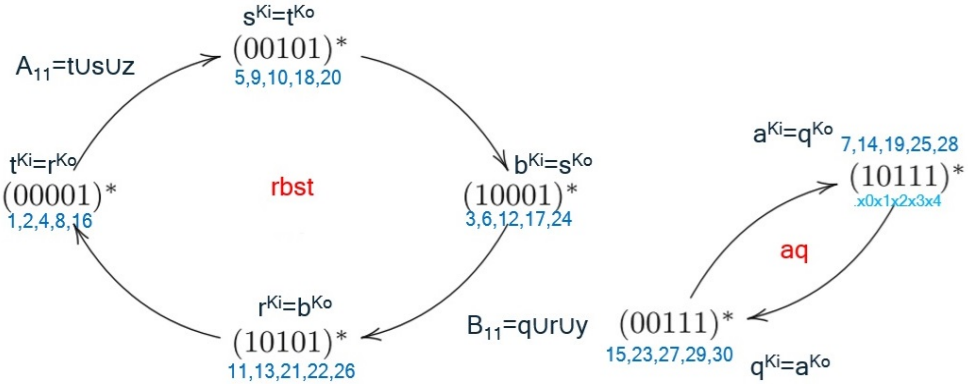

Fig. 5. Canonical states with Kecak S-box cf. [ScDa24] vs. notations of this paper.

Scalabilty of LSS. It might seem that the two cyclic matrices above will also work for larger odd sizes such as $k = 7, 9, \ldots$. They do, however they still produce just 11 solutions. This contrasts very much with what happens for even sizes. In Section 5.1 of [ScDa24] we discover that the χ S-box in even size k can be fully linearized by an affine function (in precise LSS sense) for as many as $2^{k/2}$ points. More about this in Section 8.6 where we conjecture that the maximum LSS size for χ_{2k+1} grows exponentially with k.

7.3 Counting LSS properties

The reader might wonder how many LSS exist. In the table below we count all LSS properties currently known to authors. For a majority of large size (or close to the maxium size of 11) among all LSS properties in existence, the matrix A involved is unique and invertible. There are however many exceptions to this rule, especially at smaller sizes. In this table and in subsequent table we do NOT display LSS properties U which are contained inside larger a LSS property $U \subsetneq V$ and which could be easily re-added by considering all possible subsets of any "proper" or "maximal" LSS property listed here.

size=	LSS-5	LSS-6	LSS-7	LSS-8	LSS-9	LSS-10	LSS-11
sets	0	131344	36032	7680	1760	160	32
rank(A)=5	0	49820	29372	5250	970	130	30
rank(A)<5	0	263202	38020	5470	790	30	2

Table 11. Counting LSS properties and matrices involved in LSS properties in Ascon/Keccak.

Some Observations. The smallest size for a proper LSS property in Ascon/Keccak is 6. This means that any LSS-5 set is always contained in some LSS-6 set.

size=	LSS-5	LSS-6	LSS-7	LSS-8
sets	0	349184	15872	0
matrices	0	367517	15872	0
rank(A)=5	0	173333	9672	0
rank(A)<5	0	194184	6200	0

Table 12. Counting maximal LSS properties in Fides-5 APN S-box.

7.4 Counting LSS properties in AES

For AES the table is quite similar, beta version, not final:

size=	LSS-15	LSS-16	LSS-17	LSS-18
sets V	$\geq$ 31965	$\geq$ 8710	568	15
$0 \in V$	$\geq$ 31534	$\geq$ 8630	568	15
matrices	$\geq$ 32226	$\geq$ 8769	568	15
with c=99	$\geq$ 31731	$\geq$ 8630	568	15
rank(A)=8	$\geq$ 21346	$\geq$ 7192	548	15
rank(A)=7	$\geq$ 9776	$\geq$ 407	0	0
rank(A)=6	$\geq$ 1112	$\geq$ 1197	0	0
rank(A)<6	0	0	0	0

Table 13. Counting LSS properties and matrices involved in LSS with AES S-box.

Key Observations. In AES an overwhelming proportion of large size LSS properties are of the type $F[x] = A \cdot x + b$ with one single value $b = 99$ dominating very strongly. Any other value $b \neq 99$ occurs very rarely. In AES an overwhelming proportion of large size LSS properties contain 0.

The smallest size for proper LSS property in AES S-box in 9. This means that any LSS-8 set is always contained in some LSS-9 set.

More Observations. All properties of size 18 are disjoint except for zero, i.e. their intersections always have size equal to 1 exactly. All properties in LSS-17 and LSS-18 always contain 0, and hence intersection of two LSS sets of sizes 17 or 18 is never empty. Apart from $x = 0$, the sets of non-zero points V involved in LSS properties are very diverse and very rarely similar to each other. An intersection of LSS-17 and LSS-18 is always tiny with size of at most 3. For intersections of two distinct LSS-17, it is a set of size 1 exactly for 40 % of the time and it has size 2 for another 32 % of pairs. The probability that two sets LSS-17 have a larger intersection with 7 more points is also very small, about 0.3 %. An intersection with 8 points or larger never happens.

7.5 Counting LSS properties in Quadratic APN

It is possible to see that for some APN, for example with x^9, we have LSS-18 properties which makes them not better than AES. For a majority of APNs it seems that Max(LSS)=15.

size=	LSS-15	LSS-16
sets	$\geq$ 14060	0
matrices	$\geq$ 14060	0
rank(M)=8	$\geq$ 3136	0
rank(M)=7	$\geq$ 7872	0
rank(M)=6	$\geq$ 2762	0
rank(M)<6	290	0

Table 14. Counting LSS properties and matrices involved in LSS for a quadratic APN BL3638 with DMI=1.01115.

8 A General Result for all Quadratic S-boxes

We recall that we call LSS or Linearizable Sub Set any set of points V such that

$$F[x] = A \cdot x + a \quad \forall x \in V.$$

It is possible to see that **all** quadratic S-boxes have abundant self-similarity properties.

Theorem 3 (Translation Invariance of LSS for All Quadratic S-boxes). *Let F be a quadratic S-box on k bits. If there exists a set V forming an LSS-m property for $F[]$ for some integer $|V| = m > 0$,* **then** *for any affine constant d the shifted set $V \oplus d$ also forms another distinct LSS-m property, i.e. there exists another matrix B and vector b such that:*

$$F[x] = B \cdot x + b \quad \forall x \in V \oplus d.$$

Proof. In mathematics, the process of "polarization" of a polynomial in produces a unique symmetric bi-linear homogeneous form, from which the original polynomial can be recovered. In the case of multivariate quadratic S-boxes, also known in the literature as Dembowski-Ostrom polynomials, researchers typically study their "difference function" cf. Section 8 in [FeHAPa21] or their "polar form" cf. Appendix 1 in [PaGo98]. In this paper our polynomials are not homogeneous quadratic and our polar form has a constant term, same as in Section 2.2. in [Be20]:

$$\Delta_F(x, d) = F[0] \oplus F[x] \oplus F[d] \oplus F[x \oplus d].$$

It is easy to show that our polar form is a bi-linear multivariate function in $k + k$ variables, and when one variable is fixed, it becomes a multivariate linear function in k variables, also known as a "linearized polynomial". These well-known facts are sufficient to prove our theorem. By the initial property we have $F[x] = A \cdot x + a \quad \forall x \in V$ and we are now going to replace $F[x]$ by $\Delta_F(x, d) \oplus F[0] \oplus F[d] \oplus F[x \oplus d]$ which gives:

$$F[x \oplus d] = (\Delta_F(x, d) \oplus A \cdot x) + (a \oplus F[0] \oplus F[d]) \quad \forall x \in V$$

which is the same as:

$$F[x] = (\Delta_F(x \oplus d, d) \oplus A \cdot (x \oplus d)) + (a \oplus F[0] \oplus F[d]) \quad \forall x \in V \oplus d$$

and given that when d is fixed our $\Delta_F(x, d)$ becomes linear, we obtain:

$$F[x] = (\Delta_F(x, d) \oplus A \cdot x) + (\Delta_F(d, d) \oplus F[0] \oplus a \oplus F[d] \oplus A \cdot d) \quad \forall x \in V \oplus d$$

which provides an explicit formula of type $F[x] = B \cdot x + b \quad \forall x \in V \oplus d$ which is exactly the explicit simultaneous linearization property we claimed. □

Applications of Thm. 3. Our similarity result of Thm. 3 holds for an overwhelming majority of 5-bit S-boxes used in applied cryptography including SHA-3, SHAKE and Ascon.

Corollary and Related Research. It follows that all the 80 of LAS-2 properties from Eurocrypt 2017 cf. [QSMG17] are also stable by arbitrary translations at the input side. Ascon S-box can be seen to be $(2, 5)$-linear in terms of Boura-Canteaut definition in [BoCa14].

Important Relaxation. In this paper we drop the requirement of sets of points V forming an affine space in [QSMG17,BoCa14], yet our LSS/LAS properties hold again and again for every coset of a given set or space respectively.

A more general therem. A related theorem which covers multiple linear approximations can be found in Section 7.1. of [CoNi25] entitled A Level Two Theorem for all Quadratic S-boxes.

More on 80 LAS-2 properties. A detailed enumeration shows that they split into two disjoint classes of 40 cosets stable by arbitrary translations and that properties spanned with $15 \cdot 8$ sets DDT() of size 4 and $5 \cdot 4$ sets DDT() of size 8, both cover uniformly the whole set of 80 LAS-2 properties several times.

Observations. Our result is somewhat very surprising, because we have first fully linearized a **non-linear** S-box on a subset, where it is equal to a multivariate linear function with a matrix and a constant, and then we claim that simple transformed variants of our linear property cover the whole space F_2^k several times, very much **as if** the original S-box $F[]$ was linear.

Output Side Relaxations. Our LAS and LSS properties span the wole input and output spaces, and they are stronger than many known Boura-Canteaut properties in symmetric cryptanalysis cf. [BoCa14], which typically only work for a linear space at the output side. In contrast Boura-Canteaut (v, w)-linearity properties which are of LAS type only, can potentially lead to better cryptographic attacks and stronger or larger size translation-invariant properties, because they may also hold when the S-box is NOT at all quadratic, as shown with Hamsi S-box in [BoCa14] which properties can be characterized and studied in terms of second order derivatives of Boolean functions, see Proposition 9 in [SaGl14]. Similar properties will hold with the S-boxes of DES.

Output Side. This type of LSS or LAS result very rarely holds with output side translations like $c \oplus S[A_{11} \oplus x]$. Moreover, with numerous cryptographic S-boxes we checked, we found that the maximum LSS size of an inverse S-box, is typically strictly lower than the best LSS size for the initial S-box. This with some exceptions when for example the inverse box is quadratic (in Ascon/Keccak the inverse is cubic). A quadratic inverse never happens for any "good" cryptographic S-boxes in size 5, or all S-boxes which are quadratic in both directions are extremely weak, as shown very clearly by comparison of Table 1 and Table 2 in [BoFiPe13].

Violations of Thm. 3. We have verified that our result is NOT true for Thakor and Icepole S-box, which are not quadratic. Therefore these two ciphers could have **better** resistance against various attacks with translations of affine spaces, and with translations of arbitrary sets, and with multiple related **simultaneous** linear approximations [BCQ04,SmAes07,CoDe08], leading simply to a bigger variety of affine spaces and sets which will be harder to connect together in some attack. In this case however, one can work on say 3/5 output space

restriction like in (v, w) Boura-Canteaut linear properties of [BoCa14] or further generalize these properties to arbitrary sets V at input side.

8.1 Ascon, Dual or More Low-MC Approximations

An interesting question is, since A_{11} and B_{11} are disjoint, is it possible to combine both these very "simple" entirely linear approximations of the whole S-box, into one single "super-approximation" true with a larger probability of 22/32. A key question is how, and what might be a reasonable way to evaluate the "quality" of such approximations which will no longer be linear. One possible measure of quality could be the MC (Multiplicative Complexity). For example we checked that for 13 points but not more, the whole S-box can be approximated by a function with MC=1. This should have some serious consequences in probabilistic variants of current attacks on Ascon with zero sums and cube distinguishers. In this paper we propose another quality metric for such approximations, and this metric is studied in later Section 12.

8.2 On Unicity of Matrices A

Following on A_{11} and all their shifts, we have checked that these matrices are unique and no more solutions exist. In general these matrices do not have to be unique at smaller sizes. The matrix unicity here is easy with 11 points, however it will not hold for 80 LAS-2 properties from [QSMG17], with 4 points only, the solutions A cannot be unique for just 4 points. Two examples of actual matrices are shown in Section 7 which section also explains that these matrices are invertible in all 30/32 cases except precisely with A_{11} and B_{11} themselves.

8.3 On Affine Dimension of Set of Matrices A

Now we are going to show that the set of matrices A here is much smaller and simpler than expected, and forms an affine space of small dimension. We start by observing that our proof technique for Thm. 3 provides an interesting bonus property and some insights. Let $k = 5$ and consider the Ascon S-box. Here potentially this space of unique matrices could span a linear space of large dimension up to 25. However, in our proof we see that every new matrix is of the form $(\Delta_F(x, d) \oplus A \cdot x)$ discarding any constants. Knowing that Δ_F is bi-linear, we can consider some basis of 5 elements for d, we obtain that our set matrices must be contained within an affine space of dimension at most 5 out of 25. In spite of this property, we found that all 32 these matrices are all distinct, for both Ascon and Keccak S-box.

8.4 On Entropy of Constant Parts in A_{11} Approximations

Let us denote by $M^{11}[d]$ a unique matrix such that:

$$F[x] = M^{11}[d] \cdot x + \phi[d] \quad \forall x \in A_{11} \oplus d.$$

where each entry of our 5x5 matrix $M^{11}[d]$ is an affine function of 5 bits of d, and for some "constant part" function $\phi : F_2^5 \to F_2^5$. For example we have seen that: $\forall x \in A_{11} = \{0, 3, 4, 12, 16, 17, 19, 20, 21, 29, 31\}$ we have:

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{bmatrix} 1\,0\,1\,1\,0 \\ 1\,0\,1\,1\,1 \\ 0\,1\,1\,1\,1 \\ 1\,1\,1\,0\,0 \\ 0\,1\,0\,1\,1 \end{bmatrix} \cdot \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Then we can observe that in 16 out of 32 cases, the constant $\phi[x] = F[0] = 4$ which is the same as with A_{11} above. This is withut doubt related to the fact the S_{in} is linear and that the constant in T_{ou} is also equal to $4 = 00100$ in binary. Needless to say a similar result hold for Keccak: in half of the cases that constant is equal to $0 = F[0]$.

This strong bias in the constant is a bit surprising and expected to lead to some zero-sum properties and should also be studied for many other properties such as all Boura-Canteaut (v, w)-linear properties.

By looking at various spaces of Dim 2 contained within some of $A_{11} \oplus d$ we can here obtain multiple properties where an affine space of Dim 2 is sent to an affine space of Dim 2, i.e. re-discover our 80 LAT-2 properties. Similarly previously known LAT and Boura-Canteaut (v, w) partial linearization results can all be studied for how they extend into a larger set of V for the same matrix and vector, plus all translated properties which will also follow.

8.5 On Almost-Maximal size LSS-10 Properties

It is easy to see that the number of LSS$-m$ properties increases as m goes down. A summary of all LSS properties of all sizes in existence can be found in Table 11 page 29.

For example, any subset of 10 inside an LSS-11 property form a valid albeit trivial LSS-10 property. In addition we found there exists exactly 5 **non-trivial** LSS-10 properties. They form 5 distinct classes or orbits w.r.t. translations of LSS-10 properties of Ascon. Each these 5 classes generates 32 distinct translations of size 10 following Thm. 3. We list these remarkable sets of 10 here together with a XOR of all elements at output side, which is a translation invariant additive property, for all 160 sets (true because 10 is even). Overall our 160 properties are freely translatable by translations and rotations which act on these properties. We have exactly 5 classes each containing 32 affine shifts:

- 0,1,3,4,5,9,16,19,20,24 $\sum_{in}(V)$=5
- 0,3,4,15,16,17,19,20,21,28 $\sum_{in}(V)$=7

– 0,4,8,11,17,21,23,24,25,27 $\sum_{in}(V)$=14
– 0,2,3,4,12,14,16,19,20,31 $\sum_{in}(V)$=15
– 0,2,4,12,14,16,17,20,21,29 $\sum_{in}(V)$=25

Note. The set of 5 sums $\sigma_{ou}(V)$ we obtain here is not quite a random set of 5 points, and later we will denote this set by $r^i = \{5, 7, 14, 15, 25\} \subset B_{11}$.

Related Research. In [KaEA21] Kaleyski studies a special case: $\sum_{ou}(V)$ when set V adds to zero (i.e. $\sum_{in}(V) = 0$. With LAS-2 which are also a special case of LSS this would be the case, but not for general LSS.

On Matrices Involved. This class of properties leads to $5 \cdot 32 = 160$ distinct approximation matrices. Most of these matrices are invertible however a large number of these are not invertible and of rank 4.

Rotation Invariance of all 160 of non-trivial LSS-10.
It is possible to see that all internal rotations due to embedded Keccak S-box such as say ρ^2_{in}, cf. Table 1, will act on these 5 classes with their shifts, transforming sets of 10 into different sets of 10 from another class. Hence we have obtained 160 LRS-10 properties. It goes without saying that the sums of all elements in these sets are also preserved by rotations, and the set of $r^i = \{5, 7, 14, 15, 25\}$ is stable by our rotations such as type ρ_{in} and their combinations.

Rotation Invariance of LSS-11. Moreover, rotations always transform sets of type LSS-11 into another sets of type LSS-11. Moreover exactly 2/32 properties of type $A_{11} \oplus b$ obtained from Thm. 2 are themselves invariant by rotations (we obtain the exact same set). These 2/32 are also exactly those where approximation matrices are not invertible, cf. Section 7.

On Dual Action and Group Theory Questions. We have a set of 160 sets of 10 points and 32 sets of 11 points where two transformations: arbitrary translations such as ρ^2_{in}, and arbitrary translations with a XOR with a constant, both act and stay within the same set of 160 properties. It is important to note that shift rotations also act at the output side, but translations typically do not work on the output side. We leave for future research the study of the groups generated by these two operations for various sets of points of any size including the full space. It goes without saying that there are many remarkable identities there to be discovered, and this work can be extended to other S-boxes based on Daemen Chi family. This is related to multiple group-theoretic works on iterated permutations χ see [KrKy24,ScDa24] to name only a few.

8.6 Known Linearization Results for χ in Even Sizes

Linearization results on χ type S-boxes are not new. Here we give a short summary how our results compare to some known results.

In Section 5.1 of a recent paper [ScDa24] Schoone and Daemen show that for every even integer, the χ S-box can be fully linearized (LAS and LSS property) on two special disjoint sets S_0 and S_1 of exactly $2^{n/2}$ points each, which fix half of bits to zero, and which are affine spaces. Their matrices are not invertible and having a Kernel being the union of 0^n and 1^n. This in spite of the fact that χ S-box is not invertible in even dimensions. Our Thm. 3 also applies to their LAS/LSS properties and leads to further pairs of disjoint LAS/LSS properties.

Our Thm 2 works for both Ascon and Keccak and also offers disjoint pairs of (much less) regular solutions at size 11+11. This is instead of $16 = n/2$ which we might want to extrapolate from even dimensions, but in fact there is no hope such solution could exist and our results of size 11 are optimal. We obtain two disjoint sets of 11 points where the matrices are also non-invertible and their kernels are essentially the same (like 0^n vs. 1^n). These points fix some bits in both Ascon and Keccak, and are included in an affine space on the output side for both, and this affine space is defined by $\beta = \rho_{ou}^{-1}[1^n]$. Our two canonical matrices are also not invertible and having the same Kernel mapping to union of 0^n and 1^n. Our properties are NOT transformable by rotations by our two canonical sets A_{11} and B_{11} are invariant by rotations. All this in spite of the fact that χ S-box being invertible in odd dimensions. For pure LAS properties we work with 4 points only. LSS properties are stronger with 11 points. We also exhibited a large class of 160 non-trivial LSS-10 properties with 10 points, cf. Section 8.5 which are freely translatable by translations and rotations.

Scalability and Open Problems. An interesting question is how does our LSS property scale to larger odd sizes. How would the Table 11 in page 29 look like in larger dimensions? What is the maximum LSS size for χ_{257} which can fully linearize the large S-box of Subterranean 2.0, see [DCMR20]. Let $f(k)$ denote the maximum size of an LSS property for odd k. We conjecture that this function $f(k)$ grows exponentially with k and that $DMI \to 0.5 \cdot k$ for χ_k for odd k when $k \to \infty$. These results are directly applicable to security of Subterranean 2.0 cf. [DCMR20] and Rasta cipher, see Section 3.2. in [Ras18].

8.7 Known Holographic Linearization Results in PK Cryptography

In [SaGl14] and in [ACKSS23] we find a similar result which is also a special case of our Thm. 3 which also works in a larger finite field. The authors show that for any coset of the so called Oil space in UOV multivariate public key cryptosystem, the public key of UOV is equal to an affine function which fully linearizes the PK. Again this happens as many as q^n times covering the whole input space. Our Thm. 3 is however more general, if for example we could extend this space by more points (LSS type) then also each of the q^n shifted approximations will be extended accordingly with the same matrix and vector.

9 Order Emerging From Chaos

It is time to define a new classification of points in Ascon S-box and introduce some new notations. The same exact things could be done for the Keccak S-box. It is important to see that S-boxes in odd size have some interesting properties, and more specifically at size 5 we can have some very unique properties which other S-boxes of size say 6 will not have, or things will be very different there. We claim that there exists a **unique** and natural partitioning of all the 32 points into 6 sets of 5 points and two sets of size 1. The easiest way to define our partitioning is to consider that, buried inside Ascon S-box, is the Keccak S-box which is stable by simultaneous rotation on both sides, i.e. it is SI or Shift Invariant, see Def. 6.1 page 111 in [DaPhD95].

We can always partition a set of 32 points into sets stable by rotation. However, with Ascon things are less simple than with Keccak. We have two rotation functions different on each side of the S-box, which were specified in Table 1. At the end we get pairs of rotation-stable sets which are different on each side of the Ascon S-box, which however are the same from the point of view of the Keaccak S-box hidden inside. In the following table we report the input of the Ascon S-box, the output, and two "internal" values: corresponding to input of Keccak S-box which is called Ki, and the output of the Keccak S-box, which is called Ko. In addition in the middle column we report also the XOR of these two values Ki and Ko which quantity we call Kx. To summarize we obtain the following partitioning of Ascon set of 32 states into:

name	Ascon input	Keccak Ki	Kx=$Ki \oplus Ko$	Keccak Ko	Ascon output
a	6,13,18,24,27	7,14,19,25,28	1,2,4,8,16	15,23,27,29,30	3,7,9,16,25
b	1,2,8,23,28	3,6,12,17,24	1,2,4,8,16	11,13,21,22,26	11,22,24,27,31
q	9,10,11,22,30	15,23,27,29,30	1,2,4,8,16	7,14,19,25,28	5,8,15,17,18
r	5,7,14,15,25	11,13,21,22,26	5,9,10,18,20	1,2,4,8,16	2,6,12,21,28
s	3,20,21,29,31	5,9,10,18,20	5,9,10,18,20	3,6,12,17,24	0,10,13,20,23
t	4,12,16,17,19	1,2,4,8,16	1,2,4,8,16	5,9,10,18,20	14,19,26,29,30
y	0	0	0	0	4
z	26	31	0	31	1

Table 15. Our special space partitioning in Ascon which corresponds to classes of points stable by rotation for the embedded internal Keccak S-box.

If there is no ambiguity we will denote these sets by single letters a, b, q, r, s, t, y, z. At places where it is not clear on which side of the S-box we are, or when we mix both sides, we will use the exponents such as a^i and a^o to distinguish input and output sides, which is a common notation used in [CARG19] in the study of non-linear invariant attacks on block ciphers. For example we write

$$t^i = \{4, 12, 16, 17, 19\}$$

If there is no ambiguity we denote by the $S[]$ the Ascon S-box, and we have by definition and by construction:

$$S[a^i] = a^o \quad S[t^i] = t^o \quad \text{etc..}$$

Likewise we can define Keccak variants of some sets and write things like:

$$t^{Ki} = \{1, 2, 4, 8, 16\} \qquad and \qquad t^{Ko} = \{5, 9, 10, 18, 20\}.$$

9.1 Initial Observations About Our Sets

All our sets come in pairs, however the pairing is not the same on both sides:

$$a^i = b^i \oplus 26 \oplus 0 \qquad t^i = q^i \oplus 26 \oplus 0 \qquad s^i = r^i \oplus 26 \oplus 0 \qquad y^i = z^i \oplus 26 \oplus 0$$

$$a^o = r^o \oplus S[26] \oplus S[0] \;\; t^o = b^o \oplus S[26] \oplus S[0] \;\; s^o = q^o \oplus S[26] \oplus S[0] \;\; y^o = z^o \oplus S[26] \oplus S[0]$$

In Section 7.1 we discovered that lines in Ascon DDT table can be partitioned into disjoint sets of lines with highly uniform content and properties:

1. 1 line with $\delta_{in} \in y^i$ which contains zeros and 32.
2. 11 lines $\delta_{in} \in z^i \cup q^i \cup a^i$ which contain only 2s.
3. 15 lines $\delta_{in} \in s^i \cup r^i \cup b^i$ which contain only 4s.
4. 5 lines $\delta_{in} \in t^i$ which contain only 8s.

It is further possible to see that content of $DDT()$ sets in various lines can be bijectively mapped to the content of other $DDT()$ sets in other lines of the same class by more or less arbitrary translations of sets by a constant.

We observe that the set $t^i = \{4, 12, 16, 17, 19\}$ which are exactly five input differentials δ_{in} in Ascon of maximum strength (DDT=8). These 5 are also half of 10 known "undisturbed bits" UDB properties in [TeAs16,TeDi19]. There are two special singleton sets y and z stable by translation $0, 26$ which were those involved in flipping between A_{11} and B_{11}. They correspond to two fixed points in Keccak. These two points play an important role and are involved in countless remarkable identities. For example sum of all five values in t^i is $26{=}z^i$. Sum of all five values in t^o is $4{=}y^o$. Similar properties hold for all six classes and on both input and output sides. We can relate these sets to our pair of a unique special maximum size linear spaces called L^i/L^o and discovered in Property B in Section 5, a special and unique partitioning of type 6+10 leading to LIO-63. Then we observe that s is the complement of A_{11} inside L^i excluding the special point. One can define a as a unique set of 5 which is a complement of $S[A_{11}]$ inside our special linear space L^o of dim 4, this at the output side, i.e. $S[a]$ and $S[A_{11}]$ forms a linear space which linear space is actually the same as our L^o with $\beta = 11$.

9.2 Further Remarkable Identities

Here we define multiple derived sets. For example we write $z^i \oplus b^i$ to enumerate all possible XORs on 5 bits of an element from z on input side, with an element from b on the input side. Given that the origins of these sets is the study of various mappings in Ascon which match other when we translate them by a

constant and in relation to various affine spaces, we frequently obtain sets which can also be obtained in a different way.

Fact 9.1. The reader can verify that the following identities hold:

$$\begin{array}{ll} a^i = z^i \oplus y^i \oplus b^i & a^o = z^o \oplus y^o \oplus r^o \\ q^i = z^i \oplus y^i \oplus t^i & b^o = z^o \oplus y^o \oplus t^o \\ r^i = z^i \oplus y^i \oplus s^i & q^o = z^o \oplus y^o \oplus s^o \end{array}$$

Fact 9.2. There are also many identities which involve basic sets of 11 points.

$$\begin{array}{ll} s^i \cup q^i \cup y^i = b^i \oplus b^i & r^i \cup t^i \cup z^i = a^i \oplus b^i \\ b^i \cup q^i \cup y^i = s^i \oplus s^i & a^i \cup t^i \cup z^i = r^i \oplus s^i \\ s^i \cup b^i \cup y^i = q^i \oplus q^i & r^i \cup a^i \cup z^i = t^i \oplus q^i \end{array}$$

which bears some similarity and could be compared to:

$$\begin{array}{ll} A^i_{11} = s^i \cup t^i \cup y^i & B^i_{11} = q^i \cup r^i \cup z^i \\ A^o_{11} = s^o \cup t^o \cup y^o & B^o_{11} = q^o \cup r^o \cup z^o \end{array}$$

Fact 9.3. Many more such identities exist with XORs of 3 elements, many of which also give sets of exactly 11 points:

$$\begin{array}{ll} a^o \cup s^o \cup y^o = t^o \oplus t^o \oplus y^o & b^o \cup q^o \cup z^o = r^o \oplus r^o \oplus z^o \\ a^o \cup t^o \cup y^o = s^o \oplus s^o \oplus y^o & b^o \cup r^o \cup z^o = q^o \oplus q^o \oplus z^o \\ t^o \cup s^o \cup y^o = a^o \oplus a^o \oplus y^o & r^o \cup q^o \cup z^o = b^o \oplus b^o \oplus z^o \end{array}$$

$$\begin{array}{ll} a^i \cup r^i \cup z^i = t^i \oplus t^i \oplus z^i & s^i \cup q^i \cup y^i = b^i \oplus b^i \oplus y^i \\ a^i \cup t^i \cup z^i = r^i \oplus r^i \oplus z^i & s^i \cup b^i \cup y^i = q^i \oplus q^i \oplus y^i \\ r^i \cup t^i \cup z^i = a^i \oplus a^i \oplus z^i & b^i \cup q^i \cup y^i = s^i \oplus s^i \oplus y^i \end{array}$$

$$\begin{array}{ll} r^o \cup b^o \cup z^o = q^o \oplus q^o \oplus z^o & s^o \cup a^o \cup y^o = t^o \oplus t^o \oplus y^o \\ q^o \cup b^o \cup z^o = r^o \oplus r^o \oplus z^o & s^o \cup t^o \cup y^o = a^o \oplus a^o \oplus y^o \\ r^o \cup q^o \cup z^o = b^o \oplus b^o \oplus z^o & t^o \cup a^o \cup y^o = s^o \oplus s^o \oplus y^o \end{array}$$

Many more similar identities exist with longer XORs of various sets.

Fact 9.4. For example here are some identities involving 16 points:

$$\begin{array}{ll} a^i \oplus a^i \oplus a^i = r^i \cup t^i \cup a^i & a^o \oplus a^o \oplus a^o = t^o \cup s^o \cup a^o \\ r^i \oplus r^i \oplus r^i = a^i \cup t^i \cup r^i & r^o \oplus r^o \oplus r^o = q^o \cup b^o \cup r^o \\ s^i \oplus s^i \oplus s^i = q^i \cup b^i \cup s^i & s^o \oplus s^o \oplus s^o = t^o \cup a^o \cup s^o \\ b^i \oplus b^i \oplus b^i = s^i \cup q^i \cup b^i & b^o \oplus b^o \oplus b^o = q^o \cup r^o \cup b^o \\ q^i \oplus q^i \oplus q^i = s^i \cup b^i \cup q^i & q^o \oplus q^o \oplus q^o = r^o \cup b^o \cup q^o \\ t^i \oplus t^i \oplus t^i = r^i \cup a^i \cup t^i & t^o \oplus t^o \oplus t^o = a^o \cup s^o \cup t^o \end{array}$$

Fact 9.5. More remarkable identities: with A_{11} and B_{11} there are more substantial differences about what happens on each side [input/output].

$$A^i_{11} = s^i \cup t^i \cup y^i \qquad B^i_{11} = q^i \cup r^i \cup z^i$$
$$A^o_{11} = s^o \cup t^o \cup y^o \qquad B^o_{11} = q^o \cup r^o \cup z^o$$
$$A^i_{11} \cup s^i \cup t^i = B^i_{11} \oplus z^i \qquad B^i_{11} \cup q^i \cup r^i = A^i_{11} \oplus z^i$$
$$A^i_{11} \oplus A^i_{11} = F_2^5 \setminus 26 \qquad B^i_{11} \oplus B^i_{11} = F_2^5 \setminus 26$$
$$A^o_{11} \oplus A^o_{11} = a^o \cup s^o \cup t^o \cup y^o \qquad B^o_{11} \oplus B^o_{11} = a^o \cup s^o \cup t^o \cup y^o$$
$$A^o_{11} \cup a^o = A^o_{11} \oplus A^o_{11} = B^o_{11} \oplus B^o_{11}$$
$$B^o_{11} \cup b^o = B^o_{11} \oplus a^o = A^o_{11} \oplus B^o_{11}$$

There are many other interesting observations about these 6 sets of 5 points.

Fact 9.6.i. At input side sets t^i, a^i, r^i form a basis for the whole space and the other three do not form a basis and their sum is 0. Together, these 16 elements $s^i + b^i + q^i + y^i$ form a linear space of maximum dimension 4 known as L^i in Property B of Section 5, with $\alpha = 20 = \rho_{in}[31]$, with $L^i_B = \{0, 1, 2, 3, 8, 9, 10, 11, 20, 21, 22, 23, 28, 29, 30, 31\}$ and its elements are exactly all the unions of 0,2 or 4 distinct elements of t^i.

Fact 9.6.o. At output side sets b^o, r^o, q^o form a basis for the whole space and the other three do not form a basis and their sum is $S[0]$ and together these 16 elements $a^o + s^o + t^o + y^o$ form a linear space of maximum dimension 4 known as L^o in Property B of Section 5 with $\beta = 11 = \rho^{-1}_{ou}[31]$, with $L^o_B = \{0, 3, 4, 7, 9, 10, 13, 14, 16, 19, 20, 23, 25, 26, 29, 30\}$ and its elements are exactly all the unions of 1,3 or 5 distinct elements of t^o.

We now consider some remarkable identities which are mixing "apples and oranges": sets and affine spaces from both I/O sides of the S-box. This type of properties could be exploited in some invariant attacks (yet to be discovered). We call I the intersection of the 2 remarkable spaces of dimension 4 above.

Fact 9.7. We have $I = \{0, 3, 9, 10, 20, 23, 29, 30\}$. This space I creates a partitioning of the whole set of 32 points into 4 affine spaces or cosets which are:

name	either side	
I	0,3,9,10,	20,23,29,30
J	1,2,8,11,	21,22,28,31
K	4,7,13,14	16,19,25,26
L	5,6,12,15	17,18,24,27

It is hard to imagine that these sets of 8 can align well with various sets of 5, while mixing both sides of the S-box, however there are still some remarkable identities at size 16 to report:

$$L^i_B = I \cup J = s^i \cup q^i \cup b^i \cup y^i \qquad K \cup L = a^i \cup t^i \cup r^i \cup z^i$$
$$S[I] \cup S[J] = s^o \cup q^o \cup b^o \cup y^o \qquad S[K] \cup S[L] = a^o \cup t^o \cup r^o \cup z^o$$
$$L^o_B = I \cup K = a^o \cup s^o \cup t^o \cup y^o \qquad J \cup L = b^i \cup q^i \cup r^i \cup z^i$$

9.3 Keccak Only Section

We should also observe the following. Keccak S-box has two fixed points y^{Ki} and z^{Ki} whcih are also exactly those states which have period 1 by cyclic rotation, i.e. $31 = 1^5$ and $0 = 0^5$. The remaining 32 points can be ordered in two cycles of size 2 and 4, cf. Thm. 2 in [ScDa24]. Moreover they are operating on sets of 5 bits as follows, which corresponds to 2+4 canonical states and their cyclic rotations.

$$a^{Ki} \mapsto q^{Ki} \mapsto a^{Ki} \qquad b^{Ki} \mapsto r^{Ki} \mapsto t^{Ki} \mapsto s^{Ki} \mapsto b^{Ki}$$

$$\{7, 14, 19, 25, 28\} \mapsto \{15, 23, 27, 29, 30\} \mapsto \{7, 14, 19, 25, 28\}$$

$$\{3, 6, 12, 17, 24\} \mapsto \{11, 13, 21, 22, 26\} \mapsto \{1, 2, 4, 8, 16\} \mapsto \{5, 9, 10, 18, 20\} \mapsto \{3, 6, 12, 17, 24\}$$

To summarize we have 20 points having order 4, and 10 points of order 2, and these points are grouped in 6 sets of 5 canonical states with their rotations. These cycles were shown in Fig. 5 which we reproduce here for convenience.

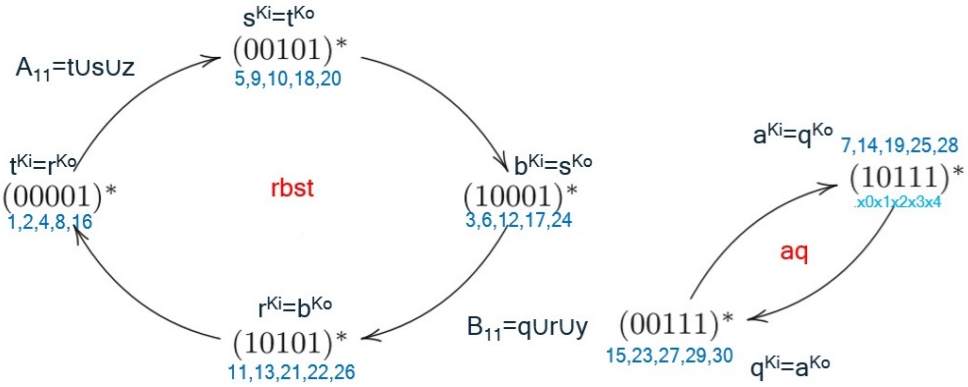

Fig. 6. State diagram for Kecak S-box excluding 2 special points, with 20 points of order 4 and 10 points of order 2, grouped into sets of 5 canonical states with all their roations, based on Fig. 1 page 1394 in [ScDa24] and annotated with special distinguished sets defined in this paper.

In cotrast, the Ascon S-box does no longer have any comparable cycling properties. However we can construct multiple derived functions such as

$$x \mapsto S^{-1}[\, S[\, x\,] \oplus 5\,]$$

and such functions have short cycles operating on sets of size 5. This is due to the fact that $T_{ou}[31] \oplus T_{ou}[0] = 4 \oplus 1 = 5$ and flipping all bits (with 31) preserves orbits for cyclic bit rotations at the Keccak output level.

9.4 Invariant Affine Spaces and Fixed Points

In [KB22] we read that the AES S-box has no inavriant affine space of dimension 2, zero fixed points and one invariant affine space of dimension 1 swapping two points ($a \to b \to a$ type). In this paper we find it has 85 properties where an affine space of dimension 2 is mapped into another affine space, i.e. 85 of LAS-2 properties. This shows that none of the 85 are identical on both sides, which shows the importance of the matrix choice in the AES S-box, and the fact that the AES S-box is not an involution.

10 Comparison to Other Permutations on 5 Bits

There exist exactly 75 affine equivalence classes of (quadratic) permutations on 5 bits, cf. Table 1 in [BoBiSa17], and Ascon/Keccak S-box belongs to class 68. Is the Ascon/Keccak S-box A_{11} property a typical case or is it a particularly weak quadratic permutation? We have run our SAT solver tool on all 75 classes from [BoBiSa17] and looked at the maximum size set LSS. In most cases these S-boxes are substantially weaker than Ascon and Keccak, direct comparison makes little sense, and no one would agree to use such S-boxes in cryptography. Therefore we restrict our study to all classes out of 75 where $DMI(\delta_{in};\delta_{ou}) \leq 2.2$ bits knowing that in Ascon/Keccak we have $DMI = 1.91$ and $DMI = 1.12$ in the best case. The LSS results obtained with two independent implementations of our tool are reported below in Table 16. These results on the maximum possible LSS size are optimal and cannot be improved. In addition, we also report the maximum DDT size and the Di-Size for Dillon's sets as defined in Section 3.3.

Other S-boxes. We have not included Thakor and Icepole because these S-boxes are not quadratic. For these S-boxes LSS=9/10, and Di-size=Ci-size=32, same as in Ascon/Keccak.

classes	also known as	best LSS size	Di-size	max DTT size	MC	$DMI(\delta_{in};\delta_{ou})$
28		15	8	16	3	2.16
36		13	29	16	5	2.16
52		11	16	16	5	2.06
56,58		11	30	16	5	2.06
59		11	31	8	5	1.94
47		10	16	16	5	1.97
61,62,65		11	32	8	6	2.06
68	Ascon,Keccak	11	32	8	5	1.91
53,70		11	32	8	5	1.88
57		10	32	8	6	1.78
69		10	32	8	6	1.72
63,64		9	32	8	6	1.69
71		10	32	4	5	1.69
72		9	32	4	5	1.59
66		10	32	8	6	1.56
67		9	32	8	6	1.56
73		8	32	4	6	1.41
75		7	32	2	7	1.12
74	Fides-5,Primates	7	32	2	8	1.12

Table 16. Maximum size LSS properties for all cryptographically "not too weak" equivalence classes of quadratic permutations on 5 bits according to [BoBiSa17].

Barely Secure Enough. Ascon is an S-box which is just barely secure enough compared to other S-boxes. Any S-box with DDT() size of 16 should be considered as flawed. Any S-box with Di-size smaller than 32 should be considered as flawed. We see that all 5-bit S-boxes with $DMI \geq 1.94$ bits are flawed

w.r.t. the set of their Dillon's sums and except for class 59 they all have very strong differentials with probability 16/32.

Further Analysis. In Table 16 it is possible to see that the maximum LSS property size behaves in a highly regular way and could be actually quite reliably predicted from the value of DMI. Likewise, the number of LAS and LSS properties of any size are expected to be predictable from DMI, see Table 1. We observe that LSS-11 property occurs for all S-boxes with $1.88 \leq \text{DMI} \leq 1.94$. We must conclude that LSS-11 property is **a typical property of an S-box with** $DMI \approx 1.9$.

Remark. In spite of the fact Ascon S-box has some rotational symmetries which other S-boxes do not have, and the value Kx inside this S-box is biased, not uniformly distributed and always has a small HW, the results in Table 16 do not confirm the idea that Ascon S-box would be very special or weaker than other comparable S-boxes.

Comparison with DDT Sets. An interesting transition occurs in our table: in the upper part, best LSS properties are not as large as best DDT() sets, which means that we **can hardly expect**, even in the worst case, that very large sets obtained from DDT can be fully linearized in a LSS property. We can eventually hope that, as observed above, DDT() sets are supersets [or disjoint unions] of LSS sets. We then expect that for such (weaker) S-boxes, traditional DC with single differentials will perform well. Then for most of cryptographically significant S-boxes, roughly in the lower half of Table 16, best LSS size can be substantially larger than best DDT() size. Here, while there is hope to break these ciphers by simple DC, the attacker should try to exploit combinations of LSS properties and also some closely related [weaker] differential properties which can be sometimes amplified by such subsets, cf. Section 12.

Can We Do Better? Ascon and Keccak cipher could be made arguably and measurably more secure against simultaneous linear approximation attacks, by changing the S-box to an S-box in classes 74/75, or rather a well-chosen affine equivalent with a low ASIC implementation depth.

10.1 On Implicit Equations in 5-bit S-boxes

In the table below we report the number of iMQ, also known as quadratic LIO equations which are a special case of "implicit equations" in mathematics, a.k.a. "I/O equations" or MQ equations cf. [CoDe08] and [CDG03], and which are sometimes called "indirect polynomials" and can be computed in sagemath with `.polynomials()`.

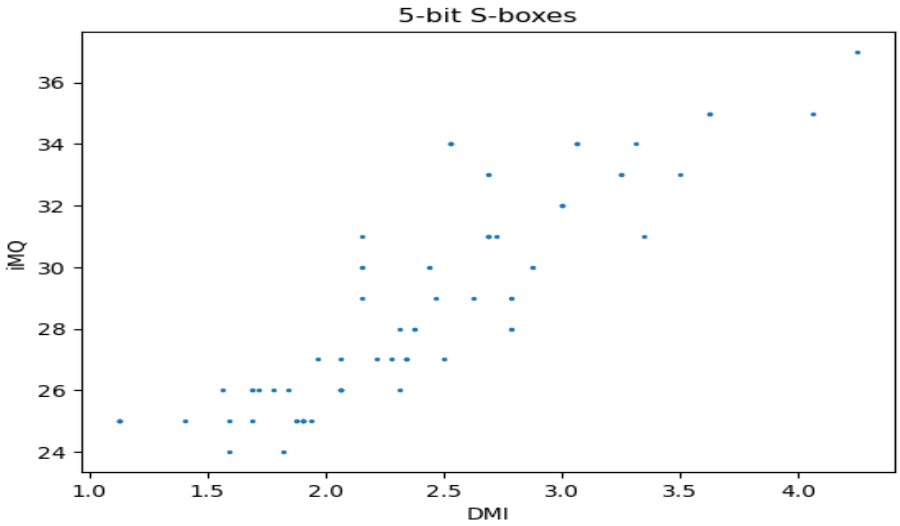

Fig. 7. Number of implicit iMQ equations for a selection of 5-bit S-boxes.

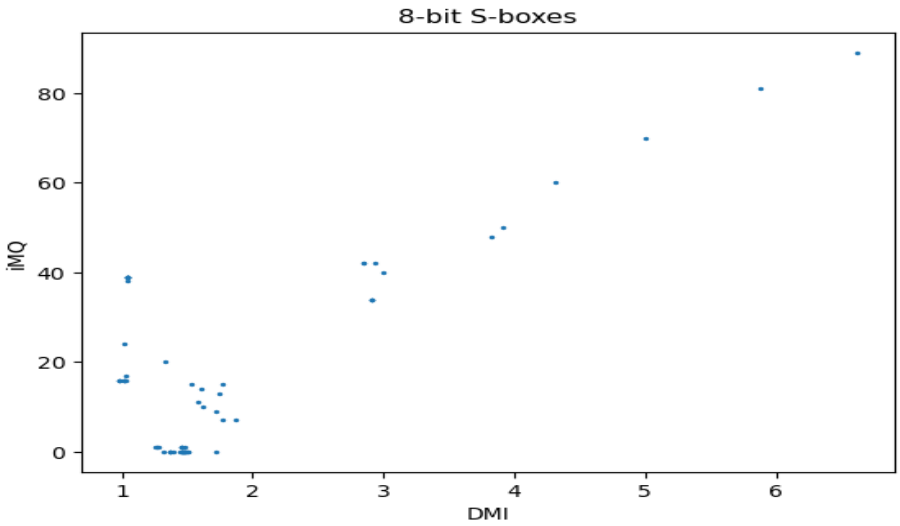

Fig. 8. Number of implicit iMQ equations for a selection of 8-bit S-boxes.

10.2 On Thickness Spectrum in 5 and 8-bit S-boxes

In table below we report tN_2 or the second moment of the thickness spectrum $N_2 = \#$ of affine spaces of dimension 2 inside the set WZ of Walsh Zeros, cf. [CCP22] which is implemented inside github.com/lpp-crypto/sboxU/ library.

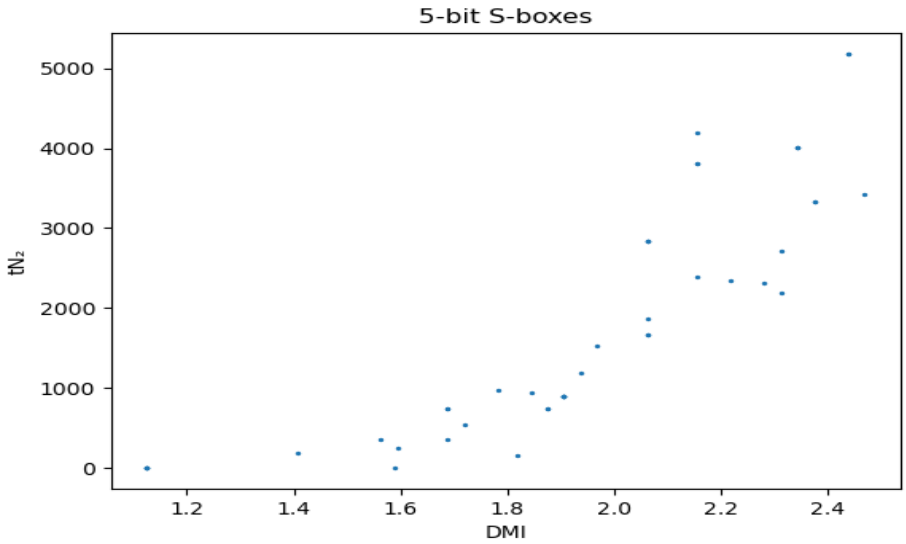

Fig. 9. Thickness Spectrum tN2 for a selection of 5-bit S-boxes.

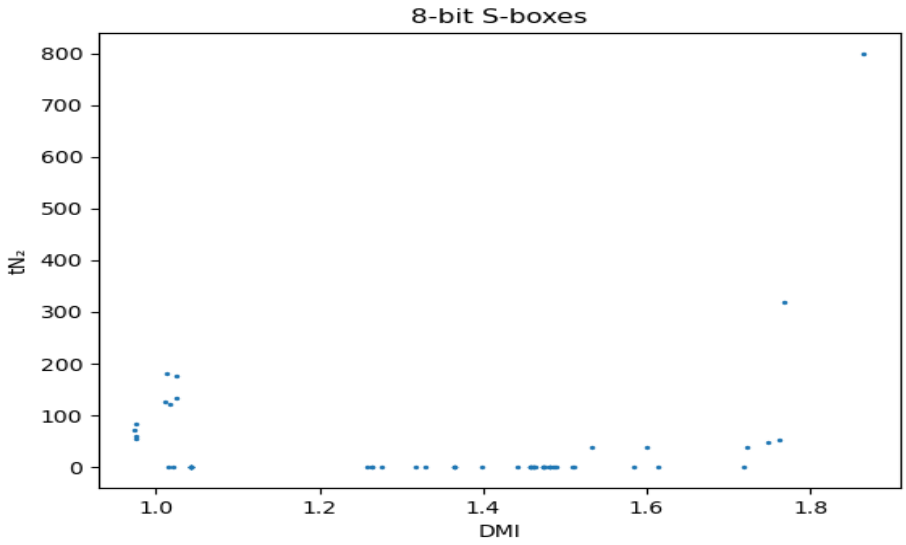

Fig. 10. Thickness Spectrum tN2 for a selection of 8-bit S-boxes.

10.3 On Thickness Classification and WZ Discriminating Ability

In Fig. 11 table below we show that the second moment tN_2 of thickness spectrum or/and DMI are NOT precise enough to tell apart the 6 main disjoint and quite distant clusters of APN functions in existence, cf. Fig. 12. Hence the best or primary tool for the study of APNs should be WZ stats where tN_2 is an accessory figure which in our opinion does not qualify at all as a good APN quality reporting or classification tool.

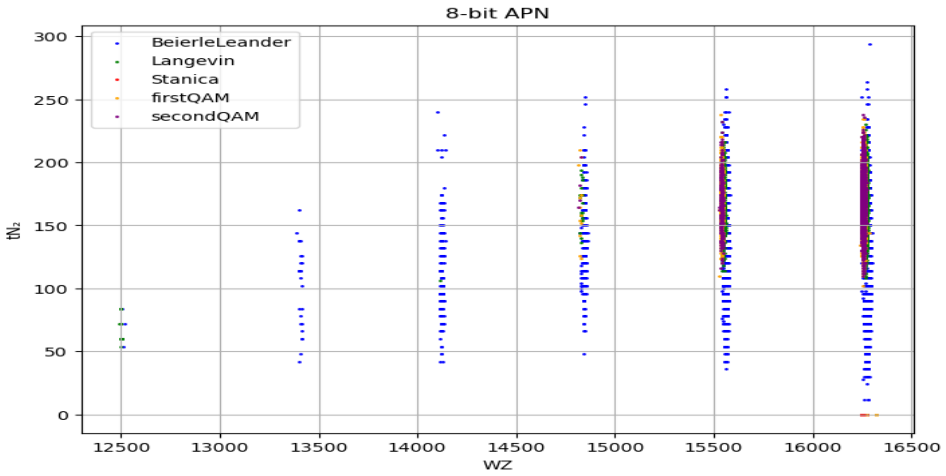

Fig. 11. Thickness Spectrum tN2 and WZ for various 8-bit APNs.

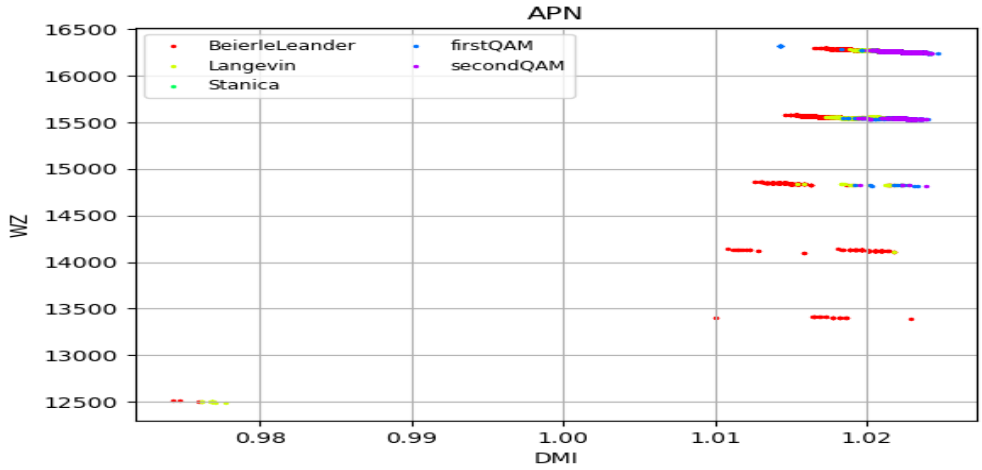

Fig. 12. DMI and Walsh Zeros in major known families of APNs on 8-bits.

More On WZ and Walsh/LAT Types in APNs

Following an online database at boolean.wiki.uib.no/Walsh_spectra_of_all_known_APN_functions_over_GF(2%5E8) there would be exactly 3 types of Walsh spectra for known APN functions with $k = 8$. To simplify, if we look at absolute Walsh spectra we would always be in one of 3 cases:

- WZ=16320, $|\pm 16| = 43520$ - a.k.a. Gold class.
- WZ=15600, $|\pm 16| = 44544$ - a.k.a. type 1.
- WZ=14880, $|\pm 16| = 45568$ - a.k.a. type 2.

This claim is not quite correct or not up to date in several ways. In fact there are 6+1 and not 3 very distinct sets of APNs for $k = 8$ as clearly demonstrated here and the numbers of WZ and $|\pm 16|$ vary very slightly w.r.t. figures above.

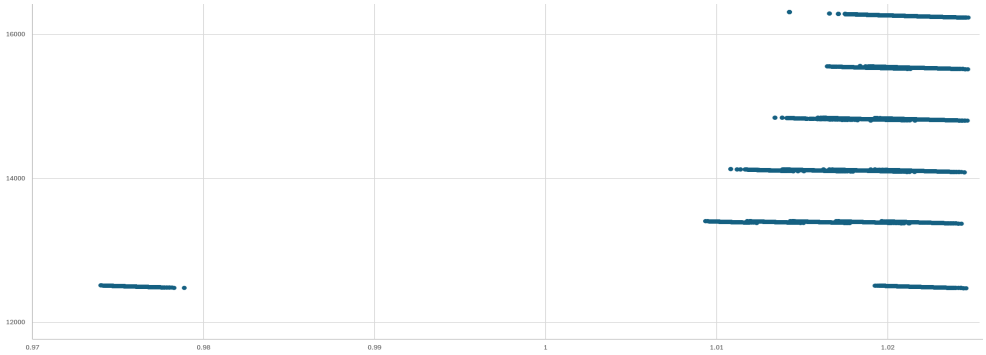

Fig. 13. 6+1 Sets of APNs with Focus on Lower DMI and Lower WZ Values.

Likewise we have 6+1 layers if we count entries at ± 16 in the LAT table.

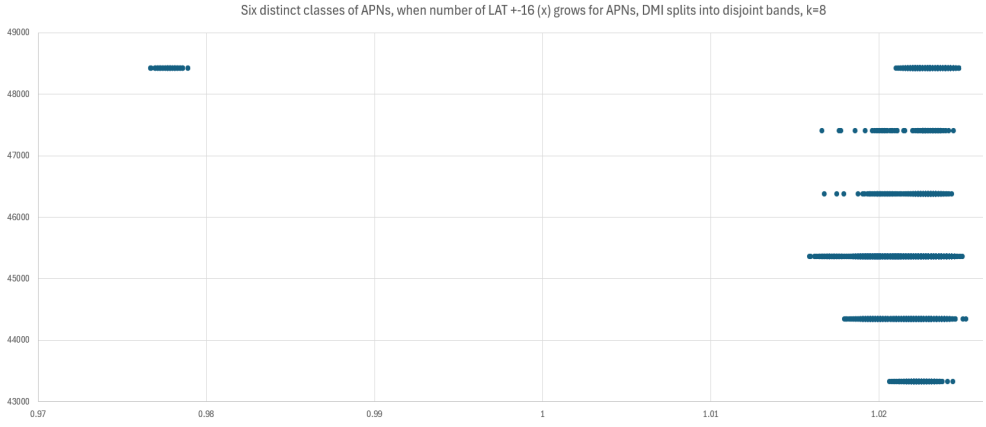

Fig. 14. Six Layers of APNs with Focus on Lower DMI and LAT entries at ± 16.

Needless to say, the 6 classes for WZ are in bijection with classes for ± 16:

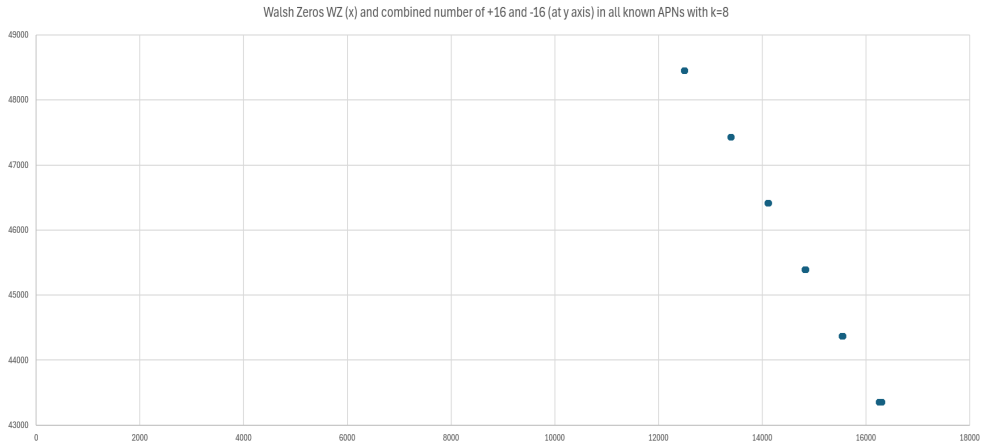

Fig. 15. Walsh Zeros vs. ± 16 values in Walsh in 6 layers of all known APNs.

On Low DMI and Kaleyski Gamma Rank in APNs

We have studied 28 different classes of APNs classified by their Gamma rank values between 11818 (smallest known) and 15358 (largest ever observed). Only for 4 out of 28 known values of Gamma we ever found any APN with DMI<1.0. These values are 14032, 14034, 14036 and 14042. Numerous specific examples of these appear in datasets of Beierle-Leander and Langevin *at al.*.

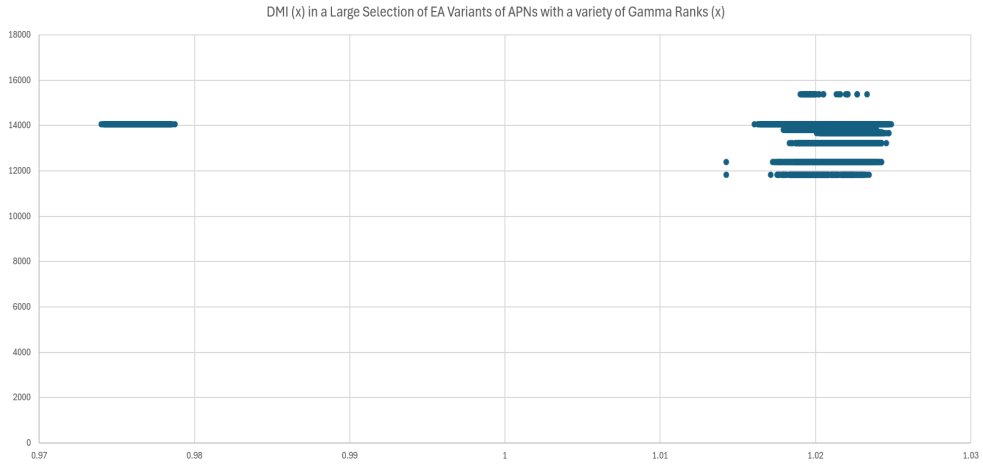

Fig. 16. DMI and LSS size of a matrix for multiple EA variants of APNs

11 ASIC Implementation Considerations

In this paper we intend to study and evaluate several S-boxes not only on security w.r.t. some new precise differential and linear security metric, but also on an ASIC implementation cost metric. This as a first approximation and [for now] without considering side channels or/and threshold implementations [BGST23].

In Table 16 we already studied the MC (Mutliplicative Complexity) metric [BoPePo00] which is expected to "heuristically" approximate the overall ASIC cost, cf. [BoPe10]. Another important heuristic is the number of high degree monomials, cf. [BGST23]. Here we do not have any high degree monomials therefore we will try to approximate ASIC cost by the overall number of homogeneous quadratic monomials with multiplicity. For example if a monomial say x_2x_3 appears twice in say both y_0 and y_2, we count it twice. We call "hom-MQ size" this total number of quadratic terms in any given S-box.

In the table which follows we are going to study whole classes of S-boxes and also compare them to existing individual S-boxes. In order to avoid comparison of apples and oranges, we write *68* for a whole class of S-boxes known as 68 in [BoBiSa17], with two stars which correspond to our ability to add two arbitrary affine transformations on the input and output side. Finally we write just Ascon/Fides if there is no modification to an existing S-box.

classes	best LSS	min(hom-MQ size)	MC	DMI
28	13	4	5	2.16
36	13	5	5	2.16
52	11	6	5	2.06
62	11	8	6	2.06
56,*58*	11	8	5	2.06
61	11	6	6	2.06
65	11	7	6	2.06
47	**10**	**6**	5	1.97
59	11	8	5	1.94
Ascon	11	11	5	1.91
Keccak	11	**5**	5	1.91
68,*Ascon*,*Keccak*	**11**	**10**	5	1.91
53	11	10	5	1.88
70	11	12	5	1.88
63	9	11	6	1.69
71	10	12	6	1.69
72	9	12	5	1.59
66	10	11	6	1.56
67	9	10	6	1.56
73	8	12	6	1.41
75	7	14	7	1.12
74,*Fides*,*Inv*	7	15	8	1.12
Fides	7	18	8	1.12

Table 17. Comparison of LSS, MC and our [simplified] ASIC implementation cost metric for various S-boxes on 5 bits and full classes with affine equivalence

At this moment we only report results where $DMI(\delta_{in};\delta_{ou}) \leq 2.2$ bits. We conjecture that the combination of best LSS size, our hom-MQ metric and the classical MC metric allow to predict and anticipate with great precision the actual GE (Gate Equivalent) cost of silicon with best available tools. We also conjecture that the GE cost is roughly proportional to our hom-MQ metric, while the older MC metric is simply not precise enough to be used in practice as a tool for predicting the ASIC cost, and in rare cases it is inaccurate and misleading. Our ASIC implementation results are presented in Table 17.

Note. Our ASIC cost metric is not an ideal one, it only applies to quadratic S-boxes. We can of course count cubic monomials like in [BGST23], or degree 4 monomials in S-boxes from [MeBi19], however it is hard with this method to compare the ASIC implementation cost for two S-boxes which do not have the same degree. One [imperfect] solution to this problem is to consider the "latency complexity" metric defined in [LatRa22] which considers that all outputs are distinct functions which makes that it is efficiently computable for all small S-boxes. In all cases, it makes sense to optimize an S-box in the order of decreasing algebraic degree cf. Paar-Turan heuristics in Section 3 of [PaTuH24] and also numerous works alongside MC and the Peralta heuristics [BoPe10].

class or S-box	hom-MQ size	specification (best known example in each singleton or class)
36	5	2,0,1,3,26,25,27,24,14,12,13,15,6,5,7,4,20,22,23,21,8,11,9,10,16,18,19,17,28,31,29,30
47	6	0,17,21,4,18,3,23,6,25,24,12,13,10,11,15,14,19,2,7,22,5,20,1,16,8,9,28,29,31,30,27,26,
52	6	16,30,22,26,18,28,20,24,25,23,15,3,11,5,29,17,6,12,8,0,4,14,10,2,13,7,19,27,31,21,1,9
61	6	1,10,19,25,9,2,27,17,6,5,16,18,12,15,26,24,31,20,13,7,22,29,4,14,8,11,30,28,3,0,21,23
65	7	8,21,10,19,2,23,16,1,9,6,11,0,3,4,17,18,7,27,5,29,13,25,31,15,22,24,20,30,28,26,14,12
Ascon	11	4,11,31,20,26,21,9,2,27,5,8,18,29,3,6,28,30,19,7,14,0,13,17,24,16,12,1,25,22,10,15,23
Keccak	**5**	0,9,18,11,5,12,22,15,10,3,24,1,13,4,30,7,20,21,6,23,17,16,2,19,26,27,8,25,29,28,14,31
67	10	0,8,3,10,2,14,1,12,16,24,21,28,19,31,22,27,5,13,20,29,11,7,26,23,17,25,6,15,30,18,9,4
71	12	23,7,13,28,27,25,21,22,9,17,19,10,1,11,15,4,12,20,6,31,16,26,14,5,24,8,18,3,0,2,30,29
72	12	25,21,13,3,26,2,31,5,7,27,18,12,0,8,4,14,16,29,6,9,19,10,20,15,22,11,1,30,17,24,23,28
73	12	16,30,21,25,19,29,4,8,18,24,6,14,1,11,7,15,17,13,28,2,22,10,9,23,3,27,31,5,20,12,26,0
74	15	0,28,7,9,29,4,16,27,15,18,12,3,26,2,19,25,23,8,24,21,11,17,14,6,10,20,1,13,30,5,31,22
75	14	0,13,30,17,11,14,5,2,20,26,3,15,27,29,28,24,16,9,31,4,25,8,6,21,12,22,10,18,1,19,23,7

Table 18. Examples of S-boxes inside their class which minimize min(hom-MQ size).

Note. It is possible to see that the Keccak χ_5 S-box is an outlier regarding the low complexity of each output bit. This can also be observed in the work on 8 bits in [LatRa22] and we know extremely few such S-boxes.

Disclaimer. These examples should be regarded as a proof by example that it is possible to reduce the cost of the S-boxes in each class, rather than a recommendation. It is simply dangerous to optimize an S-box just on the implementation cost. It should first and foremost satisfy various security and quality criteria, studied in the cryptographic literature since 1970s, cf. [T310S,LMCFW23,DaPhD95] to name only a few. It in fact foolish to consider that the cheapest S-box will be the best choice. In fact, several authors observed that Keccak S-box is extremely weak, cf. Example 2 in Section 2. Many later ciphers such as Ascon or Icepole [IceP14] can be viewed as distinct attempts to remedy these problems. In spite of all this, serious signs of weakness remain in Ascon cf. Section 3.4 or Table 6 or Table 22 and at many other places in this paper.

12 Study of DDT and DMI Limited to Specific Sets

In Section 7.1 we discovered that our subsets of 11 points have exceptionally large coverage of 6/8 of the points which should interest the attacker the most: those involved in DDT() sets of maximum size 8/32. In this section we define a **more precise** methodology in order to compare quality of arbitrary sets of point from a purely differential cryptanalysis point of view. This inspiration for this work is the Table 16 above, where we see that DMI seems to a very decent overall measure of a quality of an S-box, and reproduces the ordering of S-boxes of Table 1 in [BoBiSa17] in a very predictable and near-monotonous way from weaker to stronger. In comparison the authors [BoBiSa17] have classified the S-boxes by 3 key parameters based on 3 most prominent notions of "non-linearity", which are [the sizes of] DDT,LUT and MC, cf. [BoFiPe13] and we obtain a simpler yet very similar classification.

It is easy to see that the classical mathematical definition of DMI which can be found in wikipedia [wikiMI] applies also to arbitrary subsets of points, see Definition 11 in page 8. This amounts to the we study DDT restricted all pairs x, x' inside a subset S. We get therefore a yet more precise measure of overall quality of an S-box w.r.t. differential cryptanalysis provided that the attacker is indeed able to achieve a certain peculiar probability distribution for a set of points. This leads to a new refined optimization problem in cryptanalysis which is very closely related to almost everything we do in this paper.

Definition 15 (DMI Profile of an S-box for Arbitrary Probability Distributions). *For every $H \leq 5$ bits, what is a probability distribution with input Shannon entropy H, which maximizes the value $MI(\delta_{in}; \delta_{ou})$ and what is the highest possible value of DMI? We call a graph representing all possible pairs H, DMI_{max} the DMI Profile of an S-box.*

The computation of the profile defined above is extremely difficult and can only be approximated. We define:

Definition 16 (Simplified DMI Profile of an S-box with Arbitrary Subsets:). *For every $n \leq 2^5$, and assuming uniform distribution inside a set S of n values out of 2^k, we call the "Simplified DMI Profile" a graph made with pairs (n, DMI_{max}) where DMI_{max} is the maximum of*

$$MI(\delta_{in}; \delta_{ou} | x; x' \in S)$$

over all possible sets of size $\#S = n$.

In the table below we display some values which compare the Simplified DMI Profile of Fides and Ascon/Keccak S-boxes.

LSS size	7-9	10-15	16-19	20-22
Chi/Ascon/Keccak	4.375	4.806	3.808	2.999
Fides/Primate/InvGF32	4.357	4.046	3.124	2.482

Table 19. Simplified DMI Profile for two S-boxes.

Interpretation of Results. We claim that the Simplified DMI Profile provides a precise and robust measure of vulnerability of an S-box against Differential Cryptanalysis when restricted to sets of n distinct values. It is also possible to see that for each S-box there exists an optimal size which maximizes this quantity. This does not however mean that this is the value on which the attacker should concentrate his algorithmic efforts. It is not easy to see if 2.99/5.0 of bits of shared information between input and output differences for a property which is true for a large fraction of 22/32 of the whole space, would advantage the attacker more; than the very impressive 4.8/5.0 bits of shared information for a property concerning a substantially smaller fraction of 11/32 the whole space.

Open Problems. The values reported in this table were obtained in a long GPU computation on the best effort basis and are not guaranteed to be optimal at the moment. It is an open problem to design an efficient optimized algorithm to compute the Simplified DMI Profile of an S-box. In practice it can be approximated by some heuristics such as for example considering unions of various affine spaces, DDT() sets, LSS sets, sets related to LAT, etc.

13 Conclusion

The main topic of this paper is the discovery of the strongest possible simultaneous linear approximations of a cryptographic S-box. We focus in particular on the peculiar S-box called Chi and used in Ascon, Keccak and many other ciphers, and few other comparable S-boxes. At Eurocrypt 2017 researchers have constructed collisions on up to 6 rounds of Keccak with so called "connectors"'. These connectors exploit a set of 80 partial linearization LAS properties with 4 points which form an affine space of dim 2 on both sides. The same LAS properties also work for Ascon.

In this paper, starting from observations about further remarkable sets involved in a variety of simultaneous linear, differential and translation properties, we have generalized and extended previous work in several ways. The notion of of functional approximations (LAS) within affine spaces was extended to consider to sets of arbitrary points (LSS) and to the study of arbitrary linear I/O relations which are no longer functions (LIO). In this new framework, the classical question of repeated affine spaces with applications in cryptanalysis is extended and becomes the study of pairs of sets equivalent by a translation with a constant. We have done an exhaustive study of how pairs of maximum dimension affine spaces can approximate the Ascon S-box and exhibit several interesting sets and spaces which maximize the number of linear I/O relations. Our new functional approximations are optimal and the strongest possible.

13.1 Summary of Results

It is possible to be surprised by how frequently different sets of points such $DDT()$ sets and all sets related to LAT are related to each other by translation, cf. Table 5, Table 20 and Table 22. Some of these properties are explained by internal hidden rotational symmetry of the Chi/Keccak S-box. Some other are related to our LSS-k translation property of Thm. 3. Other are not LSS, and will still be similar to (fewer) other sets. All these suggest that cryptanalysis with arbitrary subsets of points is in fact less complex than it seems with 5-bit S-boxes. For random S-boxes, even at size 5, translation properties happen very rarely or only by coincidence, and typically not all, cf. Table 7 and Table 16. Zero translation similarities like those those studied here occur with the AES S-box (size 8).

In our research we have studied many natural ways of approximating our S-box by linear functions, or LIO relations, or by linearity when restricted to subsets, and found many new properties at sizes 6,8,10,11 and 12 which are stronger, more general and **larger** than those previously studied. Our best result is a class of "maximal" linear approximation properties with as many as 11 points, cf. Thm. 2, which is optimal. We have shown that arbitrary translations of such sets give equally strong full matrix+vector linearization LSS-11 properties. This result holds for **all** quadratic S-boxes cf. Thm. 3, and therefore for an overwhelming majority of 5-bit S-boxes used in practical applications [Ascon,SHA3,SHAKE,etc..], but not in Thakor/Icepole.

In Section 12 we observed that mutual intersections between various translations of our maximum size set are in a one to one correspondence with the set of all lines of the DDT table of Ascon classified by relative strength.

13.2 Is there Anything Wrong with Ascon or Keccak?

A detailed analysis based on a classification of all quadratic permutations, shows however that it is not possible to claim that Ascon or Keccak S-box are special or dangerous or worse than other quadratic S-boxes. In fact Table 16 demonstrates very clearly that they behave like typical quadratic S-boxes with $DMI \approx 1.9$ and a cubic inverse.

The only reasonable way to make these ciphers **arguably and measurably more secure** is to use an S-box with lower DMI. Here $DMI \approx 1.1$, and LSS=7 is the best property we can hope for. This is optimal, and we confirmed that APN are the strongest possible S-boxes, here with respect to a new "global" security notion based on simultaneous linear approximations of type LSS. In Table 17 we show that though reducing max LSS size very slightly to 10 is possible with classes 68 and 47. Then we see that reducing it further is impossible with quadratic permutations at size 5, without increasing the expected ASIC area / implementation cost by up to 50%. In addition this paper provides new arguments which suggest to mistrust all quadratic S-boxes in general, see Thm. 3, and switch to the likes of Thakor and Icepole or to 4-17 quartic type. An open problem is to find yet better S-boxes with LSS$<$ 11 and similar or better ASIC implementation cost, cf. Table 17 in Section 11.

13.3 Future Research

These properties are expected to lead many new ways of attacking Ascon with a mix of linear and differential properties where the S-boxes will be (under certain conditions) modeled by linear equations and where differentials act in predictable ways [or/and with quite large probabilities]. It is possible to see that best known attacks on Ascon,Keccak and many other ARX ciphers are differential-linear attacks, cf. for example [TeDi19,HaDeEi24,TeDiLi23]. We postulate that attackers need to construct and find new forms of invariant differential-linear attacks, combined with state partitioning with sets of certain type, which will be very different than any currently known differential-linear attacks [using connectors or in [TeDiLi23]] where differential and linear properties occurred in different and disjoint parts of the cipher.

One example of how an invariant attack with affine spaces can work together with predictable differentials at every round inside one single attack can be found in [CoQi20]. Future work should try to construct similar attacks which would no longer use perfect affine spaces but rather more irregular sets.

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A More Translation Invariance with 961 Pairs in LAT

In this paper we have seen that one major weakness of Ascon S-box was the fact that the A_{11} property does not exist in isolation, but is stable by arbitrary translations. This is our **first holographic property** cf. Thm. 3. For example, an attacker is trying to manipulate the state of one S-box in order to make sure it will be one of the 11 values. Then we have an applications of a secret constant such as a cryptographic key, or a public constant such as the round constant in Ascon equal to 0x4 at one S-box. In spite of this, we obtain a related set with a similar property. Finally we will maybe even obtain the exact same configuration after a few rounds in some [hypothetic] periodic invariant attack. Now and again, with Thm. 3 this property is less exceptional than it seems, and it actually holds for all LSS properties of any size, and for all quadratic S-boxes.

Another interesting set of translation similarities was studied in Section 4 and in Table 5. We saw that in 336+40 cases and in all cases without exception which are unbalanced, i.e. not of type 8+8, two pairs of sets are equivalent by [just one] translation constant. This is our **second holographic property**. We have also seen that this did not happen **at all** for Thakor. We are going to see that such undesirable properties happen for all small size quadratic S-boxes used in cryptography (but not for larger sizes). Following Section 4 we recall the main motivation of this exercise:

Key Question: Is it possible that an S-box sends **a "large" subset** of some (maximum size) affine space of dim 4 at the input side, to a "large" subset of another affine space of dim 4 at the output side?

This leads to different ways of partitioning of 32 elements into 2+2 disjoint sets like 6+10+6+10 or similar. It would also be interesting to see how many entries of different type we see, which is essentially about relative frequencies of different entries in the LAT table. We show some further detailed results of this type in Table 20 below.

S-box	all	8+8	7+9	6+10	5+11	4+12	0+16	DMI
28	961	702	0	192	0	**128**	0	2.16
inp-shifted		474	0	192	0	**128**	0	
unrelated		228	0	0	0	0	0	
47	961	552	0	384	0	48	0	1.97
inp-shifted		174	0	384	0	48	0	
unrelated		378	0	0	0	0	0	
60	961	648	0	256	0	56	0	2.34
inp-shifted		366	0	256	0	56	0	
unrelated		282	0	0	0	0	0	
68,Ascon	961	585	0	336	0	40	0	1.91
inp-shifted		240	0	336	0	40	0	
unrelated		345	0	0	0	0	0	
70,69,71	961	585	0	336	0	40	0	1.88,
inp-shifted		240	0	336	0	40	0	1.72,
unrelated		345	0	0	0	0	0	1.69
67,66	961	585	0	336	0	40	0	1.56
inp-shifted		240	0	336	0	40	0	
unrelated		345	0	0	0	0	0	
72	961	525	0	416	0	40	0	1.59
inp-shifted		120	0	**416**	0	40	0	
unrelated		405	0	0	0	0	0	
73	961	501	0	448	0	24	0	1.41
inp-shifted		72	0	**448**	0	4	0	
unrelated		429	0	0	0	0	0	
74,75,Fides	961	465	0	496	0	0	0	1.12
inp-shifted		0	0	**496**	0	0	0	
unrelated		465	0	0	0	0	0	
Icepole	961	435	220	240	36	30	0	1.82
inp-shifted		120	10	160	6	20	0	
unrelated		315	210	80	30	10	0	
Thakor	961	270	420	196	60	15	0	1.59
inp-shifted		0	0	0	0	0	0	
unrelated		**270**	**420**	**196**	**60**	**15**	0	

Table 20. Pairs of spaces of the same size which happen to be equivalent by translation [vs. just unrelated sets of the same size] in several S-boxes and their affine equivalents.

B Focus on Stronger S-boxes - Not Quadratic

In Table 7 in [MeBi19] we find a list of 17 particularly strong quartic (degree 4) permutations on 5 bits. Here below we analyze their vulnerability to translation self-similarity properties with pairs of hyperplanes, as studied above and in Section 4. We recall some of the strongest S-boxes in this set and to avoid confusion with previous class numbers, we pre-pend 4- to the numbers used in Table 7 in [MeBi19].

sbox	specification
4-11	0,1,2,3,4,6,8,12,5,11,16,24,22,26,9,19,7,23,10,13,31,18,20,29,27,30,28,15,14,17,21,25
4-12	0,1,2,3,4,6,8,12,5,13,16,23,17,18,24,11,7,29,21,27,25,9,22,10,31,14,15,20,19,30,28,26
4-13	0,1,2,3,4,6,8,12,5,14,16,26,10,27,23,31,7,24,11,28,20,17,9,18,25,21,13,30,15,22,29,19
4-14	0,1,2,3,4,6,8,12,5,16,13,23,25,21,26,14,7,17,20,28,29,19,11,9,15,10,31,24,27,18,30,22
4-15	0,1,2,3,4,6,8,12,5,16,21,26,31,22,18,10,7,24,17,13,30,14,19,27,20,9,23,25,11,29,15,28
4-16	0,1,2,3,4,6,8,16,5,10,20,29,7,31,27,13,9,25,15,18,19,14,22,26,21,17,11,12,30,28,23,24
4-17	0,1,2,4,3,6,8,16,5,10,15,27,19,29,31,20,7,18,25,21,12,14,24,28,26,11,23,13,30,9,17,22
Fides	1,0,25,26,17,29,21,27,20,5,4,23,14,18,2,28,15,8,6,3,13,7,24,16,30,9,31,10,22,12,11,19

Table 21. The top or the strongest quartic S-boxes according to Table 7 in [MeBi19].

More Translation Properties - 8li8 and 8ci8. In Table 22 below, we introduce and study new types of translation properties, which will sometimes denote by a shorthand notation **8li8** and **8ci8** and which exist only in size 8, and which are different than 8+8 similarity studied so far. In essence, the 8+8 property was to see how many times for different pairs of hyperplanes out of 961, we get two sets in one diagonal inside Def. 14, like for example:

$S^{-1}[L^o] \cap L^i$

$$co - S^{-1}[L^o] \cap co - L^i$$

which are related by a translation and were studied in Thm. 1 page 18. In contrast, the 8li8 property occurs when two sets in the 1st column are related by a translation, which sets are exactly:

$S^{-1}[L^o] \cap L^i$

$co - S^{-1}[L^o] \cap L^i$

Furthermore we denote by 8ci8 the number of pairs of hyperplanes out of 961 when this happens in the 2nd column of our table of Def. 14.

Note. These events 8li8 and 8ci8 happen only for balanced partitions where the sizes of all 8 sets are the same. We do not report when it happens that two sets in the same line are related by translation. This happens less frequently in general, and this information can be entirely deduced from 8+8 and 8ci8 and 8co8 values displayed below (next page).

S-box	best LSS	8ci8	8li8	8+8	7+9	6+10	5+11	4+12	0+16	DMI
28	13	702	702	702	0	**192**	0	**128**	0	2.16
inp-shifted		**702**	**702**	**474**	0	**192**	0	**128**	0	
unrelated		0	0	228	0	0	0	0	0	
68,Ascon	11	585	585	585	0	336	0	40	0	1.91
inp-shifted		585	585	240	0	336	0	40	0	
unrelated		0	0	345	0	0	0	0	0	
73	8	501	501	501	0	448	0	24	0	1.41
inp-shifted		501	501	**72**	0	448	0	4	0	
unrelated		0	0	448	449	0	0	0	0	
74,75,Fides	7	465	465	465	0	496	0	0	0	1.12
inp-shifted		465	465	**0**	0	**496**	0	0	0	
unrelated		0	0	465	0	0	0	0	0	
Icepole	11	435	435	435	220	240	36	30	0	1.82
inp-shifted		435	315	120	10	160	6	20	0	
unrelated		0	120	315	210	80	30	10	0	
Thakor	11	270	270	270	420	196	60	15	0	1.59
inp-shifted		61	62	0	0	0	0	0	0	
unrelated		209	208	270	420	**196**	**60**	**15**	0	
4-13	9	245	245	245	410	236	140	0	0	1.36
inp-shifted		29	31	0	0	**16**	0	0	0	
unrelated		216	214	245	410	220	**140**	0	0	
4-11	10	215	215	215	435	250	122	0	0	1.30
inp-shifted		12	13	0	0	0	0	0	0	
unrelated		203	202	215	435	250	122	0	0	
4-15	9	215	215	215	435	250	122	0	0	1.30
inp-shifted		16	19	0	0	0	0	0	0	
unrelated		199	196	215	435	250	122	0	0	
4-16	8	205	205	205	430	276	100	0	0	1.24
inp-shifted		29	31	0	0	**16**	0	0	0	
unrelated		176	174	205	430	260	100	0	0	
4-17	**9**	155	155	155	465	310	62	0	0	1.12
inp-shifted		**0**	**0**	0	0	0	0	0	0	
unrelated		**155**	**155**	**155**	**465**	310	**62**	0	0	

Table 22. New quartic S-boxes compared to some 5-bit S-boxes previously studied.

B.1 Translation Similarity with 6-Bit and 8-bit S-boxes

We have examined all top 8 or the so-called "strong" 6-bit quadratic permutations listed in Table 15 in [MeBi19]. They all have the same characteristics with $DMI = 2.06$ and max $DDT()$ size of 8. All these permutations give the same results: we either have 16+16 situation with 3024/3969 cases of translation similarity, or 12+20 situation with guaranteed translation similarity in 1008 cases out of 1008. These are the only two types of partitioning which actually happen.

AES S-box. We have finally also looked at what happens with the AES S-box. We found that there is a great variety of 65025 pairs of hyperplanes, and yet translation similarities between pairs of sets of the same size NEVER happen at all for the AES S-box.

B.2 Some LSS Results for 6-Bit S-boxes

S-boxes at size 6 are well studied for classification and partial linearity, cf. also Appendix B in [CARG19] for DES. For a 6-bit S-box one single output will be constant on a set of 8 points forming an affine space of dimension 3 for some 3/4 of all Boolean functions in 6 variables, and yet weaker otherwise, cf. Theorem 5.2 in [CARG19]. However linearizing all 6 outputs at the same time is much harder and very strong S-boxes exist in dimension 6.

classes	NL	δ	MC	DMI	LAS-2	best-LSS
2256-2258	24	4	6-8	2.0625	336	≥ 15
2259	24	4	5-8	2.0625	336	≥ 15
2260	24	4	5-8	2.0625	336	≥ 15
2261	24	4	8	2.0625	336	≥ 15
2262	24	4	6-8	2.0625	336	≥ 15
2263	24	2	≤ 9	2.0625	336	≥ 10
SMS34	24	4	-	2.0625	336	≥ 10
SMS26	24	4	-	2.0625	336	≥ 10
SarkarS6 Cubic	24	4	8	2.0625	336	≥ 10
Sarkar Gold	24	4	8	2.0625	336	≥ 10
SarkarS6' MQ	24	4	8	2.0625	336	≥ 10
BipBip	24	4	-	1.6406	192	≥ 10
Kim-S6	24	4	9	1.6406	192	≥ 11
SC2000-6	24	4	10-19	1.1396	21	≥ 10
Fides-6	24	2	9-29	1.0781	0	9
APN-6	24	2	≤ 198	1.0781	0	9

Table 23. Some LSS results for the very top "strong" quadratic permutation classes on 6 bits out of 2263 following their classification in [MeBi19].

A list of the 13 different 6-bit quadratic APN functions was proposed by Banff can be found in Table 6 in [CCP22]. A further list of 20 non-quadratic APNs was produced by Langevin cf. [CCP22] and their work was later extended by many authors for 8 bits etc, cf. [APN425]. A big fraction of APNs in these recent works are quadratic and not bijective.

Here we observe very substantial differences between S-boxes which have exactly the same DMI value. For example clearly 2263 is much stronger and not EA-equivalent compared S-boxes 2256-2262 which have larger LSS-15. We also see that BipBip and Kim-S6 are not EA-equivalent because their best LSS result is not the same.

Most of our figures Multiplicative Complexity (MC) come from [JeBaKi24]. However it is an open problem to estimate the MC for many of these S-boxes for example in the case of Dillon-Wolfe a.k.a. APN-6, cf. [DiWo09]. All known implementations of Dillon-Wolfe APN-6 have a very large area and MC is extremely large and close to 200, cf. [BIK24]. This shows that better implementations are likely to exist.

C Some Results on 8-bit S-boxes

C.1 How to Compute LSS Properties at 8 Bits

The notion of LSS is essentially the same vector nonlinearity NL_v which is defined by a distance to all affine functions for the whole S-box and all outputs simultaneosly. According to [Nagy22] computing NL_v or equivalently finding large size LSS properties is very difficult. A summary of all LSS properties of all sizes in Ascon was given in Table Table 11 page 29. A summary of all LSS properties of all sizes in AES was given in Table 13 page 30.

A good proportion of LSS properties in this paper have been computed using a SAT solver. Thiks approach works well for 5,6 bits, and becomes problematic at 7 and 8 bits.

For 8-bit boxes, we also used a simplistic brute force method with GPU which tries 2^{64} matrices and recovers the vector. In a companion paper currently not yet public, we present a simple algorithm to find LSS properties by partial guessing and filtering, for S-boxes on 8 bits which runs on CPU and without a GPU. These methods are not deterministic and not quite complete. They can find routinely within 8 hours all known properties of sizes up to 17, and also all larger properties where one can no longer guarantee that larger properties do not exist. They may still exist.

There exist an upper bound on LSS developed by Carlet, Nagy and others cf. [Nagy22]. Namely we always have:

$$Max(LSS) = 2^k - NL_v \leq \sqrt{\delta} \cdot 2^{k/2} + \frac{1}{2}$$

where δ is the maximum value in the DDT table. There exists also a lower bound conjectured on LSS, therefore not truly guaranteed to be correct; see Appendix C.5.

$$Max(LSS) \geq 2^{k/2} + 1$$

In this paper we only report concrete LSS results which are therefore guaranteed to be correct as a lower bound (one can still maybe improve). There is always a huge gap between the two bounds (lower and upper). For AES we have for example $Max(LSS) = 2^k - NL_v \leq 16\sqrt{4} + 0.5 \leq 32.5$. sfor AES and typically even worse for any other S-box for example for Lilliput we have $23 \leq Max(LSS) \leq 45$.

When this paper was first written we believed that many of our results are already optimal. However 6 months later we found an LSS-19 property for Fuller-Millan and yet few months later, a new paper presented at Eurocrypt 2026 was presented [BiTuUd26] together with a github cf. [BiTuUd2G]. This algorithmic work allows to improve more than half of our earlier results, cf. Table 24 below.

The reader might find it hard to believe that LSS properties of sizes LSS-18 and beyond might at all exist. For example there is approximately $\binom{256}{18} \approx 2^{91}$ possible sets of 18 points in AES and there is 2^{64} possible matrices for any S-box on 8 bits. For LSS-19 it is already unthinkable to be sure if they might at all

exist, unless it is guaranteed by some theorem. For this reason we publish here concrete examples of the matrix A for each of these properties. From these it is easy to recover the LSS constant c and the set V by majority vote and counting the solutions to $S[x] = A \cdot x + c$.

We do not think that a single LSS-19 property exists in AES, more details about this will be found in a companion paper and this question was also covered during a talk at Eurocrypt 2026 cf. [BiTuUd26].

For all AES-like S-boxes we have examined the LSS-18 properties as one-sided pure sets V of size 18. These V are also 2-thin sets cf. [Nagy22] and these V are also Sidon sets of maximum size. Sidon sets of size 18 are optimal, cf. Table 2 in [CzPo24]. Sidon sets normally occur with APNs, here they occur without APNs.

In practice our algorithms more or less NEVER find some very rare but already known properties such as LSS-18 for AES or SMS4 in a reasonable time, though we find a plethora of similar properties of size say 15-17. In order to find more LSS-18 and count such properties we also used an alternative method: translate one of the LSS-18 results in Table 24 and apply two matrices S_1 and S_2 using the recent very efficient affine equivalence algorithm described in [HZLY24].

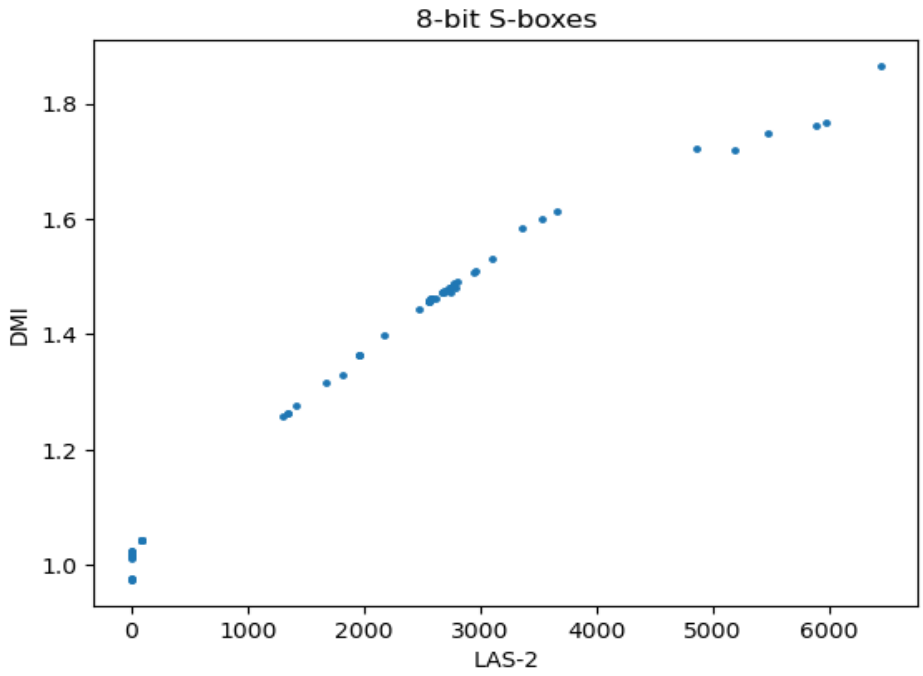

Fig. 17. DMI and LAS-2 count for a selection of 8-bit S-boxes with closer zoom at low DMI range. See also Fig. 3 page 11.

C.2 On LSS and DMI in Selected 8 bit S-boxes

DMI and LSS values should be considered as a major proxy tool for an early simplified classification of S-boxes on 8-bits. The famous and key question here to study, is that there is a near continuous space of S-boxes where the DMI and LAS-2 go down, and LSS likewise.

sbox	DMI	LSS matrix	size
Skinny	2.9043	2010010802408004	47
PIPO	1.9046	C60D539848D59698	29
Fantomas	1.8656	9B9BDD100F90C554	≥ 24
$S^{8,8,N}_{3917,353,16}$	1.4922	982BF0385F040DDB	≥ 20
SB^8_1	1.6634	993979131B747370	≥ 18
SB^9_5	1.4805	54256E228BE28A1C	≥ 18
SB^4_3	1.4594	5631C86CFCF4D18A	≥ 18
Clav37	1.5656	46D8F7968CC20E1D	≥ 20
Clav40	1.5308	ED6965F25E560732	≥ 18
Whirlpool	1.4618	7DDEC9BF1ADA2EE0	21
Kuznechik	1.3650	D3BEF7E373163B95	20
BelT	1.3169	3113562F484620A6	20
Kalyna_{pi0}	1.2756	501FDEC17BF64813	20
$x^{181}+x^{173}+x^{109}$	1.2141	1722D2BAC13CDE8A	21
FullerMillan	1.1282	DD8C2559EAE4B2DE	19
AES	1.0429	507EBDAACB7DFD67	18
Camellia	1.0429	DCDD72BE95FACF5F	18
Chiasmus	1.0429	1E382A149F42D986	18
DBLOCK	1.0429	ADF8BA7A4B47BD2E	18
SMS4	1.0429	6D4749E59B74CC4C	18
Aria-S2	1.0429	5E403717F0DBEF0D	18
SEED-S0	1.0429	633A433D758F0539	18
CLEFIA-S1	1.0429	3605E3016499DA85	17
ZUC-S1	1.0429	7A0DADAA4656FF24	17
Hierocrypt3	1.0429	2A9EA7B11801C28F	17
SEED-S1	1.0429	B461C27A49A564AB	17
Halka	1.0429	B222FA030ECB2527	17
LFT035	1.0429	EBBDE9687CDC2BE5	17
Pentanomial	1.0429	3618C7CA09FF685B	17
LFT216	1.0428	7F67F1B672D97C3C	17
Wuhan3	1.0428	D3A1828D2280E6BD	17

Table 24. A Selection of Large Size LSS Properties for Major 8-bit S-boxes

Our work also gives some new fresh and refined answers and insights about what S-boxes can be considered as best possible in terms of cryptographic quality, and the answers are not quite or not exactly as expected by many researchers.

When we put the $\geq$ sign in Table 24 this means these examples have not yet been fully explored by a deterministic algorithm which is guaranteed to

output the best solution, and we are not sure if our result is yet final or optimal. Typically in these cases the space possible solutions is particularly large. For example the number of possible sets of size 19 is $\binom{256}{19} \approx 2^{95}$. and at size 29 we have $\binom{256}{29} \approx 2^{127}$. The properties listed here were found after many months of efforts on a PC by using a very simple heuristic algorithm which first finds (larger) solutions for a reduced number of outputs. Such algorithms can however exhibit some blind spots and one can do better than this.

New Results - Eurocrypt 2026

A recent paper presented at Eurocrypt 2026 is devoted to the stochastic and also exhaustive enumeration of LSS properties at all sizes, cf. [BiTuUd26]. Several methods to compute LSS properties are proposed, studied in implemented, and it comes with a very nice publicly available github cf. [BiTuUd2G]. This new paper is a true game changer in LSS study. It allows to improve multiple results which we have earlier included in our table: namely for Kalyna, BelT, Kuzneychik and Whirlpool, cf. Table 24 above, previously known as Table 22 in earlier versions.

On Vectorial Linearity of Permutations and APNs

Interestingly this very recent paper completely misses to consider a number of well-known S-boxes. As a result the paper makes a mistaken claim or observation, to the effect that the AES-type inverse S-boxes would be optimal S-boxes and that permutations cannot do any better in terms of vectorial nonlinearity.

It is a cliché to believe that the AES S-box would be the best in all categories. The authors fall into a trap of wishful thinking here. In page 15 or eprint version of 8 May 2026 they write; *"We observe that for 8 bits, the strongest permutation is the AES S-box.* This affirmation is not at all correct.

Let us first explain why authors used here the word "permutation". It was previously observed many authors that for many APNs which are not permutations including simple Gold x^3 function, we will only achieve LSS-15, i.e. we have VecLin=15=Max(LSS). However it is a big **mistake** to consider that intermediate S-boxes with VecLin=16,17=Max(LSS) cannot be permutations and that AES would be the strongest S-box. In fact in our Table 24 above, previously known as Table 22, there are not less than 6 examples which violate the naive belief that the AES S-box would be the best or optimal. These are CLEFIA-S1, ZUC-S1, Hierocrypt3, SEED-S1, LFT216, Wuhan3 and more recently we also added three more counterexamples which are Halka from [Halka]; LFT035 from [LFT35], and Pentanomial from [DB524].

As a conclusion, we see that Max(LSS)=17 is achievable for numerous permutation S-boxes, the study of which was simply omitted or ignored so far in this latest and otherwise truly excellent algorithmic work of [BiTuUd26] and inside latest datasets found inside the github [BiTuUd2G].

C.3 Open Problems regarding LSS in permutations vs. APNs

It is an open problem if Max(LSS)=16 is achievable for some permutation S-boxes. We conjecture that such S-boxes do not exist.

One level below, we could have Max(LSS)=15 which would be achievable only for APNs. One of them in BL3638 listed below.

On average APNs seem to achieve lower LSS sizes than permutations. However, it should be noted that APNs with LSS-18 do exist, and hence not all APNs are at the same level of cryptographic strength or quality.

We further conjecture that APNs with LSS-19 most likely do not exist.

For some APNs there are gaps: for example in some cases maximum size properties LSS-17 do not exist at all, yet LSS-18 properties exist. In other words, each LSS-17 property in existence is a subset of an LSS-18 property. This happens for example withx^9 non-bijective APN on 8 bits with DMI=1.01426.

sbox	DMI	DDT#0s	NL	WZ	LSS matrix	Max(LSS)
AES	1.04291	33150	112	**4080**	507EBDAACB7DFD67	18
LFT216	1.04283	33148	112	4096	7F67F1B672D97C3C	17
Wuhan3	1.04283	33148	112	4096	D3A1828D2280E6BD	17
BL0105	1.02392	32895	112	16246	67A085F44AF2F83A	15
x^{57}	1.01426	32895	112	16320	9E8ADBDE886E3618	16
BL3638	1.01115	32895	96	14133	62D5D82DB73B37D8	15
BL0231	**0.97429**	32895	64	12516	DE0FC53E14763239	16

Table 25. Top-class 8-bit S-boxes vs. APNs which are Not Permutations

We recall here the specification of some eminent APNs listed here. For example BL0231 comes from Beierle-Leander vast collection of known quadratic APNs inside github.com/lpp-crypto/sboxU/. It is:

```
BL0231=0,242,242,242,2,217,140,165,2,101,185,44,2,76,197,121,8,206,136,188,130,109,126,99,91,8,146,51,211,169,102,238,8,124,174,40,179,
238,105,198,34,195,205,222,155,83,8,50,8,72,220,110,59,82,147,8,115,166,238,201,66,190,163,173,203,227,166,124,166,167,183,68,157,32,
185,246,242,102,170,204,178,174,173,67,87,98,52,243,181,60,227,152,82,242,120,42,115,221,74,22,167,32,226,151,13,54,125,180,219,201,
215,55,2,152,73,33,94,237,105,40,45,34,47,210,115,85,13,217,171,220,227,102,242,172,198,106,102,132,103,119,61,246,64,121,2,65,56,137,
211,185,149,13,158,72,237,201,77,178,66,79,114,131,110,109,146,74,242,216,151,243,194,84,117,56,92,227,211,22,189,138,187,87,169,183,
103,55,64,226,13,116,86,221,107,198,188,227,93,217,246,128,242,202,108,166,198,215,36,199,179,42,22,125,13,189,212,150,123,119,151,105,
199,226,87,128,2,41,129,88,141,143,114,130,179,13,121,53,62,169,136,237,210,205,35,206,213,227,88,156,50,184,138,242,55,148,243,162
```

```
BL0105=0,105,150,0,189,71,134,131,126,73,139,67,0,164,88,3,198,254,249,62,7,172,149,193,215,177,139,18,213,32,36,46,201,246,253,61,235,71,114,33,11,106,
92,194,234,24,16,29,0,110,157,12,94,163,110,108,173,157,83,156,48,147,99,63,130,23,244,158,44,42,247,14,248,51,237,217,149,205,45,138,214,18,9,50,4,83,118,
222,195,89,127,26,210,219,195,53,170,105,126,66,155,203,226,77,108,241,219,185,158,144,132,117,241,99,140,225,188,189,108,146,88,148,70,117,214,137,101,
197,65,248,43,109,244,222,51,230,28,251,21,13,106,30,206,69,85,189,150,129,156,231,242,118,103,209,199,142,109,72,96,186,233,6,33,49,195,191,166,37,8,185,
163,237,225,195,231,58,242,76,147,210,164,137,104,186,124,156,126,97,233,154,70,202,0,69,138,48,166,112,129,168,89,66,176,84,60,180,120,15,134,146,165,78,92,
219,210,170,176,250,240,69,169,112,68,98,73,90,97,141,112,240,245,138,172,225,231,85,86,136,176,145,192,130,65,252,133,84,169,135,74,86,168,75,204,67,131,243
```

```
BL3638=0,36,68,132,219,120,41,110,187,206,216,73,123,137,174,184,116,176,222,254,254,189,226,69,16,133,157,236,129,147,186,76,212,254,112,190,33,
140,51,122,254,133,125,226,16,236,37,61,180,126,254,208,16,93,236,69,65,218,44,83,254,226,37,221,9,30,47,220,170,58,58,78,229,163,228,70,93,156,234,207,197,
50,13,30,55,71,73,221,246,80,25,91,31,62,70,131,239,246,41,212,98,252,18,104,146,218,115,223,4,203,83,120,55,206,31,2,235,149,117,239,149,61,154,214,82,125,
235,32,9,188,30,79,157,175,60,234,106,142,90,90,229,134,99,228,87,2,174,31,146,64,221,235,235,239,53,213,53,182,93,58,165,30,82,13,31,35,94,134,87,189,135,
137,246,155,144,25,239,180,246,73,4,216,171,147,194,200,252,18,50,191,186,211,9,143,124,30,229,228,38,195,61,234,111,92,202,154,46,154,239,137,116,246,82,
179,127,122,4,51,184,107,162,18,168,252,151,31,2,110,85,90,118,157,50,235,128,189,235,181,239,85,101,13,30,146,246,25,59,48,31,38,67,158,151,41,125,39
```

Some Interesting S-boxes

One interesting and very unique S-box was obtained from pentanomial: $x^{191} + x^{223} + x^{239} + x^{247} + x^{254}$ in F_{256} in [DB524]. It is one of numerous S-boxes listed here above which are similar to the AES S-box with NL-112 but are not an [extended] affine equivalent (EA) of AES, or they would have an LSS-18 property, the best we obtain here is LSS-17.

For completeness and easier verification we also give its truth table:

```
Pentanomial=0,1,83,168,213,39,251,230,44,150,220,152,46,179,139,156,210,97,29,133,242,190,223,59,2,76,17,87,184,199,64,228,48,102,33,30,165,
129,122,92,240,72,80,65,113,93,42,161,208,204,160,235,183,141,74,27,31,126,24,35,178,149,91,243,36,26,61,70,147,78,38,15,186,205,229,81,203,
58,68,146,32,127,63,194,86,94,225,164,197,28,23,53,157,4,37,75,224,128,22,21,118,18,202,222,142,195,247,125,239,103,234,43,117,73,84,211,105,
173,67,107,66,111,158,231,88,137,163,50,95,16,185,193,101,136,253,237,209,187,112,123,143,62,145,169,207,77,69,79,8,120,5,138,10,221,121,13,
45,155,130,41,192,198,7,131,181,140,212,14,25,176,148,47,151,154,233,114,200,252,246,175,248,54,232,189,19,170,159,250,12,9,188,115,196,132,
182,180,171,119,40,6,217,56,52,20,89,100,34,177,116,241,135,57,60,218,215,214,85,96,110,106,153,11,206,3,90,191,108,167,201,236,99,166,172,
255,55,226,49,238,124,174,98,109,227,249,144,82,216,134,219,71,104,254,245,162,244,51
```

Another interesting S-box is LFT035 from [LFT35]:

```
LFT035 =
AC,76,A6,5F,CE,02,55,00,FA,C8,FB,77,37,F6,9D,73,87,5C,F9,A3,60,4E,41,65,61,88,E6,CA,53,1A,24,82
,39,49,D4,5B,06,7D,4C,7B,AD,CF,BA,B0,5A,71,2F,C2,4A,AB,D9,FC,EE,56,9F,AF,34,EB,F7,93,E8,8E,BB,0D
,66,89,5E,45,90,80,30,4D,9E,2A,23,6E,DC,46,20,A0,4B,0A,1D,14,A7,9C,C5,E5,D7,52,42,81,6D,91,9B,05
,DF,BF,48,B5,16,44,84,75,EA,1E,B6,08,35,03,2D,E3,E0,33,68,70,01,86,54,E7,E9,EC,DA,72,40,E2,1B,74
,AE,E4,3E,C1,B2,83,3F,69,D5,64,DD,D6,85,19,3B,F1,D2,22,EF,26,0C,A2,AA,59,94,6A,BE,17,8D,67,CD,0F
,38,51,FF,18,13,E1,F0,CC,29,A1,B4,79,7F,28,6F,F4,11,2C,D3,FE,DB,5D,3A,36,2B,C9,32,3D,50,8A,1F,B3
,15,21,25,F5,B9,B1,47,7E,96,F3,D8,C7,B8,3C,27,F2,8F,A8,92,2E,C6,63,99,4F,07,78,1C,6B,0B,57,6C,F8
,ED,CB,04,95,A9,A4,A5,FD,7A,12,DE,43,D0,B7,09,58,0E,98,8C,31,97,C4,C3,7C,BD,D1,8B,9A,10,62,C0,BC
```

Here is Halka from [Halka]:

```
Halka =
0x24,0x2c,0x20,0xdc,0x26,0x73,0xd8,0x91,0x25,0xb7,0x8f,0x9c,0xda,0x1f,0xfe,0xe9,
0x9f,0xa4,0xd5,0x6d,0xc3,0x71,0x32,0x78,0x96,0xdb,0x55,0xb9,0x4c,0x49,0x6e,0x42,
0x9a,0xf9,0x1d,0x64,0x03,0x5c,0xa0,0x00,0x4a,0xd7,0xe3,0x8e,0x75,0xaf,0x0b,0x0a,
0x7d,0x4d,0x5b,0x1a,0x1c,0xe7,0x6a,0x74,0x10,0x06,0x92,0x29,0x81,0x79,0x17,0x40,
0x07,0x7b,0x69,0xca,0xc8,0xb8,0xef,0x84,0xc2,0x37,0x3a,0x98,0xdf,0x66,0x12,0xb6,
0x13,0x08,0x5d,0xfc,0x47,0x31,0xf1,0x21,0x8c,0x14,0xe1,0x51,0x33,0x19,0xb3,0x65,
0x88,0x4e,0x90,0x70,0x1b,0xa8,0x3b,0xcc,0x38,0x15,0x45,0xa7,0x83,0x39,0x0c,0xde,
0xa1,0x3e,0xc1,0xb5,0xeb,0x7f,0xac,0xa2,0x01,0x76,0x9b,0x8a,0xb4,0xbd,0x99,0x16,
0x35,0xd4,0x8b,0x4f,0x02,0x54,0x53,0xbe,0x52,0xc7,0xea,0x09,0x41,0xc6,0xf4,0xb1,
0x58,0x57,0x6b,0x2d,0xf8,0xab,0x87,0x7a,0xf6,0x59,0xa3,0x85,0x61,0x3f,0x9e,0xed,
0x63,0xbf,0xfd,0xb2,0xe8,0x18,0xd2,0x48,0x7c,0x95,0x0f,0x2e,0x44,0xce,0x5f,0xa6,
0xf0,0x8d,0x3c,0xf5,0x46,0x23,0x1e,0xd0,0x2f,0xee,0xba,0x34,0x6f,0x5a,0x04,0x5e,
0xc5,0xf2,0xc4,0x11,0xe2,0x7e,0xe0,0x0e,0xdd,0xbb,0x9d,0x62,0x80,0x2b,0xae,0x50,
0xaa,0x97,0xbc,0xc9,0x94,0x72,0xe5,0xd3,0x77,0x86,0x2a,0xcd,0xb0,0x05,0xd9,0xd1,
0xe6,0xe4,0xa9,0xad,0xd6,0x56,0x6c,0x30,0x43,0xff,0x89,0xcb,0x60,0xf7,0x67,0xcf,
0xa5,0x36,0xc0,0x0d,0x93,0xfb,0x82,0xf3,0x27,0xec,0x4b,0x68,0x22,0xfa,0x28,0x3d
```

C.4 Can we do Better than AES?

This might come as a big surprise for the reader, but there exist S-boxes which are strictly better than the AES S-box, by a very small margin. One example is the LFT216 S-box is based on a fractional linear transformation of type $(216x + 15)/(20x + 157)$ in F_{256} with one position modified, see [LFT216]. We obtain an S-box which is just slightly more secure than the AES S-box. The DMI is definitely lower: 1.04283 instead of 1.04291. It achieves the lowest number of zeros in DDT table ever seen in a cryptographic S-box, namely 33148, or two less than 33150 of AES. This is achieved at the price of having two DDT sets of size 6. We recall this S-box here for easier verification.

```
LFT216=96,90,12,89,55,191,98,61,242,8,91,69,52,24,211,123,60,97,25,137,42,77,196,31,223,81,148,236,200,237,251,134,
167,224,18,51,72,103,58,1,3,17,195,115,233,43,207,6,26,65,59,141,80,162,193,120,118,78,174,93,62,76,128,9,
32,106,235,70,75,221,149,243,29,104,166,160,34,228,2,152,100,38,225,178,63,254,99,110,220,16,163,116,68,30,48,244,
161,218,212,108,37,219,114,154,217,133,198,158,171,83,194,35,165,41,227,249,205,201,192,11,213,87,215,202,180,84,105,95,
232,156,79,147,155,112,101,13,85,216,179,140,0,19,184,5,36,71,57,125,157,248,15,150,238,49,170,252,175,183,56,131,
92,67,119,127,126,222,169,153,21,204,240,74,230,109,234,199,129,146,23,27,245,143,189,173,151,117,28,255,190,66,107,182,
113,208,44,136,187,176,111,185,124,214,210,10,122,197,135,164,121,144,226,39,168,142,14,138,130,102,45,54,177,253,20,229,
7,94,206,159,250,50,139,46,132,73,82,181,86,22,40,47,145,64,4,241,247,188,172,186,88,246,33,203,53,231,209,239
```

Not everything is ideal however with this S-box. It offers a degraded number of zeros in the LAT table which is 4096 instead of 4080 in AES. The DDT table has in fact a disturbing defect: it has two entries which are at 6. Namely $DDT(17, 183)$ and $DDT(75, 185)$ have 6 elements. All other DDT sets are of sizes 0,2,4 like in AES. These two DDT sets have 4 out of 6 elements in common forming a dual super-property with 2 variants of size 6. Arguably just two outliers are not a problem, and are outweighted by the number of four's in DDT going down by 6 (from 255 to 249) and the average of all non-zero DDT entries also goes down. Another very similar S-box Wuhan3 can be found in [YMWH11]. In fact we found both are AE variants of the same quite exceptional S-box.

```
Wuhan3=5,75,163,235,242,229,110,54,0,33,17,234,9,88,250,223,246,195,115,204,184,172,148,40,3,231,36,206,80,146,116,93,167,30,198,
180,181,177,47,64,1,237,114,243,157,19,232,145,211,53,217,133,197,156,155,112,244,241,96,168,56,68,233,14,79,209,8,230,251,125,73,
150,143,46,119,190,182,52,120,129,91,4,89,253,22,220,175,201,82,21,128,81,236,102,213,189,147,113,173,226,117,205,136,15,245,51,121,
174,191,164,219,84,99,71,193,255,76,153,183,247,104,62,27,35,109,23,67,154,212,111,149,200,179,160,65,202,70,83,152,134,58,185,208,
130,124,131,63,118,37,60,196,248,38,227,105,20,139,31,252,78,249,50,188,210,107,28,95,98,166,140,11,228,207,100,41,161,132,29,225,218,
86,171,101,66,44,240,141,224,45,42,127,69,94,138,221,142,108,26,165,18,123,238,24,239,61,159,216,43,59,199,103,186,203,92,6,214,72,
135,57,170,87,137,254,222,55,39,10,2,32,85,194,192,187,178,144,106,12,162,16,74,169,158,77,7,90,151,122,215,48,176,34,13,97,49,126,25,
```

C.5 On Liu-Mesnager-Chen Conjecture (LMCC)

Some essential related work can be found in [Nagy22] and [LiMeCh17]. Consider an arbitrary function $F_2^n \to F_2^m$. The following inequality is known as Liu-Mesnager-Chen Conjecture (LMCC):

$$Max(LSS) = 2^n - NL_v \geq 2^n - \left(1 - \frac{1}{2^m}\right)\left(2^n - 2^{n/2}\right) = 2^{n/2} + 2^{n-m} - 2^{n/2-m}$$

and in particular when $n = m$ then $Max(LSS) = 2^n - NL_v \geq 2^{n/2} + 1 - 2^{-n/2}$ and since the result is an integer, we also have:

$$Max(LSS) \geq 2^{n/2} + 1$$

and these bounds are (wrongly) considered to be always true in the current litterature. When $n = m = 5$ we get $Max(LSS) \geq 4\sqrt{2} + 1 - 2^{-2.5} \approx 6.48$ which is confirmed by all S-boxes in Table 16 and with Fides-5 we attain $Max(LSS) = 7$. When $n = m = 6$ we get $Max(LSS) \geq 8 + 1 - 2^{-3} = 8.875$ and we get 9 with Fides-6.

It seems that the LMCC conjecture is not always true. When $n = m = 8$ we get $Max(LSS) \geq 16 + 1 - 2^{-4} = 16.9375$ which would imply that we always have $Max(LSS) \geq 17$ as confirmed by the AES S-box and many other but violated by Wuhan3 and LFT216. However it is hard to be sure, because a systematic check of all 2^{64} matrices have not been done yet.

Some further results of this type can be found in Table 1 in recent BFA 2025 paper 16, work in progress, cf. [BiTuUd25].