Self-Orthogonal Minimal Codes From p-ary Plateaued Functions

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Abstract

In this article, we study some linear codes based on a subclass of p-ary plateaued functions, for a prime p, which includes all the explicitly known examples of weakly and non-weakly regular plateaued functions. This construction of linear codes is referred to in the literature as the first generic construction. First, we partition the class of p-ary plateaued functions into three collections $\mathscr{C}_1, \mathscr{C}_2$, and $\mathscr{C}_3$, according to the behavior of their dual function f^*. Using these classes, we refine the results presented in a series of articles [36, 38, 44, 47, 57, 58]. Namely, we derive the full weight distributions of codes stemming from all s-plateaued functions for $n+s$ odd (parametrized by the weight of the dual $wt(f^*)$), whereas for $n+s$ even, the weight distributions are derived from the class of s-plateaued functions in $\mathscr{C}_1$ parametrized using two parameters (including $wt(f^*)$ and a related parameter Z_0). Additionally, we provide further results on the different weight distributions of codes stemming from functions in subclasses of the three classes. The exact derivation of such distributions is achieved by using some well-known equations over finite fields to count certain dual preimages. In order to improve the dimension of these codes, we study the vectorial case, thus providing the weight distributions of a few codes associated to known vectorial plateaued functions and obtaining codes with parameters $[p^n-1, 2n, p^n-p^{n-1}-p^{(n+s-2)/2}(p-1)]$. For the first time, we provide the full weight distributions of codes based on the first generic construction from (a subclass of) vectorial p-ary plateaued functions. This class includes all known explicit examples in the literature. The obtained codes are minimal and self-orthogonal in nearly all cases. Notably, we show that this is the best that can be achieved—there are no q-ary self-dual minimal codes for any prime power q, except for the ternary tetracode and the binary repetition code.

Keywords—linear codes, vectorial functions, plateaued functions, minimal codes, self-orthogonal codes, finite fields.

1 Introduction

There are a vast number of methods for constructing linear codes—constructions based on p-ary functions are among the most renowned methods. In their pioneering work, Carlet, Charpin and Zinoviev [5] showed the first explicit connection between almost bent (AB) functions (and almost perfect nonlinear (APN) functions) and linear codes, based on an earlier construction of

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Wolfmann [59], later studied by Delsarte [14]. Soon after, Carlet et al. [7] studied codes based on perfect nonlinear mappings. Since then, many authors have addressed the construction of linear codes using p-ary functions. In particular, highly structured functions have been employed, such as bent and plateaued functions, characterized by the property that the square of the absolute value of every Walsh coefficient has one of only three possible values: $0, \pm p^{n+s}$, for some integer s. The word plateaued comes from the fact that the nonzero Walsh coefficients all lie on the same "plateau" (same absolute height). This spectral regularity translates directly into linear codes with highly regular weight distributions and special properties, such as minimality [37] and self-orthogonality, which can be used to build quantum codes and LCD codes [58]. Constructions of codes from weakly and non-weakly regular bent functions have been widely studied, as well as, codes from weakly and non-weakly regular plateaued functions, see, for instance, [17, 36, 38, 40, 48, 60, 66].

Roughly speaking, known constructions of linear codes based on either bent or plateaued functions can be split into two types: the so-called first generic construction, for which a basis of the code is the set of linear functions together with the truth table of the given function, or constructions based on a set associated to the given function, the so-called second generic construction or the defining-set method. The latter method was initiated by Ding and Niederreiter in [20] and developed by Ding in [15] using bent and semi-bent functions. Several works have presented construction of linear codes based on the defining-set method from weakly regular bent and plateaued functions, e.g., [9,10,12,17,22,23,26,40,41,48–51,53,60,63,65–68]. For the first generic construction, Boolean bent and semi-bent functions were considered in [16]. Constructions based on weakly regular bent functions were introduced in [36], the plateaued case was first covered in [38] and later employed in [28, 55]. Some methods using non-weakly regular bent and plateaued functions are analyzed in [29, 43, 45, 47, 57, 58].

In this work, we address the construction of p-ary codes from (vectorial) plateaued functions using the first generic construction. Research works that are directly related to this research line are [36, 38, 47, 57, 58], where the authors studied codes from: weakly regular bent functions [36], weakly regular plateaued functions [38], and non-weakly regular plateaued functions whose duals satisfy special regularity conditions [47,57,58]. As shown in these works, explicitly deriving the full weight distribution of such codes is a complicated task for the general case.

Elaborating on previous results from [36,38,44,47,57,58], we present the full weight distribution of subclasses of weakly and non-weakly regular s-plateaued functions $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ yielding three-weight codes and three-weight or five-weight codes, for $n+s$ odd and $n+s$ even, respectively. In particular, we provide full weight distributions in general settings where previously only partial results were known. Our results are obtained by using well-known solutions of equations over cyclotomic fields, which are field extensions of the rational numbers by adding the complex p-th root of unity. These solutions are then used to compute the cardinalities of preimages of suitable dual functions that allow the exact derivation of their Walsh distributions and then the weight distributions of associated codes. The parameters of the obtained codes are $[p^n-1, n+1, (p-1)p^{n-1} - p^{\frac{n+s-1}{2}}]$ and $[p^n-1, n+1, p^n - p^{n-1} - p^{(n+s-2)/2}(p-1)]$, for $n+s$ odd and $n+s$ even respectively. In order to obtain linear codes with a larger dimension, we study the vectorial case. The vectorial case has been addressed in the literature for planar mappings [7, 61], quadratic mappings and vectorial bent functions [19,62], vectorial Boolean plateaued functions [32] or vectorial ternary weakly regular plateaued functions [31]. Despite these efforts, little is known about the general case. This is mainly due to the fact that there are only a few examples of families of vectorial

plateaued functions. Based on such examples, we extract general properties of these functions to obtain the weight distribution of codes stemming from another class of vectorial plateaued functions yielding three weight codes with parameters $[p^n-1, 2n, p^n-p^{n-1}-p^{(n+s-2)/2}(p-1)]$ and the first explicit examples of linear codes from a subclass of vectorial p-ary plateaued functions.

In another direction, a combinatorial structure known as (cutting) blocking sets is closely related to the notion of minimality, originally studied in [21], which means that there are no linearly independent codewords that cover each other. We recall that a sufficient condition for minimality of a linear code $\mathcal{C}$ over $\mathbb{F}_q$, the so-called Ashikhmin-Barg condition [1], is given as $w_{min}/w_{max} > (q-1)/q$, where w_{min} and w_{max} respectively denote the minimum and maximum nonzero Hamming weights in $\mathcal{C}$. In odd characteristic, infinite families of minimal linear codes over $\mathbb{F}_q$, with q odd, for which the Ashikhmin-Barg condition does not hold were first given in [2]. The employment of blocking sets for the purpose of constructing minimal linear codes violating this condition was then addressed in [3]. An upper bound on the length of minimal codes was provided in [24], while a full characterization of projective minimal linear codes as cutting (strong) blocking sets was given in [54]. More precisely, both primary and secondary constructions of minimal linear codes over $\mathbb{F}_q$ were obtained in [54], and some infinite families of minimal linear codes not satisfying the Ashikhmin-Barg condition were obtained. Since then, many works have explored constructions of minimal codes either satisfying the Ashikhmin-Barg condition or not, see [31] and the references therein. We show that the codes presented in this article are minimal and self-orthogonal, which makes these codes quite interesting also from a practical point of view. Interestingly, this is the best we can expect since we prove that there are no q-ary self-dual minimal codes for any prime power q, except for trivial examples, i.e., the ternary tetracode and the binary repetition code. The main contributions of this article are listed as follows.

I) A classification of plateaued functions into three classes based on properties of their dual functions is obtained (Lemma 3). These classes are denoted by $\mathscr{C}_1, \mathscr{C}_2$, and $\mathscr{C}_3$.

II) The full weight distributions of some linear codes from subclasses of plateaued functions are derived. Precisely, we obtain the full weight distributions of codes stemming from all s-plateaued functions for $n+s$ odd (Theorem 1 and Theorem 2), parametrized by the weight of the dual $wt(f^*)$, whereas for $n+s$ even, the weight distributions are derived from the class of s-plateaued functions in $\mathscr{C}_1$ parametrized using the dual weight $wt(f^*)$ and a related parameter Z_0 (Theorem 3).

III) Notably, we present a class of 2-weight linear codes from plateaued functions (Theorem 2).

IV) This study is then extended to vectorial plateaued functions, obtaining codes with improved dimension (Theorem 6 and Construction A). It is proven that the obtained codes are minimal and self-orthogonal in most cases (Theorem 7 and Theorem 8).

V) It is shown that there are no q-ary self-dual minimal codes except for a few special cases (Theorem 9).

Overall, these results unify and generalize several earlier contributions on linear codes derived from bent and plateaued functions using the first generic construction.

The paper is organized as follows. In Section 2, we provide all the necessary background on plateaued functions and linear codes. Section 2.2 summarizes all cyclotomic relations relevant for

plateaued functions used in the paper. In Section 3, we show that the dual of a plateaued function satisfies certain specific restrictions allowing us to partition the set of plateaued functions into three classes, which are introduced thereby. Several open questions related to these classes are posed. Additionally, specific dual values are determined after imposing some restrictions on the Walsh value at zero. Section 4 presents the weight distributions of the codes based on the previously introduced classes. Section 4.1 deals with the case of $n+s$ odd, whereas Section 4.2 addresses the even case. In Section 4.3, we provide Construction A, which is a way to obtain linear codes from a non-weakly regular vectorial plateaued function. Minimality, self-orthogonality and optimality of the obtained codes are discussed in Section 5. Moreover, in this section, we show that there are no self-dual minimal codes except for the ternary tetracode and the binary repetition code. This is achieved by employing known bounds on minimal codes and known classifications of self-dual codes. In Section 6, we provide a comparative list displaying the relevant codes presented in [36, 38, 47, 57, 58]. Future work and other research directions are embedded into the conclusions (Section 7). The main properties of all the considered functions and related codes (such as the weight distribution of the latter) have been verified by a MAGMA script.

2 Preliminaries

Let $\mathbb{F}_{p^n}$ denote the finite field with p^n elements, where $n > 0$ and p is prime. Let $\mathbb{F}_p^n$ be an n-dimensional vector space over $\mathbb{F}_p$. A function F from $\mathbb{F}_{p^n}$ to $\mathbb{F}_{p^m}$ is called a vectorial p-ary function. When $p = 2$, F is simply referred to as a vectorial Boolean function. The adjective vectorial is dropped when we refer to functions mapping to the prime field $\mathbb{F}_p$ (thus $m = 1$). Such functions will be usually denoted with lowercase letters. Although most of the results can be transferred to the binary case, we will typically assume $p > 2$ throughout the paper, unless otherwise specified. We treat a function $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ and its truth table as the same object whenever there is no ambiguity. The component functions $F_a : \mathbb{F}_{p^n} \to \mathbb{F}_p$ of a vectorial function $F\colon \mathbb{F}_{p^n} \to \mathbb{F}_{p^m}$ are the mappings $x \mapsto \mathrm{Tr}_1^m(aF(x))$ for $a \in \mathbb{F}_{p^m}^\star$, where $\mathbb{F}_{p^m}^\star = \mathbb{F}_{p^m} \setminus \{0\}$ and the function Tr_1^m denotes the usual trace function from $\mathbb{F}_{p^m}$ to $\mathbb{F}_p$, i.e., $\mathrm{Tr}_1^m(x) = x + x^p + x^{p^2} + \cdots + x^{p^{(m-1)}}$. The Walsh transform of $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ at a point $b \in \mathbb{F}_{p^n}$ is the sum of characters given by

$$W_f(b) = \sum_{x \in \mathbb{F}_{p^n}} \xi_p^{f(x)+\mathrm{Tr}_1^n(bx)}, \tag{1}$$

where $\xi_p = e^{2\pi i/p}$ is the complex primitive p-th root of unity. The *inverse Walsh transform* of f is then defined by

$$p^n \xi_p^{f(x)} = \sum_{b \in \mathbb{F}_{p^m}} W_f(b) \xi_p^{-\mathrm{Tr}_1^n(bx)}. \tag{2}$$

The Walsh spectrum of f is the multi-set of values $\{* \; W_f(b) \; : \; b \in \mathbb{F}_{p^n} \; *\}$. For a vectorial function F, its Walsh spectrum is given by $\{* \; W_{F_a}(b) \; : \; (a,b) \in \mathbb{F}_{p^m}^\star \times \mathbb{F}_{p^n} \; *\}$. The set of linear functions from $\mathbb{F}_{p^n}$ to $\mathbb{F}_{p^m}$ will be denoted by $\mathcal{L}_{n,m}$, whereas the set of affine functions will be denoted by $\mathcal{A}_{n,m}$. A function $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ is called balanced when its Walsh value at zero vanishes, i.e., $W_f(0) = 0$, otherwise f is unbalanced. A p-ary s-plateaued function $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ is characterized by the property $|W_f(b)|^2 = 0$ or p^{n+s} for each $b \in \mathbb{F}_{p^n}$. If $s = 0$, then there are no

zero spectral values and we call such a function a p-ary bent function. It can be shown [27] that the nonzero Walsh values of a p-ary s-plateaued function $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ are expressed as

$$W_f(b) = \begin{cases} \pm p^{(n+s)/2} \xi_p^{f^*(b)}, & p^{n+s} \equiv 1 \pmod 4, \\ \pm \mathrm{i} p^{(n+s)/2} \xi_p^{f^*(b)}, & p^{n+s} \equiv -1 \pmod 4, \end{cases}$$

where $\mathrm{i} = \sqrt{-1}$ and $f^*\colon supp(W_f) \to \mathbb{F}_p$ is a function with domain $supp(W_f) = \left\{b \in \mathbb{F}_{p^n} : |W_f(b)|^2 = p^{n+s}\right\}$, called the *dual* of f. We can then write $W_f(b) = u_b p^{(n+s)/2} \xi_p^{f^*(b)}$ for $u_b \in \{\pm 1, \pm \mathrm{i}\}$. If $u_b = 1$ for every $b \in \mathbb{F}_{p^n}$, then we say that f is a p-ary regular s-plateaued function. More generally, if the value of u_b does not depend on b, then the function f is called p-ary weakly regular s-plateaued, and non-weakly regular s-plateaued otherwise. Furthermore, it was shown [38] that a weakly regular s-plateaued function f satisfies $W_f(b) = \epsilon_f \sqrt{p^*}^{n+s} \xi_p^{f^*(b)}$, where $\epsilon_f = \pm 1$ is called the sign of the Walsh transform of f and $p^* = \eta(-1)p$, where the function $\eta : \mathbb{F}_p \to \{-1, 0, 1\}$ denotes the Legendre symbol on $\mathbb{F}_p$ defined by

$$\eta(j) = \begin{cases} 0, & j = 0 \\ 1, & \exists i \in \mathbb{F}_p^\star, i^2 = j \\ -1, & \text{otherwise.} \end{cases} \tag{3}$$

The notation $\left(\frac{j}{p}\right)$ will be used interchangeably with $\eta(j)$. In a similar fashion, one can easily show that a non-weakly regular s-plateaued function f satisfies $W_f(b) = \epsilon_f(b) \sqrt{p^*}^{n+s} \xi_p^{f^*(b)}$, where $\epsilon_f(b) = \pm 1$ will be called the sign of the Walsh transform of f at $b \in \mathbb{F}_{p^n}$. As usually, the weight of a function f, denoted by $wt(f)$, is the number of elements in its domain that are mapped to a nonzero value, i.e., the cardinality of its support.

2.1 Linear codes from functions

Let q be a prime power. A linear $[n, k, d]$ code $\mathcal{C}$ over the alphabet $\mathbb{F}_q$ is a k-dimensional linear subspace of $\mathbb{F}_q^n$, whose minimum Hamming distance (equivalently, the minimum weight of its nonzero codewords) is d. Every code considered in this paper is a linear code, thus we will not distinguish between the terms linear code and code. The code $\mathcal{S}_n$ spanned by all linear functionals over $\mathbb{F}_q^\star$ is a $[q^n - 1, n, q^n - q^{n-1}]$ code, called the n-affine simplex code, i.e., $\mathcal{S}_n = \{(L(x))_{x \in \mathbb{F}_q^\star} : L \in \mathcal{L}_{n,1}\}$ (a pruning of the first order Reed-Muller code).

Let X_j be the number of codewords in $\mathcal{C}$ with Hamming weight j, $0 \le j \le n$. The weight distribution of a code $\mathcal{C}$ is the vector $(1, X_1, \ldots, X_n)$ and it is fully specified by its weight enumerator polynomial: $1 + X_1 z + \cdots + X_n z^n$. We say that a code with parameters $[n, k, d]$ is distance-optimal, or simply optimal, provided that there does not exist an $[n, k, d']$ linear code with $d < d'$. Two fundamental bounds for an $[n, k, d]$ code $\mathcal{C}$ are the *Singleton bound* and the *Griesmer bound*. The former is given by $d \le n - k + 1$, whereas the latter states that

$$n \ge \sum_{i=0}^{k-1} \left\lceil \frac{d}{p^i} \right\rceil. \tag{4}$$

We refer the interested reader to [34] for more properties of linear codes.

A generic method to specify linear codes from a mapping $F\colon \mathbb{F}_{p^n} \to \mathbb{F}_{p^m}$ with $F(0) = 0$, known as the first generic construction, is described as follows [14, 16, 36, 59]. For positive integers n and m, the linear code $\mathcal{C}_F \subset \mathbb{F}_p^{p^n-1}$ is defined by

$$\mathcal{C}_F = \left\{ \mathbf{c}_{a,u} : a \in \mathbb{F}_{p^m}, u \in \mathbb{F}_{p^n} \right\}, \tag{5}$$

where $\mathbf{c}_{a,u} := \left(\mathrm{Tr}_1^m(aF(x)) + \mathrm{Tr}_1^n(ux) \right)_{x \in \mathbb{F}_{p^n}^\star}$. The dimension of $\mathcal{C}_F$ is at most $n+m$ and its length is $p^n - 1$. If $F\colon \mathbb{F}_{p^n} \to \mathbb{F}_{p^m}$ has no linear components, the linear code $\mathcal{C}_F$ derived from the generic construction in (5) has dimension exactly $n+m$. Moreover, its weights can be expressed by the Walsh transform of absolute trace functions of the map $F\colon \mathbb{F}_{p^n} \to \mathbb{F}_{p^m}$ as shown in [36]. In this context, linear codes based on (5) from Boolean bent and semi-bent functions were first considered in [16]. For odd p, the weakly regular bent case was treated in [36], whereas an extension to plateaued functions was given in [38]. Though some works have addressed the non-weakly regular case (see [29, 43, 45, 47, 57, 58]), less is known about the codes in (5) when plugging a non-weakly regular bent or plateaued function. The constructions in this paper will exclusively be based on the first generic construction.

2.2 Cyclotomic relations relevant for plateaued functions

Let QR denote the set of quadratic residues modulo p and let NQR be the set of quadratic non-residues modulo p. The next lemma is easy to derive from known results on finite field characters and exponential sums, see for instance [33].

Lemma 1. *The following relations are true for the Legendre symbol, defined in (3), and ξ_p:*

1. $\sum_{j \in \mathbb{F}_p^*} \eta(j) = \sum_{j \in QR^\star} 1 + \sum_{j \in NQR} (-1) = 0$;
2. $\sum_{j \in \mathbb{F}_p^*} \xi_p^j = -1$;
3. *The set $\{\xi_p, \ldots, \xi_p^{p-1}\}$ forms an integral basis of $\mathbb{Q}(\xi_p)$. Moreover, for any $a \in \mathbb{Z}$, the integral equation in unknowns a_j, $j \in \mathbb{F}_p^\star$,*

$$\sum_{j \in \mathbb{F}_p^*} a_j \xi_p^j = \begin{cases} a\sqrt{p}, & p \equiv 1 \pmod 4; \\ \mathrm{i} a\sqrt{p}, & p \equiv 3 \pmod 4; \end{cases}$$

has a unique solution $a_j = a\eta(j) \in \mathbb{Z}$.

Note that $\mathrm{i} \notin \mathbb{Z}(\xi_p)$ since it is not a root of unity for $p \not\equiv 0 \pmod 4$ [46]. Therefore, $\sum_{i=1}^{p-1} a_i \xi_p^i = p^{\frac{\theta}{2}} \nu$ for $a_i \in \mathbb{Z}$, $\theta \in \mathbb{N}$ and $\nu \in \{1, \mathrm{i}\}$ implies that either θ is odd or $\nu \neq \mathrm{i}$. Therefore, the following hold.

Lemma 2. *[30] Let $(a_1, \ldots, a_{p-1}) \in \mathbb{Z}^{p-1}$, $\theta \in \mathbb{N}$ and $\nu \in \{1, \mathrm{i}\}$. Suppose that $\sum_{i=1}^{p-1} a_i \xi_p^i = p^{\frac{\theta}{2}} \nu$.*

1. *If $\theta \equiv 0 \pmod 2$, then $\nu = 1$;*
2. *If $\theta \equiv 1 \pmod 2$, then*

$$\nu = \begin{cases} 1, & p \equiv 1 \pmod 4; \\ \mathrm{i}, & p \equiv 3 \pmod 4. \end{cases}$$

3 Dual value distributions of plateaued functions

Using a similar notation as in [38], given $f_1 \colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ and any function $f_2 \colon supp(W_{f_1}) \to \mathbb{F}_p$, we define the sets $N_{f_2}(j) = \{x \in supp(W_{f_1}) : f_2(x) = j\}$ and the numbers $n_{f_2}(j) = \#N_{f_2}(j)$, for $j \in \mathbb{F}_p$. Following the terminology introduced in [43, 44], for a given set $S \subseteq \mathbb{F}_{p^n}$, we say that a function $f \colon S \to \mathbb{F}_p$ is *bent relative to* S if $|W_f(b)| = \#S^{1/2}$ for all $b \in \mathbb{F}_{p^n}$, where $W_f(b)$ is considered as the restriction to S of the Walsh transform of f, i.e., $W_f(b) = \sum_{x \in S} \xi_p^{f(x)+\mathrm{Tr}_1^n(bx)}$. For weakly regular plateaued functions, the dual function f^* is bent relative to $supp(W_f)$. For non-weakly regular plateaued functions, the dual may or may not be bent relative to $supp(W_f)$. There are infinitely many examples of both cases. Let $S \subseteq \mathbb{F}_{p^m}$ and let $f \colon S \to \mathbb{F}_p$ be a function such that $W_f(0) = \sum_{x \in S} \xi_p^{f(x)} = t(f)\nu p^{\frac{\mu}{2}} \xi_p^j$, where $t(f) = \pm 1$ or 0, $\nu \in \{1, \mathrm{i}\}$, $j \in \mathbb{F}_p$ for some $\mu \in \mathbb{N}$, $\mu > 0$. The number $t(f)$ will be called the *type of* f. For an s-plateaued function $f \colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ with $0 \leq s \leq n$, let $\Gamma^+(f)$ and $\Gamma^-(f)$ be the sets that partition $S = supp(W_f)$ and are given by

$$\Gamma^+(f) = \left\{b \in S : W_f(b) = \nu p^{\frac{n+s}{2}} \xi_p^{f^*(b)}\right\},\ \Gamma^-(f) = \left\{b \in S : W_f(b) = -\nu p^{\frac{n+s}{2}} \xi_p^{f^*(b)}\right\},$$

where $\nu \in \{1, \mathrm{i}\}$. Note that determining whether a nonzero Walsh value $W_f(b)$ has a positive sign or negative sign comes down to determining the sign of the value $\epsilon_f(b)\sqrt{\eta(-1)}^{n+s}$. More precisely, define the function $\alpha : \{-1, 1\} \to \{-, +\}$ given by $\alpha(1) = +$ and $\alpha(-1) = -$. For $b \in supp(W_f)$, we have

$$b \in \Gamma^{\alpha(t)}(f) \iff \epsilon_f(b)\eta^{\frac{n+s}{2}}(-1) = t, \text{ when } n{+}s \text{ is even, or } \epsilon_f(b)\eta^{\frac{n+s-1}{2}}(-1) = t, \text{ when } n{+}s \text{ is odd,} \tag{6}$$

for $t \in \{-1, 1\}$. For an s-plateaued function $f \colon \mathbb{F}_{p^n} \to \mathbb{F}_p$, define the numbers

$$A_j := \#(N_{f^*}(j) \cap \Gamma^+(f)) \text{ and } B_j := \#(N_{f^*}(j) \cap \Gamma^-(f)), \tag{7}$$

for $j \in \mathbb{F}_p$. We also define $Z_j := A_j - B_j$. If f is s-plateaued, then, for each $b \in supp(W_f)$, $W_f(b) = \epsilon_f(b)\nu p^{\frac{n+s}{2}} \xi_p^{f^*(b)}$. This implies that

$$\sum_{b \in \mathbb{F}_{p^n}} W_f(b) = \sum_{b \in supp(W_f)} \epsilon_f(b)\nu p^{\frac{n+s}{2}} \xi_p^{f^*(b)} = \sum_{j \in \mathbb{F}_p} (A_j - B_j)\xi_p^j \nu p^{\frac{n+s}{2}}. \tag{8}$$

Moreover, the inverse Walsh transform of $f(x)$ at $x = 0$, displayed in (2), yields $p^n \xi_p^{f(0)} = \sum_{b \in \mathbb{F}_{p^n}} W_f(b)$. Combining this with (8) gives

$$p^n \xi_p^{f(0)} = \sum_{j \in \mathbb{F}_p} (A_j - B_j)\xi_p^j \nu p^{\frac{n+s}{2}}. \tag{9}$$

Lemma 3. *Let $f \colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ be any s-plateaued function. Let $f(0) = j_0$. Then $A_{j_0} \neq B_{j_0}$ (i.e. $Z_{j_0} \neq 0$). The distribution values A_j, B_j associated to f satisfy exactly one of the following.*

i) $A_j \neq B_j$ for every j;

ii) The number $n - s$ is even and $A_j = B_j$ for each $j \neq j_0$. In this case, $A_{j_0} = p^{\frac{n-s}{2}} + B_{j_0}$ and, $\sum_{j \neq j_0} A_j = \sum_{j \neq j_0} B_j = \frac{p^{n-s}+p^{\frac{n-s}{2}}}{2} - A_{j_0} = \frac{p^{n-s}-p^{\frac{n-s}{2}}}{2} - B_{j_0}$;

iii) The number $n-s$ is odd and $A_j = B_j$ for $j \in \mathcal{I}$ and $A_j - B_j = 2\left(\frac{j_0-j}{p}\right) p^{\frac{n-s-1}{2}}$ for $j \notin \mathcal{I}$, where $\mathcal{I} \subset \mathbb{F}_p^\star$ and

$$\mathcal{I} = \begin{cases} j_0 - QR^\star, & \frac{Z_{j_0}}{|Z_{j_0}|} = -1; \\ j_0 - NQR, & otherwise. \end{cases}$$

In this case, $Z_{j_0} = -\left(\frac{j_0-j}{p}\right) p^{\frac{n-s-1}{2}}$ for (any) $j \in \mathcal{I}$. Moreover, if $A_{j_0} \neq 0$, then $\sum_{i \neq j_0} A_i = \frac{p^{n-s} + \left(\frac{j_0-j}{p}\right) p^{\frac{n-s+1}{2}}}{2} - A_{j_0}$, and, if $B_{j_0} \neq 0$, then $\sum_{i \neq j_0} B_i = \frac{p^{n-s} - \left(\frac{j_0-j}{p}\right) p^{\frac{n-s+1}{2}}}{2} - B_{j_0}$, for (any) $j \notin \mathcal{I}$.

Proof. Consider the equation

$$p^n \xi_p^{j_0} = \sum_{j \in \mathbb{F}_p} (A_j - B_j) \xi_p^j \nu p^{\frac{n+s}{2}}$$

derived as in (9), where we tacitly replaced $f(0) = j_0$. By using 2) of Lemma 1, it can be rewritten as

$$\sum_{j \neq j_0} (A_j - B_j - Z_{j_0}) \xi_p^j \nu p^{\frac{n+s}{2}} = p^n \xi_p^{j_0}, \tag{10}$$

which can be rearranged as

$$\sum_{j \neq j_0} (A_j - B_j - Z_{j_0}) \xi_p^{j-j_0} = p^{\frac{n-s}{2}} \nu^{-1}. \tag{11}$$

Suppose that $n-s$ is even. Thus, $\nu = 1$ by Lemma 2. We first show that $Z_{j_0} \neq 0$. On the contrary, suppose that $Z_{j_0} = 0$. The integral equation (11) has a uniquely determined solution by Lemma 1, thus, for each $j \neq j_0$, it holds that $A_j - B_j = -p^{\frac{n-s}{2}}$. Since f is plateaued, its Walsh support has size p^{n-s}, that is, $\sum_{j \in \mathbb{F}_p} (A_j + B_j) = p^{n-s}$. By expressing the first equalities in terms of A_j and replacing into the second one, we get $2\sum_{j \in \mathbb{F}_p} A_j = p^{n-s} - p^{\frac{n-s}{2}}(p-1)$, which is a contradiction since $p^{n-s} - p^{\frac{n-s}{2}}(p-1)$ is an odd number. Therefore $Z_{j_0} \neq 0$.

We will show that either *i)* or *ii)* happens, exclusively. For this, let us suppose that *i)* is not true, i.e., suppose that there is an index $j' \neq j_0$ such that $A_{j'} = B_{j'}$. From (11), we get $A_j - B_j = Z_{j_0} - p^{\frac{n-s}{2}}$ for each $j \neq j_0$. In particular, $0 = A_{j'} - B_{j'} = Z_{j_0} - p^{\frac{n-s}{2}}$, so that $Z_{j_0} = p^{\frac{n-s}{2}}$ and $A_j = B_j$ for every $j \neq j_0$, i.e., *ii)* holds. The second part of *ii)* comes from this and the fact that $\sum_{j \in \mathbb{F}_p} (A_j + B_j) = p^{n-s}$.

Suppose that $n-s$ is odd. To show that $Z_{j_0} \neq 0$, suppose the opposite. The integral equation in (11) has a unique solution by Lemma 1. Namely, it must hold

$$A_j - B_j = \left(\frac{j_0 - j}{p}\right) p^{\frac{n-s-1}{2}},$$

where we used $\nu^{-1} = \eta(-1)\nu$, which can be easily verified by using Lemma 2. Since f is plateaued, its Walsh support has size p^{n-s}, i.e., $\sum_{j \in \mathbb{F}_p} (A_j + B_j) = p^{n-s}$. Again, expressing the obtained values for $A_j - B_j$ in terms of A_j and replacing into the sum $\sum_{j \in \mathbb{F}_p} (A_j + B_j)$ yield $2\sum_{j \in \mathbb{F}_p} A_j = p^{n-s}$, which is a contradiction to the fact that p^{n-s} is odd. Hence $Z_{j_0} \neq 0$.

Suppose that i) is not true, i.e. suppose that there is an index $j' \neq j_0$ such that $A_{j'} = B_{j'}$. We will prove iii). From (11), we get

$$A_j - B_j = Z_{j_0} + \left(\frac{j_0 - j}{p}\right) p^{\frac{n-s-1}{2}}, \tag{12}$$

for each $j \neq j_0$, by an application of Lemma 1. In particular, $0 = A_{j'} - B_{j'} = Z_{j_0} + \left(\frac{j_0-j'}{p}\right) p^{\frac{n-s-1}{2}}$, so that $Z_{j_0} = -\left(\frac{j_0-j'}{p}\right) p^{\frac{n-s-1}{2}}$. Moreover, for every j such that $\left(\frac{j_0-j}{p}\right) = \left(\frac{j_0-j'}{p}\right) = -\frac{Z_{j_0}}{|Z_{j_0}|}$, it must be $A_j = B_j$. Collect these indices to form the set $\mathcal{J}$, so that

$$\mathcal{J} = \left\{ j \in \mathbb{F}_p : j \neq j_0 \text{ and } \left(\frac{j_0 - j}{p}\right) = -\frac{Z_{j_0}}{|Z_{j_0}|} \right\}.$$

Define $\mathcal{I}$ as in the statement, we claim that $\mathcal{I} = \mathcal{J}$. Indeed, if $j \in \mathcal{J}$, then $j \neq j_0$ and $\left(\frac{j_0-j}{p}\right) = -\frac{Z_{j_0}}{|Z_{j_0}|}$. This implies that $j_0 - j \in j_0 - \mathcal{I}$, equivalently, $j \in \mathcal{I}$. As $j_0 - j$ runs over $\mathbb{F}_p^\star$ as j runs over $\mathbb{F}_p \setminus \{j_0\}$, it must be that $\#\mathcal{J} = \frac{p-1}{2}$. Therefore, $\mathcal{I} = \mathcal{J}$.

Then, we have that $A_j = B_j$ for every $j \in \mathcal{I}$. From (12), we have $A_j - B_j = 2\left(\frac{j_0-j}{p}\right) p^{\frac{n-s-1}{2}}$ for each $j \notin \mathcal{I}$. The second part of iii) comes from the above and $\sum_{j\in\mathbb{F}_p}(A_j + B_j) = p^{n-s}$. □

Let $f(x)$ be a plateaued function and $g(x)$ denote its translate by a constant, i.e., $g(x) = f(x)+j$, where $j \in \mathbb{F}_p$. Then, the dual value distributions of $f(x)$ and $g(x)$ are the same, up to a translation of the indices by j. Thus, we can always assume without loss of generality that $f(0) = 0$.

Using the previous lemma, we can partition the set of plateaued functions into three classes. These classes will be denoted by $\mathscr{C}_1, \mathscr{C}_2$, and $\mathscr{C}_3$, respectively. Thus, $\mathscr{C}_1$ contains all functions that satisfy i) of Lemma 3, $\mathscr{C}_2$ corresponds to the set of functions satisfying ii) and $\mathscr{C}_3$ contains functions with the property specified in iii). First we show that $\mathscr{C}_i$, $1 \leq i \leq 3$, is non-empty (Example 1). Afterwards, we will determine the exact values of A_j, B_j for certain functions in each family $\mathscr{C}_i$.

Example 1. *Any weakly regular plateaued function whose dual is surjective belongs to $\mathscr{C}_1$ for which there are several infinite families of functions.*

To construct an infinite family inside $\mathscr{C}_2$, consider the function $f(x) = \mathrm{Tr}_1^3(x^7)$ over $\mathbb{F}_{3^3}$. This function is a non-weakly regular 1-plateaued function with zero dual, namely, $\{W_f(b) : b \in \mathbb{F}_{3^3}\} = \{0, 9, -9\}$ with distribution $\{*0^{18}, 9^6, -9^3*\}$, so $f^*(x) = 0$ for each $x \in supp(W_f)$. For any $l \in \mathbb{N}$, we consider the l-th iteration of the direct sum of f with itself, $f^l : \mathbb{F}_{3^{3(l-1)}} \times \mathbb{F}_{3^3} \to \mathbb{F}_3$, defined recursively by

$$f^1 = f;$$
$$f^l(x, y) = f^{l-1}(x) + f(y) \text{ for } l \geq 2,$$

which is an l-plateaued function defined on $\mathbb{F}_{3^{3l}}$ (upon identifying $\mathbb{F}_{3^{3l}} \cong \mathbb{F}_{3^{3(l-1)}} \times \mathbb{F}_{3^3}$) with constant zero dual (recall that the Walsh values of the direct sum of functions are directly related to the Walsh spectra of the summands).

For $\mathscr{C}_3$, consider the function $g(x) = \mathrm{Tr}_1^3(2x^4 + x^2)$ in $\mathbb{F}_{3^3}$. This function is a weakly regular 2-plateaued function with $\{W_f(b) : b \in \mathbb{F}_{3^3}\} = \{0, \mathfrak{i}3^{5/2}, \mathfrak{i}3^{5/2}\xi_3^2\}$ and distribution

$$\{*0^{24}, (\mathfrak{i}3^{5/2})^1, (\mathfrak{i}3^{5/2}\xi_3^2)^2*\}$$

For any $l \in \mathbb{N}$, consider f^l as before. The direct sum of f^l with g gives a non-weakly regular $(l+2)$-plateaued function in $\mathbb{F}_{3^{3(l+1)}}$ with

$$\{W_f(b) : b \in \mathbb{F}_{3^{3(l+1)}}\} = \{0, \mathfrak{i}3^{\frac{5+4l}{2}}, -\mathfrak{i}3^{\frac{5+4l}{2}}, \mathfrak{i}3^{\frac{5+4l}{2}}\xi_3^2, -\mathfrak{i}3^{\frac{5+4l}{2}}\xi_3^2\},$$

which belongs to $\mathscr{C}_3$.

Although the previous example (Example 1) shows that the classes $\mathscr{C}_1, \mathscr{C}_2$ and $\mathscr{C}_3$ are non-empty, it also gives rise to some existence problems. Namely, the following questions arise naturally.

Question 1. For which values (p, n, s) are the classes $\mathscr{C}_2$ and $\mathscr{C}_3$ non-empty?

Note also that every function in $\mathscr{C}_2$ is non-weakly regular. Indeed, any function in $\mathscr{C}_2$ is an s-plateaued function that satisfies $A_j = B_j$ for each $j \neq j_0$. Obviously, any such function is non-weakly regular if $A_j = B_j \neq 0$ for some $j \neq j_0$. Then, we only need to show that a zero dual plateaued function is non-weakly regular: it must hold that

$$(A_{j_0} - B_{j_0})p^{\frac{n+s}{2}} = p^n \xi_p^{j_0},$$

in this case, $\nu = 1$ and $n+s$ is even. Hence, $j_0 = 0$ as the left-hand side of this equation is an integer. By plateauedness, $A_0 + B_0 = p^{n-s}$. Therefore, a zero dual plateaued function satisfies the system of equations

$$A_0 - B_0 = p^{\frac{n-s}{2}} \text{ and } A_0 + B_0 = p^{n-s},$$

which implies $A_0 \neq 0$ and $B_0 \neq 0$.

Question 2. Are there infinite classes of functions in $\mathscr{C}_2$ whose dual is nonzero?

Question 3. Are there infinite classes of functions in $\mathscr{C}_3$ whose dual is surjective (necessarily non-weakly regular plateaued)?

Remark 1. *Example 1 shows that secondary constructions can in principle be used to obtain functions in $\mathscr{C}_1, \mathscr{C}_2$, and $\mathscr{C}_3$ suggesting a positive answer for Questions 1,2 and 3. We leave these questions as open problems for future research.*

In the following (Propositions 1-6), we determine the exact values of A_j, B_j for certain subfamilies of p-ary plateaued functions which carry enough information about the dual to derive these values. For any plateaued function, a simple equality which we will constantly use to derive the corresponding formulae is

$$W_{f^*}(0) = \sum_{x \in supp(W_f)} \xi_p^{f^*(x)} = \sum_{j=1}^{p-1} (A_j + B_j - A_0 - B_0)\xi_p^j, \tag{13}$$

where the last equality comes from 2) in Lemma 1.

Proposition 1. *Let $f : \mathbb{F}_{p^n} \to \mathbb{F}_p$ be an s-plateaued function in $\mathscr{C}_1$ with $f(0) = 0$. Suppose that $W_{f^*}(0) = t(f^*)\nu' p^{\frac{\theta}{2}}$ for some $\nu' \in \{1, \mathfrak{i}\}$ and an integer θ such that $1 < \theta \leq 2(n-s)$. For $j \in \mathbb{F}_p^\star$, the numbers A_j, B_j are either zero or depend on A_0 and B_0, respectively. Moreover, $A_0 + B_0 = p^{n-s-1}$ when θ is odd and $A_0 + B_0 = p^{n-s-1} + t(f^*)(p-1)p^{\frac{\theta}{2}-1}$ for θ even. The values of A_j, B_j are displayed in Table 1 for different parities of $n+s$ and θ.*

Proof. Suppose that f is s-plateaued and $W_f(b) = \epsilon_f(b)\nu p^{\frac{n+s}{2}} \xi_p^{f^*(b)}$, for each $b \in \mathbb{F}_{p^n}$. From (9), we know that

$$p^n = \sum_{j \in \mathbb{F}_p} (A_j - B_j)\xi_p^j \nu p^{\frac{n+s}{2}}.$$

Working out this equation gives

$$\sum_{j \in \mathbb{F}_p} (A_j - B_j)\xi_p^j = \sum_{j \in \mathbb{F}_p^\star} (A_j - B_j - A_0 + B_0)\xi_p^j = \nu^{-1} p^{\frac{n-s}{2}}. \tag{14}$$

Suppose that $n+s$ is even. By Lemma 2, $\nu = 1$. By Lemma 1, the values of the integral equation (14) are uniquely determined. Then,

$$A_j - B_j = A_0 - B_0 - p^{\frac{n-s}{2}}, \tag{15}$$

for each $j \in \mathbb{F}_p^\star$. Suppose that θ is even, too. By hypothesis, $W_{f^*}(0) = t(f^*)\nu' p^{\frac{\theta}{2}}$. By Lemma 2, $\nu' = 1$. From this and (13), we get $t(f^*)p^{\frac{\theta}{2}} = W_{f^*}(0) = \sum_{j=1}^{p-1}(A_j + B_j - A_0 - B_0)\xi_p^j$. Therefore, for each nonzero j,

$$A_j + B_j = A_0 + B_0 - t(f^*)p^{\frac{\theta}{2}} \tag{16}$$

by Lemma 1. From (15) and (16), one can obtain the values of A_j, B_j in terms of A_0, B_0 respectively. Since the Walsh support of a plateaued function has size p^{n-s}, it follows that

$$p^{n-s} = \sum_{j=0}^{p-1}(A_j + B_j) = (p-1)(A_0 + B_0 - t(f^*)p^{\frac{\theta}{2}}) + A_0 + B_0.$$

Thus, $A_0 + B_0 = p^{n-s-1} + t(f^*)(p-1)p^{\frac{\theta}{2}-1}$. Assume now that θ is odd (and $n+s$ even). Again, from hypothesis and (13), $t(f^*)\nu' p^{\frac{\theta}{2}} = W_{f^*}(0) = \sum_{j=1}^{p-1}(A_j + B_j - A_0 - B_0)\xi_p^j$. Thus,

$$A_j + B_j = A_0 + B_0 + \left(\frac{j}{p}\right) t(f^*)p^{\frac{\theta-1}{2}}, \tag{17}$$

for each nonzero j, by Lemma 1. From (15) and (17), one can obtain the values of A_j, B_j in terms of A_0, B_0 respectively. Lastly,

$$p^{n-s} = \sum_{j=0}^{p-1}(A_j + B_j) = p(A_0 + B_0)$$

Thus, $A_0 + B_0 = p^{n-s-1}$.

For the case when $n+s$ is odd, a similar reasoning is applied. In this case, the value of ν^{-1} in (17) is equal to 1 or $-\mathfrak{i}$ depending on whether 4 divides $p-1$ or not. Thus, we obtain, for each $j \in \mathbb{F}_p^\star$,

$$A_j - B_j = A_0 - B_0 + \left(\frac{-j}{p}\right) p^{\frac{n-s-1}{2}}, \tag{18}$$

by 3) of Lemma 1. Combining this with (16) and (17) yields the desired result. □

Table 1: Values of A_j, B_j and $A_0 + B_0$ in Proposition 1 for different parities of $n + s$ and θ, where the pairs stand for $(n + s \pmod 2, \theta \pmod 2)$.

	A_j	B_j	$A_0 + B_0$
$(0,0)$	0, or, $A_0 - \frac{p^{\frac{n-s}{2}} + t(f^*)p^{\frac{\theta}{2}}}{2}$	0, or, $B_0 + \frac{p^{\frac{n-s}{2}} - t(f^*)p^{\frac{\theta}{2}}}{2}$	$p^{n-s-1} + t(f^*)(p-1)p^{\frac{\theta}{2}-1}$
$(0,1)$	0, or, $A_0 + \frac{\left(\frac{j}{p}\right)t(f^*)p^{\frac{\theta-1}{2}} - p^{\frac{n-s}{2}}}{2}$	0, or, $B_0 + \frac{\left(\frac{j}{p}\right)t(f^*)p^{\frac{\theta-1}{2}} + p^{\frac{n-s}{2}}}{2}$	p^{n-s-1}
$(1,0)$	0, or, $A_0 + \frac{\left(\frac{-j}{p}\right)p^{\frac{n-s-1}{2}} - t(f^*)p^{\frac{\theta}{2}}}{2}$	0, or, $B_0 + \frac{-\left(\frac{j}{p}\right)p^{\frac{n-s-1}{2}} - t(f^*)p^{\frac{\theta}{2}}}{2}$	$p^{n-s-1} + t(f^*)(p-1)p^{\frac{\theta}{2}-1}$
$(1,1)$	0, or, $A_0 + \frac{\left(\frac{j}{p}\right)t(f^*)p^{\frac{\theta-1}{2}} + \left(\frac{-j}{p}\right)p^{\frac{n-s-1}{2}}}{2}$	0, or, $B_0 + \frac{\left(\frac{j}{p}\right)t(f^*)p^{\frac{\theta-1}{2}} - \left(\frac{-j}{p}\right)p^{\frac{n-s-1}{2}}}{2}$	p^{n-s-1}

Remark 2. *Proposition 1 extends the results of [36, 38, 44, 47, 57, 58]. Namely, in [36], the value distribution of the dual of a weakly regular bent function f was studied. Then the extension to weakly regular plateaued functions was given in [38]. In [44], the case of f being a non-weakly regular bent function whose dual is bent with respect to $supp(f)$ was investigated. Later, in [47], the authors presented the case of non-weakly regular s-plateaued functions f whose dual is bent with respect to $supp(f)$, which was further analyzed in [57, 58]. Therefore Proposition 1 is the most general result of this kind.*

Remark 3. *Proposition 1 covers the value distributions of all known instances of weakly and non-weakly regular bent functions. To the best of our knowledge, there are no known examples of plateaued functions for which $n - s$ and θ have different parities (see the second and third rows in Table 1), where θ is defined through $W_{f^*}(0) = t(f^*)\nu' p^{\frac{\theta}{2}}$. We leave it as an open problem to either provide such examples or to give an impossibility result.*

Example 2. *Working in $\mathbb{F}_{3^6}$, consider the non-weakly regular bent function $f(x) = \mathrm{Tr}_1^6(\omega^7 x^{98})$, where ω is a primitive element of $\mathbb{F}_{3^6}$ satisfying $\omega^6 + \omega^4 + \omega^3 + \omega + 1 = 0$. Its Walsh spectrum is*

$$\{*-27^{57}, 27^{168}, (-27\xi_3)^{84}, (27\xi_3)^{168}, (-27\xi_3^2)^{84}, (27\xi_3^2)^{168}*\},$$

which means that $f \in \mathscr{C}_1$, $\nu = 1$ and $n+s = 6$ is even. The dual values are then $A_0 = A_1 = A_2 = 168$ and $B_0 = 57, B_1 = B_2 = 84$. Moreover, the dual function $f^* : \mathbb{F}_{3^6} \to \mathbb{F}_3$ is not bent and it satisfies $W_{f^*}(0) = -27$. With reference to Proposition 1, $\theta = 6$ is even, $t(f^*) = -1$ and $\nu' = 1$. As $n + s$ and θ are even, the dual values are determined by the first row in Table 1. Indeed, $A_0 + B_0 = 3^5 - 2 \cdot 3^2 = 225$ and $A_j = A_0 + \frac{3^3 + (-1)3^3}{2} = 168$ and $B_j = B_0 + \frac{3^3 - (-1)3^3}{2} = 84$.

Example 3. *Working in $\mathbb{F}_{3^3} \times \mathbb{F}_{3^3}$, we now consider the non-weakly regular 1-plateaued function $f(x, y) = \mathrm{Tr}_1^3(x^2) + \mathrm{Tr}_1^3(y^7)$. Its Walsh spectrum is*

$$\{*0^{486}, (-\mathfrak{i}3^3\sqrt{3})^{54}, (\mathfrak{i}3^3\sqrt{3})^{27}, (-\mathfrak{i}3^3\sqrt{3}\xi_3)^{72}, (\mathfrak{i}3^3\sqrt{3}\xi_3)^{36}, (-\mathfrak{i}3^3\sqrt{3}\xi_3^2)^{36}, (\mathfrak{i}3^3\sqrt{3}\xi_3^2)^{18}*\},$$

which means that $f \in \mathscr{C}_1$, $\nu = \mathfrak{i}$ and $n + s = 7$ is odd. The dual values are then $A_0 = 27, A_1 = 36, A_2 = 18$ and $B_0 = 54, B_1 = 72, B_2 = 36$. Moreover, the dual function $f^* : \mathbb{F}_{3^6} \to \mathbb{F}_3$ is not bent and it satisfies $W_{f^*}(0) = \mathfrak{i}3^3\sqrt{3}$. With reference to Proposition 1, $\theta = 7$ is odd, $t(f^*) = 1$ and $\nu' = \mathfrak{i}$. As $n + s$ and θ are odd, the dual values are determined by the last row in Table 1. Indeed, $A_0 + B_0 = 3^4 = 81$; $A_1 = A_0 + \frac{3^3 - 3^2}{2} = 27 + 9 = 36$ and $B_1 = B_0 + \frac{3^3 + 3^2}{2} = 54 + 18 = 72$; $A_2 = A_0 + \frac{-3^3 + 3^2}{2} = 27 - 9 = 18$ and $B_2 = B_0 + \frac{-3^3 - 3^2}{2} = 54 - 18 = 36$.

Next, we analyze three subclasses of functions in $\mathscr{C}_2$ and determine the values of A_j, B_j. The necessary conditions to derive these values are imposed on $W_{f^*}(0)$, namely, we study the cases $W_{f^*}(0) = t(f^*)p^{\frac{\theta}{2}}$ for some $\theta \in \mathbb{N}$ with $1 < \theta \leq 2(n-s)$. The case of θ even is treated in Proposition 2, whereas Proposition 4 deals with the case θ odd. The subclass of zero dual functions is further analyzed in Proposition 3. For a function in $\mathscr{C}_2$, a more precise derivation than (13) can be obtained. Namely, it holds

$$W_{f^*}(0) = \sum_{x \in supp(W_f)} \xi_p^{f^*(x)} = \sum_{j=1}^{p-1} (A_j + B_j - A_0 - B_0)\xi_p^j = \sum_{j=1}^{p-1} (2A_j - 2A_0 + p^{\frac{n-s}{2}})\xi_p^j, \tag{19}$$

where the last equality comes from $ii)$ of Lemma 3.

Proposition 2. *Let $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ be an s-plateaued function in $\mathscr{C}_2$ such that $f(0) = 0$. Suppose that $W_{f^*}(0) = t(f^*)p^{\frac{\theta}{2}}$ for some even integer θ with $1 < \theta \leq 2(n-s)$. Then, $A_0 = \frac{p^{n-s-1} + p^{\frac{n-s}{2}} + t(f^*)(p-1)p^{\frac{\theta}{2}-1}}{2}$, $B_0 = \frac{p^{n-s-1} - p^{\frac{n-s}{2}} + t(f^*)(p-1)p^{\frac{\theta}{2}-1}}{2}$. Moreover, for $j \in \mathbb{F}_p^\star$, $A_j = B_j = A_0 - \frac{p^{\frac{n-s}{2}} + t(f^*)p^{\frac{\theta}{2}}}{2}$.*

Proof. By hypothesis and from (19), we have $t(f^*)p^{\frac{\theta}{2}} = W_{f^*}(0) = \sum_{j=1}^{p-1}(2A_j - 2A_0 + p^{\frac{n-s}{2}})\xi_p^j$. Using Lemma 1, all coefficients in the RHS must be equal to $-t(f^*)p^{\frac{\theta}{2}}$. Then,

$$A_j = A_0 - \frac{p^{\frac{n-s}{2}} + t(f^*)p^{\frac{\theta}{2}}}{2}, \tag{20}$$

for each nonzero j. From Lemma 3, the value of $\sum_{j=0}^{p-1} A_j$ is known, namely, $\frac{p^{n-s} + p^{\frac{n-s}{2}}}{2} = \sum_{j=0}^{p-1} A_j$. On the other hand,

$$\sum_{j=0}^{p-1} A_j = (p-1)(A_0 - \frac{p^{\frac{n-s}{2}} + t(f^*)p^{\frac{\theta}{2}}}{2}) + A_0.$$

Equalizing these two values for $\sum_{j=0}^{p-1} A_j$ leads to

$$A_0 = \frac{p^{n-s-1} + p^{\frac{n-s}{2}} + t(f^*)(p-1)p^{\frac{\theta}{2}-1}}{2} \text{ and } B_0 = \frac{p^{n-s-1} - p^{\frac{n-s}{2}} + t(f^*)(p-1)p^{\frac{\theta}{2}-1}}{2},$$

where we used the fact that $A_0 = B_0 + p^{\frac{n-s}{2}}$ (see Lemma 3). □

Proposition 3. *Let $f \in \mathscr{C}_2$ be a plateaued function with $f(0) = 0$ such that $W_{f^*}(0) = t(f^*)p^{\frac{\theta}{2}}\nu$, for some even integer θ such that $1 < \theta \leq 2(n-s)$, and $\nu \in \{1, \mathrm{i}\}$. Then f^* is the constant zero function if and only if $\nu = 1$, $\theta = 2(n-s)$ and $t(f^*) = 1$.*

Proof. Suppose that f^* is the constant zero function. Compute

$$t(f^*)p^{\frac{\theta}{2}}\nu = W_{f^*}(0) = \sum_{x \in supp(W_f)} 1 = p^{n-s}.$$

Hence, $\nu = 1$, $\theta = 2(n-s)$ and $t(f^*) = 1$. Conversely, suppose that $W_{f^*}(0) = p^{n-s}$. By Proposition 2, $A_0 = \frac{p^{n-s} + p^{\frac{n-s}{2}}}{2}$, $B_0 = \frac{p^{n-s} - p^{\frac{n-s}{2}}}{2}$ and, for each $j \in \mathbb{F}_p^\star$, $A_j = B_j = \frac{p^{n-s} + p^{\frac{n-s}{2}}}{2} - \frac{p^{\frac{n-s}{2}} + p^{n-s}}{2} = 0$. Then, the function f^* is the constant zero function. □

Proposition 4. *Let $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ be an s-plateaued function in $\mathscr{C}_2$ such that $f(0) = 0$. Suppose that $W_{f^*}(0) = t(f^*)\nu' p^{\frac{\theta}{2}}$ for some $\nu' \in \{1, \mathrm{i}\}$ and some odd integer θ with $1 < \theta \leq 2(n-s)$. Then, it holds $A_0 = \frac{p^{n-s-1}+p^{\frac{n-s}{2}}}{2}$, $B_0 = \frac{p^{n-s-1}-p^{\frac{n-s}{2}}}{2}$. Moreover, for $j \in \mathbb{F}_p^\star$, $A_j = B_j = A_0 - \frac{p^{\frac{n-s}{2}}+\left(\frac{j}{p}\right)t(f^*)p^{\frac{\theta-1}{2}}}{2}$.*

Proof. By hypothesis, $W_{f^*}(0) = t(f^*)\nu' p^{\frac{\theta-1}{2}}\sqrt{p}$. Equalizing this with (19) gives

$$\sum_{j=1}^{p-1}(2A_j - 2A_0 + p^{\frac{n-s}{2}})\xi_p^j = t(f^*)\nu' p^{\frac{\theta-1}{2}}\sqrt{p}.$$

By Lemma 1, the coefficients of the LHS are determined, so that

$$2A_j = 2A_0 - p^{\frac{n-s}{2}} + \left(\frac{j}{p}\right) t(f^*)p^{\frac{\theta-1}{2}} \tag{21}$$

for each nonzero j. On the one hand, the value of $2\sum_{j=0}^{p-1} A_j$ is known from Lemma 3. Namely,

$$2\sum_{j=0}^{p-1} A_j = p^{n-s} + p^{\frac{n-s}{2}}. \tag{22}$$

On the other hand, from (21), we infer

$$2\sum_{j=0}^{p-1} A_j - p^{\frac{n-s}{2}} = A_0 - p^{\frac{n-s}{2}} + 2(p-1)A_0 - (p-1)p^{\frac{n-s}{2}} + \sum_{j\in\mathbb{F}_p^\star}\left(\frac{j}{p}\right) t(f^*)p^{\frac{\theta-1}{2}}.$$

Since the last term vanishes, it must be $2\sum_{j=0}^{p-1} A_j - p^{\frac{n-s}{2}} = 2pA_0 - p^{\frac{n-s}{2}+1}$. Equalizing with (22), we get $A_0 = \frac{p^{n-s-1}+p^{\frac{n-s}{2}}}{2}$. The value for B_0 is readily obtained from Lemma 3. □

Similarly as for $\mathscr{C}_2$, we analyze two subclasses of functions in $\mathscr{C}_3$, depending on whether θ is even or odd, where $W_{f^*}(0) = t(f^*)p^{\frac{\theta}{2}}$.

Proposition 5. *Let $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ be an s-plateaued function in $\mathscr{C}_3$ such that $f(0) = 0$. Let the set $\mathcal{I}$ be defined as in iii) of Lemma 3. Suppose that $W_{f^*}(0) = t(f^*)p^{\frac{\theta}{2}}$ for some even integer θ with $1 < \theta \leq 2(n-s)$. Then,*

$$A_0 = \frac{p^{n-s-1} + t(f^*)(p-1)p^{\frac{\theta}{2}-1} - \left(\frac{-i}{p}\right)p^{\frac{n-s-1}{2}}}{2}$$

and

$$B_0 = \frac{p^{n-s-1} + t(f^*)(p-1)p^{\frac{\theta}{2}-1} + \left(\frac{-i}{p}\right)p^{\frac{n-s-1}{2}}}{2}$$

for any $i \in \mathcal{I}$. Moreover, for $j \in \mathcal{I}$, $A_j = B_j = \frac{p^{n-s-1}-t(f^*)p^{\frac{\theta}{2}-1}}{2}$ and, for $j \notin \mathcal{I}$,

$$A_j = \frac{p^{n-s-1} - t(f^*)p^{\frac{\theta}{2}-1}}{2} - \left(\frac{-j}{p}\right)p^{\frac{n-s-1}{2}}$$

and $B_j = \frac{p^{n-s-1}-t(f^*)p^{\frac{\theta}{2}-1}}{2} + \left(\frac{-j}{p}\right)p^{\frac{n-s-1}{2}}$.

Proof. From (13) and the hypothesis, it follows $t(f^*)p^{\frac{\theta}{2}} = W_{f^*}(0) = \sum_{j=1}^{p-1}(A_j + B_j - A_0 - B_0)\xi_p^j$. The coefficients of the RHS in this integral equation are completely determined, thus

$$A_j + B_j = A_0 + B_0 - t(f^*)p^{\frac{\theta}{2}} \tag{23}$$

for each nonzero j. Since f is plateaued, we get $p^{n-s} - A_0 - B_0 = \sum_{j=1}^{p-1}(A_j + B_j)$. On the other hand, using (23), we infer

$$\sum_{j=1}^{p-1}(A_j + B_j) = (p-1)(A_0 + B_0 - t(f^*)p^{\frac{\theta}{2}}).$$

Thus, $A_0 + B_0 = p^{n-s-1} + t(f^*)p^{\frac{\theta}{2}-1}(p-1)$. By Lemma 3, we know that $A_0 - B_0 = -\left(\frac{-i}{p}\right)p^{\frac{n-s-1}{2}}$, for any $i \in \mathcal{I}$. Rewriting these two equations in terms of A_0 and B_0 yields

$$A_0 = \frac{p^{n-s-1} + t(f^*)(p-1)p^{\frac{\theta}{2}-1} - \left(\frac{-i}{p}\right)p^{\frac{n-s-1}{2}}}{2}$$

and

$$B_0 = \frac{p^{n-s-1} + t(f^*)(p-1)p^{\frac{\theta}{2}-1} + \left(\frac{-i}{p}\right)p^{\frac{n-s-1}{2}}}{2}.$$

The result now follows after computing the values for A_j and B_j, $j \in \mathbb{F}_p^\star$, by using (23) and the corresponding differences obtained in *iii)* of Lemma 3. □

Finally, let us derive the dual values of a plateaued function in $\mathscr{C}_3$ for which $W_{f^*}(0) = t(f^*)\nu' p^{\frac{\theta}{2}}$ and θ is odd. More precisely, we have the following.

Proposition 6. *Let $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ be an s-plateaued function in $\mathscr{C}_3$ such that $f(0) = 0$. Let the set $\mathcal{I}$ be defined as in Lemma 3. Suppose that $W_{f^*}(0) = t(f^*)\nu' p^{\frac{\theta}{2}}$ for some $\nu' \in \{1, \mathfrak{i}\}$ and an odd integer θ such that $1 < \theta \leq 2(n-s)$. Then*

$$A_0 = \frac{p^{n-s-1} - \left(\frac{-i}{p}\right)p^{\frac{n-s-1}{2}}}{2} \text{ and } B_0 = \frac{p^{n-s-1} + \left(\frac{-i}{p}\right)p^{\frac{n-s-1}{2}}}{2}$$

for any $i \in \mathcal{I}$. Moreover, for $j \in \mathcal{I}$, it follows $A_j = B_j = \frac{p^{n-s-1} + \left(\frac{j}{p}\right)t(f^*)p^{\frac{\theta-1}{2}}}{2}$ and, for $j \notin \mathcal{I}$,

$$A_j = \frac{p^{n-s-1} + \left(\frac{j}{p}\right)t(f^*)p^{\frac{\theta-1}{2}}}{2} - \left(\frac{-j}{p}\right)p^{\frac{n-s-1}{2}} \text{ and } B_j = \frac{p^{n-s-1} + \left(\frac{j}{p}\right)t(f^*)p^{\frac{\theta-1}{2}}}{2} + \left(\frac{-j}{p}\right)p^{\frac{n-s-1}{2}}.$$

Proof. By hypothesis, $t(f^*)\nu' p^{\frac{\theta-1}{2}}\sqrt{p} = W_{f^*}(0)$. From (13), we also have $W_{f^*}(0) = \sum_{j=1}^{p-1}(A_j + B_j - A_0 - B_0)\xi_p^j$. Thus,

$$\sum_{j=1}^{p-1}(A_j + B_j - A_0 - B_0)\xi_p^j = t(f^*)\nu' p^{\frac{\theta-1}{2}}\sqrt{p}.$$

By 3) in Lemma 1, the coefficients of this equation are determined. Namely, it holds

$$A_j + B_j = A_0 + B_0 + \left(\frac{j}{p}\right) t(f^*) p^{\frac{\theta-1}{2}} \tag{24}$$

for each nonzero j. Since f is plateaued, we have $\sum_{j\in\mathbb{F}_p}(A_j + B_j) = p^{n-s}$. On the other hand, summing up the terms in (24) gives

$$\sum_{j\in\mathbb{F}_p^\star}(A_j + B_j) + A_0 + B_0 = p(A_0 + B_0) + t(f^*)p^{\frac{\theta-1}{2}} \sum_{j\in\mathbb{F}_p^\star}\left(\frac{j}{p}\right).$$

Thus, $\sum_{j\in\mathbb{F}_p}(A_j + B_j) = p(A_0 + B_0)$ since the last term in the above equality vanishes. This computation then yields $A_0+B_0 = p^{n-s-1}$. By Lemma 3, we also know that $A_0-B_0 = -\left(\frac{-i}{p}\right) p^{\frac{n-s-1}{2}}$, for any $i \in \mathcal{I}$. Expressing these two equalities in terms of A_0 and B_0 gives

$$A_0 = \frac{p^{n-s-1} - \left(\frac{-i}{p}\right) p^{\frac{n-s-1}{2}}}{2} \text{ and } B_0 = \frac{p^{n-s-1} + \left(\frac{-i}{p}\right) p^{\frac{n-s-1}{2}}}{2}.$$

For $j \neq 0$, the values of A_j and B_j are computed by combining (24) with the values for $A_j - B_j$ obtained in *iii*) of Lemma 3. □

Remark 4. *Propositions 1, 2 and 6 cover all known examples of plateaued functions (up to now). To the best of our knowledge, there are no known examples of functions satisfying the hypotheses of Propositions 4 and 5. We leave it as an open problem to either provide examples of such plateaued functions or to show that θ and $n + s$ must have the same parity.*

Example 4. *Use the same notation as in Proposition 6. Working in $\mathbb{F}_{3^3}\times\mathbb{F}_{3^3}$, consider the function $f(x, y) = \mathrm{Tr}_1^3(2x^4 + x^2) + \mathrm{Tr}_1^3(y^7)$. This function is a non-weakly regular 3-plateaued function (so that $n + s = 9$ is odd) with Walsh spectrum*

$$\{*0^{702}, (-\mathrm{i}3^{9/2})^3, (\mathrm{i}3^{9/2})^6, (-\mathrm{i}3^{9/2}\xi_3^2)^{12}, (\mathrm{i}3^{9/2}\xi_3^2)^6 *\},$$

so that it belongs to $\mathscr{C}_3$. Using MAGMA, we have verified that $W_{f^*}(0) = -18\xi_3 - 9 = -\mathrm{i}3^{5/2}$, so that $t(f^*) = -1$ and $\theta = 5$ is odd. The set $\mathcal{I}$ defined in Lemma 3 equals $-NQR = QR^\star$ since $Z_0/|Z_0| = 1$, thus $\mathcal{I} = \{1\}$. Now, we can readily verify that the values for A_j and B_j are consistent with Proposition 6.

4 Codes from plateaued functions

In this section, we will use plateaued functions $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ to construct linear codes using (5). This approach extends the results in [36,38,47,57,58]. In order to explicitly compute the weights of the derived codes $\mathcal{C}_f$, where f is an s-plateaued function, we must count the number of elements in the preimage of a given function. We will do so by considering the possible dual value distributions studied in Section 3. In the following sections, we derive the full weight distributions of codes stemming from plateaued functions such that $f(0) = 0$, where the distributions are parametrized

by $wt(f^*)$ when $n+s$ is odd and by $wt(f^*)$ and Z_0 when $n+s$ is even, in the latter case it is also required that $f^*(0)=0$.

By adapting the proof of Theorem 2 in [38], the weights of $\mathcal{C}_f$ can easily be obtained when f is any s-plateaued function with $f(0)=0$. Namely, we have the following.

Lemma 4. *[38] Let f be an s-plateaued function. Consider the code $\mathcal{C}_f$ given in (5). For a codeword $\boldsymbol{c}_{a,u} \in \mathcal{C}_f$, we have $wt(\boldsymbol{c}_{0,0})=0$ and $wt(\boldsymbol{c}_{0,u})=p^n-p^{n-1}$ for $u \in \mathbb{F}_{p^n}^\star$. For $a \in \mathbb{F}_p^\star$, if $a^{-1}u \notin supp(W_f)$, then $wt(\boldsymbol{c}_{a,u})=p^n-p^{n-1}$. If $a^{-1}u \in supp(W_f)$, then,*

- *when $n+s$ is odd,*

$$wt(\boldsymbol{c}_{a,u})=\begin{cases} p^n-p^{n-1}, & f^*(a^{-1}u)=0; \\ p^n-p^{n-1}-\epsilon_f(a^{-1}u)\eta(-1)^{\frac{n+s+1}{2}}p^{\frac{n+s-1}{2}}\left(\frac{j}{p}\right), & f^*(a^{-1}u)=j \text{ for some } j\in\mathbb{F}_p^\star; \end{cases} \tag{25}$$

- *when $n+s$ is even,*

$$wt(\boldsymbol{c}_{a,u})=\begin{cases} p^n-p^{n-1}-\epsilon_f(a^{-1}u)\eta(-1)^{\frac{n+s}{2}}p^{\frac{n+s-2}{2}}(p-1), & f^*(a^{-1}u)=0; \\ p^n-p^{n-1}+\epsilon_f(a^{-1}u)\eta(-1)^{\frac{n+s}{2}}p^{\frac{n+s-2}{2}}, & f^*(a^{-1}u)=j \text{ for some } j\in\mathbb{F}_p^\star. \end{cases} \tag{26}$$

4.1 The weight distribution of $\mathcal{C}_f$ for $n+s$ odd

Throughout this section, $n+s$ will be odd. From (25) in Lemma 4, the possible weights of codewords in $\mathcal{C}_f$ are

$$w_1 := p^n-p^{n-1}-p^{(n+s-1)/2}, w_2 := p^n-p^{n-1} \text{ and } w_3 := p^n-p^{n-1}+p^{(n+s-1)/2}. \tag{27}$$

Moreover, counting the number of different weights in $\mathcal{C}_f$ comes down to counting the number of possible values taken by $\epsilon_f(a^{-1}u)\eta(f^*(a^{-1}u)) \in \{-1,0,1\}$ for each $(a,u)\in\mathbb{F}_p^\star\times\mathbb{F}_{p^n}$. More precisely, the weight w_1 is attained by codewords $\mathbf{c}_{a,u}$ for which $\epsilon_f(a^{-1}u)\eta(f^*(a^{-1}u))=\eta(-1)^{\frac{n+s+1}{2}}$, whereas the weight w_3 is attained by codewords $\mathbf{c}_{a,u}$ for which $\epsilon_f(a^{-1}u)\eta(f^*(a^{-1}u))=-\eta(-1)^{\frac{n+s+1}{2}}$.

Multiply both sides of $\epsilon_f(a^{-1}u)\eta(f^*(a^{-1}u)) = \pm\eta(-1)^{\frac{n+s+1}{2}}$ by $\eta(-1)^{\frac{n+s-1}{2}}$. Then, this is equivalent to

$$\eta(-1)^{\frac{n+s-1}{2}}\epsilon_f(a^{-1}u)\eta(f^*(a^{-1}u)) = \pm\eta(-1)^{n+s} = \pm\eta(-1).$$

Thus, from (6), the number $\epsilon_f(a^{-1}u)\eta(f^*(a^{-1}u))$ equals $\eta(-1)^{\frac{n+s+1}{2}}$ if and only if either

$$a^{-1}u \in N_{f^*}(j)\cap\Gamma^{\alpha(\eta(-1))}(f) \text{ for some } j\in QR^\star \text{ or } a^{-1}u \in N_{f^*}(j)\cap\Gamma^{\alpha(-\eta(-1))}(f) \text{ for some } j\in NQR. \tag{28}$$

Similarly, $\epsilon_f(a^{-1}u)\eta(f^*(a^{-1}u)) = -\eta(-1)^{\frac{n+s+1}{2}}$ if and only if either

$$a^{-1}u \in N_{f^*}(j)\cap\Gamma^{\alpha(\eta(-1))}(f) \text{ for some } j\in NQR \text{ or } a^{-1}u \in N_{f^*}(j)\cap\Gamma^{\alpha(-\eta(-1))}(f) \text{ for some } j\in QR^\star. \tag{29}$$

Therefore, it can in principle happen that (28) or (29) never occur, yielding codes with two weights. To address these cases formally, we define the following three subclasses of plateaued functions:

$$\widehat{\mathcal{P}_2} = \{f \in \mathscr{C}_3 \mid f \text{ is weakly regular}\},$$

$$\widetilde{\mathcal{P}_2} = \{f \in \mathscr{C}_1 \mid \forall i \in QR^\star A_i = 0, B_i \neq 0 \text{ and } \forall i \in NQR\ A_i \neq 0, B_i = 0\},$$

and

$$\overline{\mathcal{P}_2} = \{f \in \mathscr{C}_1 \mid \forall i \in QR^\star\ A_i \neq 0, B_i = 0 \text{ and } \forall i \in NQR\ A_i = 0, B_i \neq 0\}.$$

Define $\mathcal{P}_2 = \widehat{\mathcal{P}_2} \cup \widetilde{\mathcal{P}_2} \cup \overline{\mathcal{P}_2}$. These classes yield codes with two weights (see Theorem 2), thus they can be regarded as exceptions since every other plateaued function, with $n+s$ odd, gives rise to a 3-weight code, as shown in the following theorem, which is quite general and it does not necessarily follow from Lemma 1.

Theorem 1. *Let $n > 0$ and $0 \leq s < n$ be integers such that $n+s$ is odd. Let f be any s-plateaued function defined over $\mathbb{F}_{p^n}$ with $f(0) = 0$ such that $f \notin \mathcal{P}_2$. The code $\mathcal{C}_f$ in (5) ($m = 1$) is a three-weight code with parameters $[p^n - 1, n+1, (p-1)p^{n-1} - p^{\frac{n+s-1}{2}}]$, whose weight distribution is displayed in Table 2.*

Proof. From (27), the weights of codewords in $\mathcal{C}_f$ are $w_1 := p^n - p^{n-1} - p^{(n+s-1)/2}, w_2 := p^n - p^{n-1}$ and $w_3 := p^n - p^{n-1} + p^{(n+s-1)/2}$. Since $f \notin \mathcal{P}_2$, there are pairs $(a,u) \in \mathbb{F}_p^\star \times \mathbb{F}_{p^n}$ satisfying both (28) and (29). This implies that the code has three weights. Denote by X, Y and Z the number of codewords attaining the weights w_1, w_2 and w_3, respectively. Using the first two Pless power moments, we get the system of equations

$$X + Y + Z = p^{n+1} - 1; \tag{30}$$

$$w_1 X + w_2 Y + w_3 Z = p^n(p-1)(p^n - 1). \tag{31}$$

By Lemma 4, balanced codewords (i.e., codewords with weight $p^n - p^{n-1}$) come from three types of codewords, namely, codewords stemming from linear functions, codewords whose underlying values $(a,u) \in \mathbb{F}_p^\star \times \mathbb{F}_{p^n}$ yield a zero in the Walsh spectrum of f, and codewords whose underlying values $(a,u) \in \mathbb{F}_p^\star \times \mathbb{F}_{p^n}$ make the value $f^*(a^{-1}u)$ zero. Therefore, the number of balanced codewords is

$$Y = p^n - 1 + (p-1)(p^n - p^{n-s}) + (p-1)(p^{n-s} - wt(f^*)) = p^{n+1} - (p-1)wt(f^*) - 1,$$

and the above system of equations can be solved in terms of $wt(f^*)$. Namely, $X = \frac{(p-1)}{2}(wt(f^*) - (p-1)p^{(n-s-1)/2})$ and $Z = \frac{(p-1)}{2}(wt(f^*) + (p-1)p^{(n-s-1)/2})$. □

Example 5. *Continuing with Example 3, consider the non-weakly regular 1-plateaued function $f(x,y) = \mathrm{Tr}_1^3(x^2 + y^7)$ over $\mathbb{F}_{3^3} \times \mathbb{F}_{3^3}$ (so that $n+s = 7$ is odd). Since its dual values are $A_0 = 27, A_1 = 36, A_2 = 18$ and $B_0 = 54, B_1 = 72, B_2 = 36$, the function f belongs to $\mathscr{C}_1 \setminus \mathcal{P}_2$. Moreover, we have $wt(f^*) = 162 = 3^6 - 3^5$. The code $\mathcal{C}_f$ is a 3-weight code with weight enumerator*

$$1 + 180z^{459} + 1862z^{486} + 144z^{513},$$

which is consistent with Theorem 1 (see also Table 3).

Table 2: Weight distribution of the code $\mathcal{C}_f$, derived in Theorem 1, for an s-plateaued function $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ in $\mathscr{C}_1$ with $f(0) = 0$, when $n + s$ is odd.

Weight w	Number of codewords
$p^n - p^{n-1} - p^{(n+s-1)/2}$	$\frac{(p-1)}{2}(wt(f^*) + (p-1)p^{\frac{n-s-1}{2}})$
$p^n - p^{n-1}$	$p^{n+1} - (p-1)wt(f^*) - 1$
$p^n - p^{n-1} + p^{(n+s-1)/2}$	$\frac{(p-1)}{2}(wt(f^*) - (p-1)p^{\frac{n-s-1}{2}})$

From Theorem 1, we can now derive the weight distribution of $\mathcal{C}_f$ for s-plateaued functions with $f(0) = 0$ such that $wt(f^*) = p^{n-s} - p^{n-s-1}$. For instance, functions satisfying the hypothesis of Proposition 1, when θ and $n + s$ are odd. The corresponding values are displayed in Table 3.

Table 3: Weight distribution of the code $\mathcal{C}_f$, derived in Theorem 1, for an s-plateaued function $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ in $\mathscr{C}_1$ with $f(0) = 0$, and $wt(f^*) = p^{n-s} - p^{n-s-1}$, when $n + s$ is odd.

Weight w	Number of codewords
$p^n - p^{n-1} - p^{(n+s-1)/2}$	$\frac{(p-1)^2}{2}(p^{n-s-1} + p^{\frac{n-s-1}{2}})$
$p^n - p^{n-1}$	$p^{n+1} - (p-1)^2 p^{n-s-1} - 1$
$p^n - p^{n-1} + p^{(n+s-1)/2}$	$\frac{(p-1)^2}{2}(p^{n-s-1} - p^{\frac{n-s-1}{2}})$

As noted in the beginning of Section 4.1, it could happen that the code $\mathcal{C}_f$ is a 2-weight code. This case is addressed in detail in Theorem 2 below, but first we need the following result.

Proposition 7. *There does not exist a p-ary linear code with parameters $[p^n - 1, n + 1, d]$, where $d \geq p^n - p^{n-1}$.*

Proof. By the Griesmer bound (4), a code with minimum distance $d \geq p^n - p^{n-1}$ and dimension $k = n + 1$ must have length at least p^n since

$$\sum_{i=0}^{n} \left\lceil \frac{d}{p^i} \right\rceil \geq \sum_{i=0}^{n} \left\lceil \frac{p^n - p^{n-1}}{p^i} \right\rceil = (p-1)p^{n-1} + (p-1)p^{n-2} + \cdots + (p-1) + \left\lceil \frac{p-1}{p} \right\rceil = (p-1)\frac{p^n - 1}{(p-1)} + 1 = p^n.$$

Therefore, a $[p^n - 1, n + 1, d]$-code with $d \geq p^n - p^{n-1}$ cannot exist. □

Theorem 2. *Let $n > 0$ and $0 \leq s < n$ be integers such that $n + s$ is odd. Let f be any s-plateaued function defined over $\mathbb{F}_{p^n}$ with $f(0) = 0$. The code $\mathcal{C}_f$ in (5) ($m = 1$) is a two-weight code if and only if $f \in \mathcal{P}_2$. In this case, it must be that either $f \in \widehat{\mathcal{P}_2}, \epsilon_f = \eta(j)\eta(-1)^{\frac{n+s-1}{2}}$, $j \notin \mathcal{I}$ (as defined in Lemma 3), or, $f \in \overline{\mathcal{P}_2}, \eta(-1) = 1$, or, $f \in \widetilde{\mathcal{P}_2}, \eta(-1) = -1$. Its weight enumerator polynomial is*

$$1 + Xz^{w_1} + (p^{n+1} - X - 1)z^{w_2},$$

where w_1, w_2 are given in (27), and $X = (p-1)wt(f^*)$.

Proof. From the discussion in the beginning of Section 4.1, we know that, when $n + s$ is odd, the number of different weights in $\mathcal{C}_f$ equals the number of possible values taken by $\epsilon_f(a^{-1}u)\eta(f^*(a^{-1}u))$

for each $(a,u) \in \mathbb{F}_p^\star \times \mathbb{F}_{p^n}$ (which takes values in $\{-1,0,1\}$). Note that balanced codewords are always attained. Therefore, from (28) and (29), the code $\mathcal{C}_f$ is readily seen to be two-weight exactly when $f \in \mathcal{P}_2$. Again, consider the values w_1, w_2 and w_3 as in (27). Since $\mathcal{C}_f$ is two-weight, it must have nonzero weights either w_1 and w_2 or w_2 and w_3. Proposition 7 implies that the latter case cannot happen, thus it must be that the two nonzero weights are w_1 and w_2. Using a similar derivation as in Theorem 8, we see that the frequency of balanced codewords is $Y = p^{n+1} - (p-1)wt(f^*) - 1$. Therefore, there must be $X = (p-1)wt(f^*)$ codewords with weight w_1. To precise the weight distributions, we use the definitions of the classes $\widehat{\mathcal{P}_2}, \overline{\mathcal{P}_2}$, and $\widetilde{\mathcal{P}_2}$, and the conditions in (28). Therefore, the weight-enumerator polynomial $p(z)$ of $\mathcal{C}_f$ equals $1 + Xz^{w_1} + Yz^{w_2}$ in any of the following three cases: $f \in \widehat{\mathcal{P}_2}, \epsilon_f = \eta(j)\eta(-1)^{\frac{n+s-1}{2}}$, or $f \in \overline{\mathcal{P}_2}, \eta(-1) = 1$, or $f \in \widetilde{\mathcal{P}_2}, \eta(-1) = -1$, for $j \notin \mathcal{I}$, where $\mathcal{I}$ is defined as in Lemma 3. □

Remark 5. *The proof of Theorem 2 shows that the conditions in* (29) *can happen only when* (28) *also occur, otherwise a code with parameters* $[p^n - 1, n+1, p^n - p^{n-1}]$ *would be obtained, which does not exist by Proposition 7. Therefore, any plateaued function* $f \in \mathcal{P}_2$ *with* $f(0) = 0$ *necessarily satisfies*

$$f \in \widehat{\mathcal{P}_2}, \epsilon_f = \eta(j)\eta(-1)^{\frac{n+s-1}{2}}, \text{ or } f \in \overline{\mathcal{P}_2}, \eta(-1) = 1, \text{ or } f \in \widetilde{\mathcal{P}_2}, \eta(-1) = -1,$$

for $j \notin \mathcal{I}$, *where* $\mathcal{I}$ *is defined as in Lemma 3.*

Example 6. *Working in* $\mathbb{F}_{3^3} \times \mathbb{F}_{3^3}$, *consider the function given by* $f(x,y) = \mathrm{Tr}_1^3(2x^4 + x^2 + y)$. *This function is a weakly regular* 5*-plateaued function in* $\mathscr{C}_3$ *(thus in* $\widehat{\mathcal{P}_2}$*) with Walsh spectrum*

$$\{*0^{726}, (\mathrm{i}3^{\frac{11}{2}})^1, (\mathrm{i}3^{\frac{11}{2}}\xi_3^2)^2 *\}.$$

Note that $n+s = 11$ *is odd,* $\epsilon_f = 1$ *and* $\mathcal{I} = QR^\star = \{1\}$. *For any* $j \notin \mathcal{I}$, *it holds that* $\eta(-1)^{\frac{10}{2}}\eta(j) = 1$. *Moreover,* $wt(f^*) = 2$. *The code* $\mathcal{C}_f$ *is a linear code with parameters* $[728, 7, 243]$ *and weight enumerator*

$$1 + 4z^{243} + 2182z^{486},$$

which is consistent with Theorem 2.

4.2 The weight distribution of $\mathcal{C}_f$ for $n+s$ even

We now demonstrate that there is an essential difference when $n+s$ is even since the derived codes are either 3-weight or 5-weight. This is a non-trivial consequence of the behavior of some exponential sums of the quadratic automorphism of $\mathbb{Q}(\xi_p)$, which depends on the parity of $n+s$. Throughout this section, $n+s$ will be even. From (26) in Lemma 4, the *a priori* weights of codewords in $\mathcal{C}_f$ are

$$\begin{aligned} &w_1 = p^n - p^{n-1} - p^{(n+s-2)/2}(p-1), w_2 = p^n - p^{n-1} - p^{(n+s-2)/2}, \\ &w_3 = p^n - p^{n-1}, \\ &w_4 = p^n - p^{n-1} + p^{(n+s-2)/2}, w_5 = p^n - p^{n-1} + p^{(n+s-2)/2}(p-1). \end{aligned} \tag{32}$$

4.2.1 The case $f \in \mathscr{C}_1$

For plateaued functions in $\mathscr{C}_1$, $n+s$ even, the weight distribution of $\mathcal{C}_f$ can also be fully derived by using the parameters Z_0 and $wt(f^*)$ in the corresponding frequency counts. Note that the weakly regular case has been addressed in [38].

Theorem 3. *Let $n > 0$ and $0 \leq s < n$ be integers such that $n+s$ is even. Let $f \in \mathscr{C}_1$ be a non-weakly regular s-plateaued function defined over $\mathbb{F}_{p^n}$ with $f(0) = 0$. The code $\mathcal{C}_f$ in (5) ($m = 1$) is a five-weight code with parameters $[p^n - 1, n+1, p^n - p^{n-1} - p^{(n+s-2)/2}(p-1)]$, whose weight distribution is summarized in Table 4.*

Proof. From (32), we get that the weights of $\mathcal{C}_f$ are $w_1 = p^n - p^{n-1} - p^{(n+s-2)/2}(p-1)$, $w_2 = p^n - p^{n-1} - p^{(n+s-2)/2}$, $w_3 = p^n - p^{n-1}$, $w_4 = p^n - p^{n-1} + p^{(n+s-2)/2}$ and $w_5 = p^n - p^{n-1} + p^{(n+s-2)/2}(p-1)$. By Lemma 4, these weights are directly related to the dual values. Indeed, the number of codewords with weight w_1 is equal to

$$\#\{(a,u) \in \mathbb{F}_p^\star \times \mathbb{F}_{p^n} : f^*(a^{-1}u) = 0, \epsilon_f(a^{-1}u) = \eta(-1)^{\frac{n+s}{2}}\},$$

which equals $(p-1)A_0$ by using (6). Similarly, the number of codewords with weight w_5 is

$$\#\{(a,u) \in \mathbb{F}_p^\star \times \mathbb{F}_{p^n} : f^*(a^{-1}u) = 0, \epsilon_f(a^{-1}u) = -\eta(-1)^{\frac{n+s}{2}}\},$$

which equals $(p-1)B_0$. To get the values of A_0 and B_0, note that $A_0 + B_0 = p^{n-s} - wt(f^*)$ and $A_0 - B_0 = Z_0$, thus $A_0 = \frac{p^{n-s} - wt(f^*) + Z_0}{2}$ and $B_0 = \frac{p^{n-s} - wt(f^*) - Z_0}{2}$. Then,

$$(p-1)A_0 = \frac{(p-1)}{2}(p^{n-s} - wt(f^*) + Z_0) \text{ and } (p-1)B_0 = \frac{(p-1)}{2}(p^{n-s} - wt(f^*) - Z_0).$$

The number of codewords of weight w_2 and w_4 is equal to

$$\#\{(a,u) \in \mathbb{F}_p^\star \times \mathbb{F}_{p^n} : f^*(a^{-1}u) \neq 0, \epsilon_f(a^{-1}u) = \eta(-1)^{\frac{n+s}{2}}\}$$

and

$$\#\{(a,u) \in \mathbb{F}_p^\star \times \mathbb{F}_{p^n} : f^*(a^{-1}u) \neq 0, \epsilon_f(a^{-1}u) = -\eta(-1)^{\frac{n+s}{2}}\},$$

respectively. These values are $(p-1)\sum_{j=1}^{p-1} B_j$ and $(p-1)\sum_{j=1}^{p-1} A_j$. Let us compute them explicitly. By definition, it holds that $\sum_{j=1}^{p-1}(A_j + B_j) = wt(f^*)$. From (15), we also know that $A_j - B_j = Z_0 - p^{\frac{n-s}{2}}$ for each $j \in \mathbb{F}_p^\star$. These two equalities yield

$$(p-1)\sum_{j=1}^{p-1} A_j = \frac{(p-1)}{2}(wt(f^*) + (p-1)(Z_0 - p^{\frac{n-s}{2}}))$$

and

$$(p-1)\sum_{j=1}^{p-1} B_j = \frac{(p-1)}{2}(wt(f^*) - (p-1)(Z_0 - p^{\frac{n-s}{2}})).$$

Finally, note that there are $p^n - 1 + (p-1)(p^n - p^{n-s})$ balanced codewords (i.e. corresponding to the weight w_3) by Lemma 4. □

Table 4: Weight distribution of the code $\mathcal{C}_f$, derived in Theorem 3, for an s-plateaued function $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ in $\mathscr{C}_1$ with $f(0)=0$, when $n+s$ is even, where $Z_0 = A_0 - B_0$.

Weight w	Number of codewords
$p^n - p^{n-1} - p^{(n+s-2)/2}(p-1)$	$\frac{(p-1)}{2}(p^{n-s} - wt(f^*) + Z_0)$
$p^n - p^{n-1} - p^{(n+s-2)/2}$	$\frac{(p-1)}{2}(wt(f^*) - (p-1)(Z_0 - p^{\frac{n-s}{2}}))$
$p^n - p^{n-1}$	$p^{n+1} - (p-1)p^{n-s} - 1$
$p^n - p^{n-1} + p^{(n+s-2)/2}$	$\frac{(p-1)}{2}(wt(f^*) + (p-1)(Z_0 - p^{\frac{n-s}{2}}))$
$p^n - p^{n-1} + p^{(n+s-2)/2}(p-1)$	$\frac{(p-1)}{2}(p^{n-s} - wt(f^*) - Z_0)$

Corollary 1. *Let $n > 0$ and $0 \le s < n$ be integers such that $n+s$ is even. Let $f \in \mathscr{C}_1$ be any non-weakly regular s-plateaued function defined over $\mathbb{F}_{p^n}$ with $f(0)=0$. Suppose that $W_{f^*}(0) = t(f^*)\nu' p^{\frac{n-s}{2}}$ for some $\nu' \in \{1, \mathrm{i}\}$. The code $\mathcal{C}_f$ in (5) ($m=1$) is a five-weight code with parameters $[p^n-1, n+1, p^n - p^{n-1} - p^{(n+s-2)/2}(p-1)]$, whose weight distribution is displayed in Table 5.*

Proof. By Proposition 1,

$$A_0 + B_0 = p^{n-s-1} + t(f^*)(p-1)p^{\frac{n-s}{2}-1}$$

since $n+s$ and $n-s$ are even (see Table 1). Hence,

$$wt(f^*) = p^{n-s} - (A_0 + B_0) = p^{n-s} - p^{n-s-1} - t(f^*)(p-1)p^{\frac{n-s}{2}-1}.$$

The result follows then from Theorem 3 by plugging the obtained value for $wt(f^*)$. □

Table 5: Weight distribution of the code $\mathcal{C}_f$, derived in Corollary 1, for an s-plateaued function $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ in $\mathscr{C}_1$ with $f(0)=0$ and $W_{f^*}(0) = t(f^*)\nu' p^{\frac{n-s}{2}}$, when $n+s$ is even.

Weight w	Number of codewords
$p^n - p^{n-1} - p^{(n+s-2)/2}(p-1)$	$\frac{(p-1)}{2}(p^{n-s-1} + t(f^*)p^{\frac{n-s}{2}} - t(f^*)p^{\frac{n-s}{2}-1} + Z_0)$
$p^n - p^{n-1} - p^{(n+s-2)/2}$	$\frac{(p-1)^2}{2}(p^{n-s-1} - t(f^*)p^{\frac{n-s}{2}-1} - (Z_0 - p^{\frac{n-s}{2}})$
$p^n - p^{n-1}$	$p^{n+1} - (p-1)p^{n-s} - 1$
$p^n - p^{n-1} + p^{(n+s-2)/2}$	$\frac{(p-1)^2}{2}(p^{n-s-1} - t(f^*)p^{\frac{n-s}{2}-1} + (Z_0 - p^{\frac{n-s}{2}})$
$p^n - p^{n-1} + p^{(n+s-2)/2}(p-1)$	$\frac{(p-1)}{2}(p^{n-s-1} + t(f^*)p^{\frac{n-s}{2}} - t(f^*)p^{\frac{n-s}{2}-1} - Z_0)$

Example 7. *Continuing with Example 2, consider the non-weakly regular bent function $f(x) = \mathrm{Tr}_1^6(\omega^7 x^{98})$ over $\mathbb{F}_{3^6}$, where ω is a primitive element of $\mathbb{F}_{3^6}$ satisfying $\omega^6 + \omega^4 + \omega^3 + \omega + 1 = 0$. Its dual values are $A_0 = A_1 = A_2 = 168$ and $B_0 = 57, B_1 = B_2 = 84$, thus $Z_0 = 168 - 57 = 111$. The dual function $f^* : \mathbb{F}_{3^6} \to \mathbb{F}_3$ satisfies $W_{f^*}(0) = -27$, so that $t(f^*) = -1$ and $wt(f^*) = 504$. The code $\mathcal{C}_f$ is a 5-weight $[728, 7, 468]$-code with weight enumerator*

$$1 + 336z^{468} + 336z^{477} + 728z^{486} + 672z^{495} + 114z^{504},$$

which is consistent with Theorem 3 and Corollary 1 (see also Table 5).

4.2.2 The case $f \in \mathscr{C}_2$

Now, we consider plateaued functions in $\mathscr{C}_2$, $n+s$ even. First, note that the case when f^* is constant zero has been already addressed in [47]. Here, we provide a short description for completeness.

It is not hard to see that when $f \in \mathscr{C}_2$ and f^* is the constant zero function the code $\mathcal{C}_f$ is a three-weight code. Indeed, using (26), we see that the possible values for the weight of unbalanced codewords are

$$w_1 := p^n - p^{n-1} - p^{(n+s-2)/2}(p-1) \text{ and } w_3 := p^n - p^{n-1} + p^{(n+s-2)/2}(p-1),$$

whose frequencies come from the cardinalities (by Lemma 4),

$$\#\{(a,u) \in \mathbb{F}_p^\star \times \mathbb{F}_{p^n} : f^*(a^{-1}u) = 0, \epsilon_f(a^{-1}u) = \eta(-1)^{\frac{n+s}{2}}\}$$

and

$$\#\{(a,u) \in \mathbb{F}_p^\star \times \mathbb{F}_{p^n} : f^*(a^{-1}u) = 0, \epsilon_f(a^{-1}u) = -\eta(-1)^{\frac{n+s}{2}}\},$$

which are equal to $(p-1)A_0$ and $(p-1)B_0$, respectively. By Lemma 3, we know that

$$A_0 = \frac{p^{n-s} + p^{\frac{n-s}{2}}}{2}, B_0 = \frac{p^{n-s} - p^{\frac{n-s}{2}}}{2}.$$

Thus, the weight enumerator polynomial is given by

$$1 + \frac{(p-1)}{2}(p^{n-s} + p^{\frac{n-s}{2}})z^{w_1} + (p^{n+1} - (p-1)p^{n-s} - 1)z^{w_2} + \frac{(p-1)}{2}(p^{n-s} - p^{\frac{n-s}{2}})z^{w_3},$$

where $w_2 = p^n - p^{n-1}$. For further reference, we record this discussion in Table 6.

Table 6: Weight distribution of the code $\mathcal{C}_f$ for an s-plateaued function $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ in $\mathscr{C}_2$ such that $f(0) = 0$, whose dual f^* is constant zero (so $n+s$ is even).

Weight w	Number of codewords
$p^n - p^{n-1} - p^{(n+s-2)/2}(p-1)$	$\frac{(p-1)}{2}(p^{n-s} + p^{\frac{n-s}{2}})$
$p^n - p^{n-1}$	$p^{n+1} - (p-1)p^{n-s} - 1$
$p^n - p^{n-1} + p^{(n+s-2)/2}(p-1)$	$\frac{(p-1)}{2}(p^{n-s} - p^{\frac{n-s}{2}})$

Now we turn to analyzing the remaining cases of functions in $\mathscr{C}_2$. First, we consider the case $f \in \mathscr{C}_2$ and θ even, where $W_{f^*}(0) = t(f^*)\nu' p^{\frac{\theta}{2}}$.

Theorem 4. *Let $n > 0$ and $0 \le s < n$ be integers such that $n+s$ is even. Let $f \in \mathscr{C}_2$ be an s-plateaued function defined over $\mathbb{F}_{p^n}$ with $f(0) = 0$. Suppose that $W_{f^*}(0) = t(f^*)\nu' p^{\frac{\theta}{2}}$ for some $\nu' \in \{1, \mathrm{i}\}$ and an even integer θ with $1 < \theta \le 2(n-s)$. Suppose that f^* is not constant zero. Then, the code $\mathcal{C}_f$ in (5) ($m = 1$) is a five-weight code with parameters $[p^n - 1, n+1, p^n - p^{n-1} - p^{(n+s-2)/2}(p-1)]$, whose weight distribution is summarized in Table 7.*

Proof. Again, from (32), the weights are seen to be $w_1 = p^n - p^{n-1} - p^{(n+s-2)/2}(p-1)$, $w_2 = p^n - p^{n-1} - p^{(n+s-2)/2}$, $w_3 = p^n - p^{n-1}$, $w_4 = p^n - p^{n-1} + p^{(n+s-2)/2}$ and $w_5 = p^n - p^{n-1} + p^{(n+s-2)/2}(p-1)$.

Following a similar reasoning as in Theorem 3 and using Proposition 2 to obtain the values of A_j, B_j, for each $j \in \mathbb{F}_p$, the number of codewords with weights w_1 and w_5 are, respectively,

$$(p-1)A_0 = (p-1)\left(\frac{p^{n-s-1} + p^{\frac{n-s}{2}} + t(f^*)(p-1)p^{\frac{\theta}{2}-1}}{2}\right),$$

and

$$(p-1)B_0 = (p-1)\left(\frac{p^{n-s-1} - p^{\frac{n-s}{2}} + t(f^*)(p-1)p^{\frac{\theta}{2}-1}}{2}\right).$$

Since $A_j = B_j$, for each $j \in \mathbb{F}_p^\star$, the number of codewords with weights w_2 and w_4 equals

$$(p-1)\sum_{j=1}^{p-1} A_j = \frac{(p-1)}{2}(p^{n-s-1} - t(f^*)p^{\frac{\theta}{2}-1}).$$

Finally, there are $p^n - 1 + (p-1)(p^n - p^{n-s})$ balanced codewords by Lemma 4. □

Table 7: Weight distribution of the code $\mathcal{C}_f$, derived in Theorem 4, for an s-plateaued function $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ such that $f(0) = 0$ and $W_{f^*}(0) = t(f^*)\nu p^{\frac{\theta}{2}}$, when $n+s$ and θ are even.

Weight w	Number of codewords
$p^n - p^{n-1} - p^{(n+s-2)/2}(p-1)$	$\frac{(p-1)}{2}(p^{n-s-1} + p^{\frac{n-s}{2}} + t(f^*)(p-1)p^{\frac{\theta}{2}-1})$
$p^n - p^{n-1} - p^{(n+s-2)/2}$	$\frac{(p-1)^2}{2}(p^{n-s-1} - t(f^*)p^{\frac{\theta}{2}-1})$
$p^n - p^{n-1}$	$p^{n+1} - (p-1)p^{n-s} - 1$
$p^n - p^{n-1} + p^{(n+s-2)/2}$	$\frac{(p-1)^2}{2}(p^{n-s-1} - t(f^*)p^{\frac{\theta}{2}-1})$
$p^n - p^{n-1} + p^{(n+s-2)/2}(p-1)$	$\frac{(p-1)}{2}(p^{n-s-1} - p^{\frac{n-s}{2}} + t(f^*)(p-1)p^{\frac{\theta}{2}-1})$

The last case remaining is $f \in \mathscr{C}_2$ and θ odd, where $W_{f^*}(0) = t(f^*)\nu' p^{\frac{\theta}{2}}$. The main difference with the previous case is the use of Proposition 4 instead of Proposition 2.

Theorem 5. *Let $n > 0$ and $0 \le s < n$ be integers such that $n+s$ is even. Let $f \in \mathscr{C}_2$ be an s-plateaued function defined over $\mathbb{F}_{p^n}$ with $f(0) = 0$. Suppose that $W_{f^*}(0) = t(f^*)\nu' p^{\frac{\theta}{2}}$ for some $\nu' \in \{1, \mathrm{i}\}$ and an odd integer θ with $1 < \theta \le 2(n-s)$. The code $\mathcal{C}_f$ in (5) ($m = 1$) is a five-weight code with parameters $[p^n - 1, n+1, p^n - p^{n-1} - p^{(n+s-2)/2}(p-1)]$, whose weight distribution is displayed in Table 8.*

Proof. As before, the weights are $w_1 = p^n - p^{n-1} - p^{(n+s-2)/2}(p-1)$, $w_2 = p^n - p^{n-1} - p^{(n+s-2)/2}$, $w_3 = p^n - p^{n-1}$, $w_4 = p^n - p^{n-1} + p^{(n+s-2)/2}$ and $w_5 = p^n - p^{n-1} + p^{(n+s-2)/2}(p-1)$. Using Proposition 4 and Lemma 4, we compute the frequencies of codewords. The number of codewords with weight w_1 is

$$(p-1)A_0 = \frac{(p-1)}{2}(p^{n-s-1} + p^{\frac{n-s}{2}}).$$

The number of codewords with weight w_5 is

$$(p-1)B_0 = \frac{(p-1)}{2}(p^{n-s-1} - p^{\frac{n-s}{2}}).$$

The number of codewords of weight w_2 and w_4 is

$$(p-1)\sum_{j=1}^{p-1} A_j = \frac{(p-1)^2}{2} p^{n-s-1}.$$

Finally, there are $p^n - 1 + (p-1)(p^n - p^{n-s})$ balanced codewords. □

Table 8: Weight distribution of the code $\mathcal{C}_f$, derived in Theorem 5, for an s-plateaued function $f: \mathbb{F}_{p^n} \to \mathbb{F}_p$ such that $f(0) = 0$ and $W_{f^*}(0) = t(f^*)\nu p^{\frac{\theta}{2}}$, when $n+s$ is even and θ is odd.

Weight w	Number of codewords
$p^n - p^{n-1} - p^{(n+s-2)/2}(p-1)$	$\frac{(p-1)}{2}(p^{n-s-1} + p^{\frac{n-s}{2}})$
$p^n - p^{n-1} - p^{(n+s-2)/2}$	$\frac{(p-1)^2}{2} p^{n-s-1}$
$p^n - p^{n-1}$	$p^{n+1} - (p-1)p^{n-s} - 1$
$p^n - p^{n-1} + p^{(n+s-2)/2}$	$\frac{(p-1)^2}{2} p^{n-s-1}$
$p^n - p^{n-1} + p^{(n+s-2)/2}(p-1)$	$\frac{(p-1)}{2}(p^{n-s-1} - p^{\frac{n-s}{2}})$

4.3 The weight distribution of $\mathcal{C}_F$

In this section, we extend the results in the previous sections to the case of vectorial plateaued functions. Little is known about infinite families of vectorial non-bent non-quadratic plateaued functions [13]. Namely, the only known examples are some power functions.

Example 8. *For an integer k with $n/\gcd(n,k)$ odd, consider the functions $F: \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$ given by $F(x) = x^{(p^{2k}+1)/2}$ and $F': \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$ given by $F'(x) = x^{p^{2k}-p^k+1}$ (Kasami). Then, both $F(x)$ and $F'(x)$ are s-plateaued functions, where $s = \gcd(n,k)$, whose components have zero duals.*

Example 9. *Working in $\mathbb{F}_{3^5}$, consider the 1-plateaued function $F: \mathbb{F}_{3^5} \to \mathbb{F}_{3^5}$ defined by $F(x) = x^{\frac{3^2+1}{2}} = x^5$. Using MAGMA, we have verified that the code $\mathcal{C}_F$ is a minimal self-orthogonal code with parameters $[242, 10, 144]$, $d^\perp = 2$, where $d^\perp$ denotes the minimum distance of the dual code. The weight enumerator polynomial of $\mathcal{C}_F$ is $1+10890z^{144}+39446z^{162}+8712z^{180}$, which is consistent with Table 9 below.*

The weight distributions for these two (vectorial) examples can then be derived in general.

Theorem 6. *Let $n > 0$ and $0 \le s \le n$ be integers such that $n+s$ is even. Let $F: \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$ be a vectorial plateaued function whose all components are s-plateaued and have zero duals such that $F(0) = 0$. The code $\mathcal{C}_F$ in (5) ($m = n$) is a three-weight code with parameters $[p^n - 1, 2n, p^n - p^{n-1} - p^{(n+s-2)/2}(p-1)]$, whose weight distribution is displayed in Table 9.*

Proof. The code $\mathcal{C}_F$ can be seen as the union of the codes corresponding to the components of F, i.e., $\mathcal{C}_F = \bigcup_{a \in \mathbb{F}_{p^n}} \mathcal{C}_{F_a}$. Since the components have zero duals, the weight distributions of the codes $\mathcal{C}_{F_a}$ are given by Table 6. Each codeword that belongs to exactly one code associated to a component, contributes the numbers displayed in Table 6. Each frequency has then to be multiplied

by $\frac{p^n-1}{p-1}$, due to the number of (nonzero) components that are not a multiple of the component in question. This way we count frequencies of codewords induced by each component, except the frequency corresponding to balanced codewords stemming from linear functions, which are counted only once. The weight distribution can then be readily obtained. □

Table 9: Weight distribution of $\mathcal{C}_F$ in Theorem 6, where $F\colon \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$ is an s-plateaued function, whose all components have zero dual and $F(0) = 0$ ($n + s$ is even).

Weight w	Number of codewords
$p^n - p^{n-1} - p^{(n+s-2)/2}(p-1)$	$\frac{1}{2}(p^n-1)(p^{n-s} + p^{\frac{n-s}{2}})$
$p^n - p^{n-1}$	$p^n - 1 + (p^n-1)(p^n - p^{n-s})$
$p^n - p^{n-1} + p^{(n+s-2)/2}(p-1)$	$\frac{1}{2}(p^n-1)(p^{n-s} - p^{\frac{n-s}{2}})$

To build an infinite family of vectorial non-weakly regular plateaued function whose components belong to $\mathscr{C}_1$, consider the following simple construction.

Construction A.

1) Let $F_1(x)$ be any planar function over $\mathbb{F}_{p^n}$ whose components belong to $\mathscr{C}_1$ and are weakly regular (e.g. x^2).

2) Let $F_2(y)$ be any of the functions over $\mathbb{F}_{p^n}$ provided in Example 8 for which we set $s = \gcd(n, k)$ (then n and s have the same parity).

3) Consider the direct sum $F : \mathbb{F}_{p^n} \times \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$ given by $F(x, y) = F_1(x) + F_2(y)$. This is a vectorial plateaued function, whose components are non-weakly regular and they belong to $\mathscr{C}_1$.

4) Construct the code $\mathcal{C}_F$, which is a $[p^{2n}-1, 3n, d]$-code, where $d = p^{2n} - p^{2n-1} - p^{(2n+s-2)/2}(p-1)$ when $2n + s$ is odd, or $d = p^{2n} - p^{2n-1} - p^{(2n+s-1)/2}$ when $2n + s$ is even.

To justify 3) in Construction A, note that the function F from Construction A is indeed a non-weakly regular plateaued function since its Walsh spectrum is completely determined by the Walsh spectra of F_1 and F_2, namely, for all $(a, b, b') \in \mathbb{F}^\star_{p^n} \times \mathbb{F}_{p^n} \times \mathbb{F}_{p^n}$,

$$W_{F_a}(b, b') = W_{(F_1)_a}(b) W_{(F_2)_a}(b').$$

To see that $F_a \in \mathscr{C}_1$ for each $a \in \mathbb{F}^\star_{p^n}$, let us introduce a superscript notation for the dual preimage distribution of the component functions of F, F_1, and F_2, i.e., we will use the symbols $A_j^{F_a}, B_j^{F_a}$, $A_j^{(F_i)_a}, B_j^{(F_i)_a}$ for $i \in \{1, 2\}$ and $a \in \mathbb{F}^\star_{p^n}$ to represent the (obvious) dual sets. By Lemma 3, we know that $A_0^{(F_2)_a} \neq B_0^{(F_2)_a}$ and $A_j^{(F_2)_a} = B_j^{(F_2)_a} = 0$, for each $j \in \mathbb{F}^\star_p$, as the dual of $(F_2)_a$ is the all-zero function. Since $(F_1)_a$ is weakly regular bent, either $A_j^{(F_1)_a} \neq 0$ and $B_j^{(F_1)_a} = 0$ for all $j \in \mathbb{F}^\star_p$, or, $B_j^{(F_1)_a} \neq 0$ and $A_j^{(F_1)_a} = 0$ for all $j \in \mathbb{F}^\star_p$. Hence, either $A_j^{F_a} = A_0^{(F_2)_a} A_j^{(F_1)_a}$ or $A_j^{F_a} = B_0^{(F_2)_a} B_j^{(F_1)_a}$. Respectively, either $B_j^{F_a} = B_0^{(F_2)_a} A_j^{(F_1)_a}$ or $B_j^{F_a} = A_0^{(F_2)_a} B_j^{(F_1)_a}$. In both cases, we clearly have $F_a \in \mathscr{C}_1$.

The minimum distance of $\mathcal{C}_F$ given in 4) of Construction A can now be determined: each component F_a is a non-weakly regular s-plateaued function in $\mathscr{C}_1$, thus the code $\mathcal{C}_{F_a}$ has minimum distance either $p^{2n} - p^{2n-1} - p^{(2n+s-2)/2}(p-1)$ or $p^{2n} - p^{2n-1} - p^{(2n+s-1)/2}$, depending on whether $2n+s$, equivalently, s, is odd or even (see Tables 2 and 4). So, either

$$d = p^{2n} - p^{2n-1} - p^{(2n+s-2)/2}(p-1) \text{ or } d = p^{2n} - p^{2n-1} - p^{(2n+s-1)/2},$$

since $\mathcal{C}_F = \bigcup_{a \in \mathbb{F}_{p^n}} \mathcal{C}_{F_a}$.

Suppose that s is odd, then n must be odd for F_2 to exist. Since $F_a \in \mathscr{C}_1$,

$$wt(F_a^*) = p^{2n-s} - p^{2n-s-1}$$

by Proposition 1. From Table 2, we get the weight distribution of $\mathcal{C}_{F_a}$. Finally, each frequency must be multiplied by $\frac{p^n-1}{p-1}$ (except those that come from linear functions) to get the number of codewords of each weight in $\mathcal{C}_F$. The weight distribution of $\mathcal{C}_F$ is displayed in Table 10.

Table 10: Weight distribution of $\mathcal{C}_F$, where F is given by Construction A when s is odd.

Weight w	Number of codewords
$p^{2n} - p^{2n-1} - p^{(2n+s-1)/2}$	$\frac{1}{2}(p^n-1)(p-1)(p^{2n-s-1} + p^{\frac{2n-s-1}{2}})$
$p^{2n} - p^{2n-1}$	$p^{3n} - (p^n-1)(p-1)p^{2n-s-1} - 1$
$p^{2n} - p^{2n-1} + p^{(2n+s-1)/2}$	$\frac{1}{2}(p^n-1)(p-1)(p^{2n-s-1} - p^{\frac{2n-s-1}{2}})$

Example 10. *Let $F_1, F_2 : \mathbb{F}_{3^3} \to \mathbb{F}_{3^3}$ be given by $F_1(x) = x^2$ and $F_2(y) = y^5$. The components of the function F_1 have Walsh spectra $\{* \ \epsilon \mathrm{i} 3^{\frac{3}{2}}, \epsilon \mathrm{i} 3^{\frac{3}{2}} \xi_3, \epsilon \mathrm{i} 3^{\frac{3}{2}} \xi_3^2 \ *\}$ with multiplicities $(9, 12, 6)$, where the sign $\epsilon \in \{-1, 1\}$ depends on the component. This yields that the weight of the dual of each component has weight $18 = 3^3 - 3^{3-1}$. The (nonzero) components of the function F given in Construction A are non-weakly regular plateaued and have amplitude $3^{\frac{6+1}{2}}$. Moreover, they belong to $\mathscr{C}_1$ and satisfy the conditions in Theorem 1. Consider the code $\mathcal{C}_F$. This is a 3-weight code with parameters $[728, 9, 477]$. From Table 2, we see that the number of balanced codewords (without counting the ones coming from linear functions) is*

$$(3^3 - 1)(3^6 - 3^5 + 3^4) = 14742,$$

accounting a total of $3^6 - 1 + 14742 = 15470$ balanced codewords. There are $(3^3 - 1)(3^4 + 3^2) = 2340$ codewords of weight $3^6 - 3^5 - 3^3 = 459$, whereas there are $(3^3 - 1)(3^4 - 3^2) = 1872$ of weight $3^6 - 3^5 + 3^3 = 513$. Hence, the weight enumerator polynomial of $\mathcal{C}_F$ is

$$1 + 2340z^{459} + 15470z^{486} + 1872z^{513}.$$

The minimum distance of a linear code with parameters $[728, 9]$ is at most 483. Hence, the minimum distance of the code $\mathcal{C}_F$ is rather close to it. Besides having few weights and a large minimum distance, $\mathcal{C}_F$ enjoys some other desirable properties that will be described in Section 5.

Remark 6. *In general, the weight distributions of codes from vectorial plateaued functions are easy to derive when we know the distribution of amplitudes of components and the classes they belong to*

(under the assumption that we can compute the dual weights and the values of Z_0). The challenge then arises when there is a mix of types and possible amplitudes, so that there is no clue on how many of each type there are or the amplitude distribution. In contrast, there are several instances when one can indeed compute the weight distributions of the codes, e.g., for known planar functions, subclasses of quadratic forms, etc.

5 Properties of the obtained codes

In Table 11, we summarize the parameters and number of weights of the families of linear codes obtained in this work. Note the reparametrization in the last row, where we use the length $p^n - 1$ instead of $p^{2n} - 1$. In the following, we refer to these families by their row position at Table 11. Namely, we use $\#i$ to refer to the family of codes at the i-th row, for $1 \leq i \leq 8$.

Table 11: The $[p^n - 1, k, d]$-codes $\mathcal{C}_f$ and $\mathcal{C}_F$ studied in this work.

Tag	Reference	Dimension k	Min. dist. d	Max. weight	# Weights
#1	Theorem 1	$n+1$	$p^n - p^{n-1} - p^{\frac{n+s-1}{2}}$	$p^n - p^{n-1} + p^{\frac{n+s-1}{2}}$	3
#2	Theorem 2	$n+1$	$p^n - p^{n-1} - p^{\frac{n+s-1}{2}}$	$p^n - p^{n-1}$	2
#3	[47], Table 6	$n+1$	$(p-1)(p^{n-1} - p^{\frac{n+s}{2}-1})$	$(p-1)(p^{n-1} + p^{\frac{n+s}{2}-1})$	3
#4	Theorem 3	$n+1$	$(p-1)(p^{n-1} - p^{\frac{n+s}{2}-1})$	$(p-1)(p^{n-1} + p^{\frac{n+s}{2}-1})$	5
#5	Theorem 4	$n+1$	$(p-1)(p^{n-1} - p^{\frac{n+s}{2}-1})$	$(p-1)(p^{n-1} + p^{\frac{n+s}{2}-1})$	5
#6	Theorem 5	$n+1$	$(p-1)(p^{n-1} - p^{\frac{n+s}{2}-1})$	$(p-1)(p^{n-1} + p^{\frac{n+s}{2}-1})$	5
#7	Theorem 6	$2n$	$(p-1)(p^{n-1} - p^{\frac{n+s}{2}-1})$	$(p-1)(p^{n-1} + p^{\frac{n+s}{2}-1})$	3
#8	Construction A	$n + \frac{n}{2}, n$ even	$p^n - p^{n-1} - p^{\frac{n+s-1}{2}}$	$p^n - p^{n-1} + p^{\frac{n+s-1}{2}}$	3

A linear code $\mathcal{C}$ is said to be minimal if the supports of any two linearly independent codewords are not included in each other. Recall that Ashikhmin-Barg's condition [1] states that if the ratio between the minimum weight and the maximum weight of a p-ary code is strictly larger than $\frac{p-1}{p}$ then the code is minimal. An equivalent reformulation of this condition is given in the following lemma, for the case of "symmetric" weights.

Lemma 5. *Let t, a, x be any positive integers. Then, $\frac{a-x}{a+x} > \frac{t-1}{t}$ if and only if $\frac{a}{x} > 2t - 1$.*

Proof. The result is straightforward from the computation

$$\frac{a-x}{a+x} > \frac{t-1}{t} \iff ta - tx > (t-1)a + (t-1)x \iff a > (2t-1)x.$$

Since $x > 0$, then this is equivalent to $\frac{a}{x} > 2t - 1$. □

Theorem 7. *Let p be an odd prime. Let n be a positive integer and s be an integer such that $0 \leq s < n$. With reference to Table 11, a code in the family #1 or in the family #8 is minimal when $n - 2 > s$ or $n - 2 = s$ and $p \in \{3, 5\}$. For the family #2, the codes are minimal when $n - 1 > s$. For the families #3–#7, the codes are minimal whenever $n - 2 > s$.*

Proof. Note that the minimum and maximum weights of the families of codes #1, and #3−#8 are symmetric. It then suffices to apply Lemma 5 to show that the Ashikhmin-Barg condition holds for these families, showing minimality *a fortiori*. The number a in Lemma 5 is always equal to $(p-1)p^{n-1}$ for these families, and $t = p$.

For the families #1 and #8, we take $x = p^{\frac{n+s-1}{2}}$. Hence, $\frac{a}{x} = (p-1)p^{\frac{n-s-1}{2}}$. By a simple algebraic manipulation,

$$(p-1)p^{\frac{n-s-1}{2}} > 2p-1 \iff \frac{1}{p}((p-1)p^{\frac{n-s-1}{2}}+1) > 2,$$

where the last inequality is taken in $\mathbb{R}$. We claim that

$$\frac{1}{p}((p-1)p^{\frac{n-s-1}{2}}+1) > 2 \iff n-2 > s \text{ or } n-2 = s \text{ and } p \notin \{3,5\}. \tag{33}$$

To show the sufficiency of the claim in (33), note that if $n-1 = s$, then this becomes $\frac{1}{p}((p-1)p^{\frac{n-s-1}{2}}+1) = \frac{1}{p}((p-1)+1) = 1$. If $n-2 = s$, then $\frac{1}{p}((p-1)p^{\frac{n-s-1}{2}}+1) = \frac{1}{p}((p-1)\sqrt{p}+1)$. Replacing $p = 3$ and $p = 5$ yields

$$\frac{1}{3}(2\sqrt{3}+1) \approx 1.488 \text{ and } \frac{1}{5}(4\sqrt{5}+1) \approx 1.988,$$

respectively, so the LHS of (33) does not hold. To show the necessity of (33), note that if $s < n-2$ (equivalently, $3 \leq n-s$), then

$$\frac{1}{p}((p-1)p^{\frac{n-s-1}{2}}+1) \geq \frac{1}{p}((p-1)p+1),$$

since $p^{\frac{n-s-1}{2}} \geq p$. This last term equals $p-1+\frac{1}{p}$ which is greater than two as $p \geq 3$. Now, suppose that $s = n-2$ and $p \notin \{3,5\}$. The real function $\phi : \mathbb{R}^{\geq 0} \to \mathbb{R}$ given by

$$x \mapsto \frac{1}{x}((x-1)\sqrt{x}+1)$$

has derivative $\frac{1}{2\sqrt{x}} + \frac{1}{2x^{\frac{3}{2}}} - \frac{1}{x^2}$, which is positive in the interval $(1,\infty)$, thus ϕ is strictly increasing on this interval. Therefore,

$$\frac{1}{p}((p-1)p^{\frac{n-s-1}{2}}+1) = \frac{1}{p}((p-1)\sqrt{p}+1) \geq \frac{1}{7}(6\sqrt{7}+1) \approx 2.41 > 2.$$

For the families #3−#7, we take $x = (p-1)p^{\frac{n+s}{2}-1}$. Recall that $a = (p-1)p^{n-1}$. Hence, $\frac{a}{x} = p^{\frac{n-s}{2}}$. By a simple algebraic manipulation,

$$p^{\frac{n-s}{2}} > 2p-1 \iff p^{\frac{n-s}{2}-1} + \frac{1}{p} > 2.$$

We claim that

$$p^{\frac{n-s}{2}-1} + \frac{1}{p} > 2 \iff n-2 > s. \tag{34}$$

If $s \geq n-2$, then $p^{\frac{n-s}{2}-1} + \frac{1}{p} \leq 1 + \frac{1}{p} < 2$. Conversely, if $s < n-2$ (i.e. $3 \leq n-s$), then $p^{\frac{n-s}{2}-1} + \frac{1}{p} > \sqrt{p} + \frac{1}{p}$. It is enough to show that this number is always greater than two. Indeed, the real function $\phi : \mathbb{R}^{\geq 0} \to \mathbb{R}$ given by $x \mapsto \sqrt{x} + \frac{1}{x}$ has derivative $\frac{1}{2}x^{-\frac{1}{2}} - x^{-2}$ which is positive in the interval $(2^{\frac{2}{3}}, \infty) \approx (1.587, \infty)$, thus ϕ is strictly increasing on this interval. Therefore, $\sqrt{p} + \frac{1}{p} \geq \sqrt{3} + \frac{1}{3} \approx 2.065 > 2$, which shows (34).

Finally, we consider the codes given from the family #2. Let again $a = (p-1)p^{n-1}$ and let $x = p^{\frac{n+s-1}{2}}$. For codes in the family #2, the minimum weight is $a - x$, whereas the maximum weight is a. Then,

$$\frac{a-x}{a} > \frac{p-1}{p} \iff a > px \iff (p-1)p^{\frac{n-s-3}{2}} > 1.$$

It is easy to see that this holds whenever $n-s > 2$. Moreover, if $n-s = 2$, then $(p-1)p^{\frac{n-s-3}{2}} = \frac{p-1}{\sqrt{p}}$. The condition then becomes $1 < \frac{p-1}{\sqrt{p}}$, which happens if and only if $1 < \frac{(p-1)^2}{p} = p - 1 + \frac{1}{p}$. The last statement is true since $p \geq 3$. Note that the inequality does not hold when $n - s = 1$ since $(p-1)p^{\frac{n-s-3}{2}} = 1 - \frac{1}{p} < 1$ in this case. □

Remark 7. *By Proposition 7, a linear code with parameters $[p^n - 1, n+1]$ has minimum distance at most $p^n - p^{n-1} - 1$. Therefore, codes in families #1 and #2 can be optimal when $n + s = 1$, that is, $n = 1$ and $s = 0$. On the other hand, codes in families #3-#6 can be almost optimal only if $n + s = 2$ and $p = 3$, that is, $n = 2$ and $s = 0$. By Theorem 7, these codes are not minimal. In fact, codes with parameters $[p^n - 1, n+1, p^n - p^{n-1} - 1]$ exist: they are the well-known punctured first order Reed-Muller codes $\mathcal{RM}_{\mathbb{F}_p^n}(1,n)^\star$, formed by evaluating all multivariate polynomials of algebraic degree at most 1 in n variables over $\mathbb{F}_p^n$, and puncturing the coordinate corresponding to 0. Hence, $\mathcal{RM}_{\mathbb{F}_p^n}(1,n)^\star$ cannot be minimal as it contains all constant vectors. Note, however, that the minimum distance of the obtained codes in all families #1-#8 is, in general, relatively close to the optimal value (see, for instance, Example 10).*

Furthermore, codes stemming from plateaued functions are always self-orthogonal. In general, self-orthogonality in the case of odd characteristic p can be characterized by the following well-known property: a code C is self-orthogonal if and only if $\mathbf{c} \cdot \mathbf{c} = 0$, for all $\mathbf{c} \in \mathcal{C}$. Indeed, if $\mathbf{c} \cdot \mathbf{c} = 0$, for all $\mathbf{c} \in \mathcal{C}$, then for any $\mathbf{c}, \mathbf{c}' \in \mathcal{C}$, $\mathbf{c} + \mathbf{c}' \in \mathcal{C}$. Thus $(\mathbf{c} + \mathbf{c}') \cdot (\mathbf{c} + \mathbf{c}') = \mathbf{c} \cdot \mathbf{c} = \mathbf{c}' \cdot \mathbf{c}' = 0$, which implies $2\mathbf{c} \cdot \mathbf{c}' = 0$. Since p is odd, it holds $\mathbf{c} \cdot \mathbf{c}' = 0$ for all $\mathbf{c}, \mathbf{c}' \in \mathcal{C}$. For characteristic two, a code is self-orthogonal if and only if all weights are divisible by 4.

Before proving the main result of this section, we need the following auxiliary lemma.

Lemma 6. *Let n, s be integers such that $n > 2$ or $n = 2, s \neq 0$ and let $g : \mathbb{F}_{p^n} \to \mathbb{F}_p$ be any s-plateaued function. The sum $\sum_{x \in \mathbb{F}_{p^n}} g(x)^2$, considered over $\mathbb{Z}$, is divisible by p.*

Proof. Let $g^{-1}(j)$ denote the preimage set of g, i.e., $g^{-1}(j) = \{x \in \mathbb{F}_{p^n} : g(x) = j\}$. The value of $\sum_{x \in \mathbb{F}_{p^n}} g(x)^2$ is then determined through the sums $\sum_{j=1}^{\frac{p-1}{2}} (\#g^{-1}(j) + \#g^{-1}(-j))j^2$ since

$$\sum_{x \in \mathbb{F}_{p^n}} g(x)^2 = \sum_{i \in QR^\star} \sum_{j \in \mathbb{F}_p^\star, j^2 = i} \#g^{-1}(j) i = \sum_{j=1}^{\frac{p-1}{2}} (\#g^{-1}(j) + \#g^{-1}(-j))j^2.$$

If g is balanced, then $\#g^{-1}(j)$ is clearly congruent to 0 modulo p, for each j. If g is unbalanced, then Lemma 9 in [40] implies that $\#g^{-1}(j) \equiv 0 \pmod{p}$, for each j, whenever $n > 2$ or $n = 2, s \neq 0$, hence $\sum_{x\in\mathbb{F}_{p^n}} g(x)^2$ is divisible by p. □

Theorem 8. *Let n, s be integers such that $n > 2$ or $n = 2, s \neq 0$. Let $f : \mathbb{F}_{p^n} \to \mathbb{F}_p$ be any s-plateaued function such that $f(0) = 0$. The code $\mathcal{C}_f$ defined by (5) is included in its dual $\mathcal{C}_f^{\perp}$, i.e., $\mathcal{C}_f$ is self-orthogonal.*

Proof. Let $(a, v) \in \mathbb{F}_p^\star \times \mathbb{F}_{p^n}$ be arbitrary and set $L_v(x) = \mathrm{Tr}_1^n(vx)$. Since L_v is linear and $a \in \mathbb{F}_p^\star$, $af + L_v$ is s-plateaued, Lemma 6 implies that the sum (over $\mathbb{Z}$)

$$\sum_{x\in\mathbb{F}_{p^n}} (af(x) + \mathrm{Tr}_1^n(vx))^2$$

is divisible by p. By looking at the form of codewords in $\mathcal{C}_f$, this implies that $\mathbf{c} \cdot \mathbf{c} = 0$ for each $\mathbf{c} \in \mathcal{C}_f$, i.e., $\mathcal{C}_f$ is self-orthogonal. □

Remark 8. *A consequence of the previous approach is that codes induced by vectorial plateaued functions $F : \mathbb{F}_{p^n} \to \mathbb{F}_{p^m}$ are also self-orthogonal. The code $\mathcal{C}_F$ can be regarded as the union $\bigcup_{a\in\mathbb{F}_{p^m}^\star} \mathcal{C}_{F_a}$. Thus, any codeword $\mathbf{c}$ (not induced by a linear function) belongs to exactly one code corresponding to a component of F, so that $\mathbf{c} \cdot \mathbf{c} = 0$ by Theorem 8.*

A code that is simultaneously minimal and self-orthogonal is *the best* we can expect, namely, there are no minimal self-dual codes besides two exceptions, as shown in the following.

Let q be in $\{2, 3\}$. Self-dual q-ary codes are classified in three types [8]: a self-dual code with some codeword of weight not divisible by 4 is called singly-even or Type I; a self-dual code whose all codewords are divisible by 4 is called doubly-even or Type II; and self-dual codes over $\mathbb{F}_3$ are Type III.

Theorem 9. *There are no self-dual minimal linear codes for $q > 3$. The only self-dual minimal ternary code is the tetracode $[4, 2, 3]_3$, whereas the only self-dual minimal binary code is the repetition code $[2, 1, 2]_2$.*

Proof. Let C be a linear code with parameters $[n, k, d]_q$ with n even. If C is minimal then $k+q-2 \leq d_{min} \leq d_{max} \leq n - k + 1$, where d_{min} and d_{max} denote the minimum and the maximum distance in C, respectively. For a reference of these results, see [11]. Thus if C is self-dual and minimal, it follows

$$\frac{n}{2} + q - 2 \leq d_{min} \leq d_{max} \leq \frac{n}{2} + 1. \tag{35}$$

Hence, for $q > 3$, the result follows. Let $q = 3$, i.e., C is a Type III code. In this case, the only possibility is that $d_{min} = d_{max} = \frac{n}{2} + 1$, so that C is also a one-weight code with parameters $[n, n/2, n/2 + 1]$. It is known that n is divisible by 4. Since C is self-dual, it is self-orthogonal, so $n/2 + 1 \equiv 0 \pmod 3$, which implies $n \equiv 1 \pmod 3$. Then $n = 4(3r + 1)$ for some $r \geq 0$. For a Type III code [8], $d_{min} \leq 3\lfloor \frac{n}{12} \rfloor + 3$. Hence, $d_{min} \leq 3\lfloor r + \frac{1}{3} \rfloor + 3 = 3r + 3$. On the other hand, $d_{min} = 6r + 3$, which yields $r = 0$. Thus, C must be the tetracode $[4, 2, 3]_3$.

Let $q = 2$. By (35), we get:

$$d_{min} = d_{max} = \frac{n}{2}, d_{min} = d_{max} = \frac{n}{2} + 1 \text{ or } d_{min} = \frac{n}{2} \text{ and } d_{max} = \frac{n}{2} + 1. \tag{36}$$

Since a self-dual binary code is even (i.e., all its weights are divisible by two), it must be that $d_{min} = d_{max}$. First suppose that C is of Type II, i.e. all codewords are divisible by four. In this case, $n \equiv 0 \pmod 8$, say, $n = 8r$ for some $r \geq 1$. It is well-known [8] that for self-dual binary codes it holds $d_{min} \leq 2\lfloor \frac{n}{8} \rfloor + 2$. This yields $d_{min} \leq 2r + 2$. Since $d_{min} = 4r$, the only possibility is that C has parameters $[8, 4, 4]_2$, that is, the extended Hamming code, which contains the all one vector $\mathbf{1}$.

Now suppose that C is of Type I (there are some codewords which are not divisible by four). For Type I codes, it holds that $d_{min} \leq 2\lfloor \frac{n+6}{10} \rfloor$ for $n \notin E := \{2, 12, 22, 32\}$ (see [8]). Assume that $n \notin E$. Suppose that $n \equiv 0 \pmod 4$, say, $n = 4r$ for some $r \geq 1$. It follows that $d_{min} = \frac{n}{2}$. This yields $2r = d_{min} \leq 2\lfloor \frac{2r+3}{5} \rfloor$. So $r \leq \lfloor \frac{2r+3}{5} \rfloor$, which is true only for $r = 1$, in other words, the code C has parameters $[4, 2, 2]_2$, which can be seen to contain $\mathbf{1}$. Suppose that $n \equiv 2 \pmod 4$, say, $n = 4r + 2$ for some $r \geq 1$. It follows that $d_{min} = \frac{n}{2} + 1$. This yields $2r + 2 = d_{min} \leq 2\lfloor \frac{2r+3}{5} \rfloor$. So $r + 1 \leq \lfloor \frac{2r+3}{5} \rfloor$, which cannot happen. Using again the bound $d_{min} \leq 2\lfloor \frac{n}{8} \rfloor + 2$, we can rule out all the values of $n \in E$ except for $n = 2$. This finishes the proof. □

6 Summary of known constructions

For an s-plateaued function f, we display, in Table 12, the families of p-ary linear codes $\mathcal{C}_f$ described in the literature and in this work for which full weight distributions have been determined.

Table 12: The $[p^n - 1, n + 1, d]$-codes $\mathcal{C}_f$ that stem from plateaued functions, whose weight distributions have been determined. Here 'wr', 'nwr', 'b' and 'p' stand for weakly regular, non-weakly regular, bent, and plateaued, respectively.

Minimum distance d	# Weights	Conditions	Reference
$p^n - p^{n-1} - p^{\frac{n-1}{2}}$	3	n odd, f wrb	[36]
$p^n - p^{n-1} - p^{\frac{n+s-1}{2}}$	3	$n+s$ odd, f wrp	[38]
$p^n - p^{n-1} - p^{\frac{n+s-1}{2}}$	3	$n+s$ odd, f nwrp with bent dual	[47, 57, 58]
$p^n - p^{n-1} - p^{\frac{n+s-1}{2}}$	3	$n+s$ odd, $f \notin \mathcal{P}_2$	Theorem 1
$p^n - p^{n-1} - p^{\frac{n+s-1}{2}}$	2	$n+s$ odd, $f \in \mathcal{P}_2$	Theorem 2
$(p-1)(p^{n-1} - p^{\frac{n}{2}-1})$	3	n even, f wrb	[36]
$(p-1)(p^{n-1} - p^{\frac{n+s}{2}-1})$	3	$n+s$ even, f wrp	[38]
$(p-1)(p^{n-1} - p^{\frac{n+s}{2}-1})$	3	$n+s$ even, f nwrp with zero dual	[47]
$(p-1)(p^{n-1} - p^{\frac{n+s}{2}-1})$	5	$n+s$ even, f nwrp with bent dual	[47, 57, 58]
$(p-1)(p^{n-1} - p^{\frac{n+s}{2}-1})$	5	$n+s$ even, $f \in \mathscr{C}_1$ nwrp	Theorem 3
$(p-1)(p^{n-1} - p^{\frac{n+s}{2}-1})$	5	$n+s$ even, $f \in \mathscr{C}_2$, $W_{f^*}(0) = t(f^*)\nu' p^{\frac{\theta}{2}}$, θ even	Theorem 4
$(p-1)(p^{n-1} - p^{\frac{n+s}{2}-1})$	5	$n+s$ even, $f \in \mathscr{C}_2$, $W_{f^*}(0) = t(f^*)\nu' p^{\frac{\theta}{2}}$, θ odd	Theorem 5

7 Conclusions

In this work, we have derived the weight distributions of linear codes associated with a subclass of p-ary plateaued functions, focusing on both single-output and vectorial cases. By partitioning p-ary plateaued functions into the classes $\mathscr{C}_1, \mathscr{C}_2$, and $\mathscr{C}_3$, based on the properties of their dual function f^*, we have extended and refined the existing results from the literature. Specifically, for s-plateaued functions with $n+s$ odd, we obtained complete weight distributions parametrized by the dual weight $wt(f^*)$. For the case when $n+s$ is even, we derived weight distributions using both $wt(f^*)$ and an additional parameter Z_0 for functions in the class $\mathscr{C}_1$. Our study extends previous research by presenting new results on weight distributions from functions belonging to subclasses of the plateaued function classes. Furthermore, by transitioning to the vectorial case, we have provided the weight distributions for codes associated with known vectorial plateaued functions. This approach has led to the construction of codes with parameters $[p^n-1, 2n, p^n-p^{n-1}-p^{(n+s-2)/2}(p-1)]$, thus expanding the repertoire of linear codes with the properties of minimality and self-orthogonality. Two open problems are left for the community. First, for all p and n, provide infinite families of plateaued functions in $\mathscr{C}_2$, whose dual is nonzero, and, provide infinite families of plateaued functions in $\mathscr{C}_3$, whose dual is surjective. Second, either to construct examples of plateaued functions for which $n+s$ and θ have different parities or to show an impossibility result. Future work could explore other subclasses of plateaued functions and investigate a full characterization of linear codes from (vectorial) plateaued functions. In another direction, it would be interesting to give constructions of families of p-ary linear codes from plateaued functions, not necessarily based on the first generic construction, that are minimal and optimal.

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8 Statements and Declarations

The authors have no competing interests.

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