Noname manuscript No.
(will be inserted by the editor)

A generalization of the class of hyper-bent Boolean functions in binomial forms

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Received: date / Accepted: date

Abstract Bent functions, which are maximally nonlinear Boolean functions with even numbers of variables and whose Hamming distance to the set of all affine functions equals $2^{n-1} \pm 2^{\frac{n}{2}-1}$, were introduced by Rothaus in 1976 when he considered problems in combinatorics. Bent functions have been extensively studied due to their applications in cryptography, such as S-box, block cipher and stream cipher. Further, they have been applied to coding theory, spread spectrum and combinatorial design. Hyper-bent functions, as a special class of bent functions, were introduced by Youssef and Gong in 2001, which have stronger properties and rarer elements. Many research focus on the construction of bent and hyper-bent functions. In this paper, we consider functions defined over $\mathbb{F}_{2^n}$ by $f_{a,b}^{(r)} := \mathrm{Tr}_1^n(ax^{r(2^m-1)}) + \mathrm{Tr}_1^4(bx^{\frac{2^n-1}{5}})$, where $n = 2m$, $m \equiv 2 \pmod 4$, $a \in \mathbb{F}_{2^m}$ and $b \in \mathbb{F}_{16}$. When $r \equiv 0 \pmod 5$, we characterize the hyper-bentness of $f_{a,b}^{(r)}$. When $r \not\equiv 0 \pmod 5$, $a \in \mathbb{F}_{2^m}$ and $(b+1)(b^4+b+1) = 0$, with the help of Kloosterman sums and the factorization of $x^5 + x + a^{-1}$, we present a characterization of hyper-bentness of $f_{a,b}^{(r)}$. Further, we give all the hyper-bent functions of $f_{a,b}^{(r)}$ in the case $a \in \mathbb{F}_{2^{\frac{m}{2}}}$.

Chunming Tang and Yanfeng Qi acknowledge support from the Natural Science Foundation of China (Grant No.10990011 & No.60763009). Baocheng Wang and Yixian Yang acknowledge support from National Science Foundation of China Innovative Grant (70921061), the CAS/SAFEA International Partnership Program for Creative Research Teams.

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Keywords Boolean functions · bent functions · hyper-bent functions · Walsh-Hadamard transformation · Kloosterman sums

1 Introduction

Bent functions are maximally nonlinear Boolean functions with even numbers of variables whose Hamming distance to the set of all affine functions equals $2^{n-1} \pm 2^{\frac{n}{2}-1}$. These functions introduced by Rothaus [30] as interesting combinatorial objects have been extensively studied for their applications not only in cryptography, but also in coding theory [4,27] and combinatorial design. Some basic knowledge and recent results on bent functions can be found in [3,12,27]. A bent function can be considered as a Boolean function defined over $\mathbb{F}_2^n$, $\mathbb{F}_{2^m} \times \mathbb{F}_{2^m}$ $(n = 2m)$ or $\mathbb{F}_{2^n}$. Thanks to the different structures of the vector space $\mathbb{F}_2^n$ and the Galois field $\mathbb{F}_{2^n}$, bent functions can be well studied. Although some algebraic properties of bent functions are well known, the general structure of bent functions on $\mathbb{F}_{2^n}$ is not clear yet. As a result, much research on bent functions on $\mathbb{F}_{2^n}$ can be found in [2,7,8,10,11,13,14, 21,22,25–29,32]. Youssef and Gong [31] introduced a class of bent functions called hyper-bent functions, which achieve the maximal minimum distance to all the coordinate functions of all bijective monomials (i.e., functions of the form $\mathrm{Tr}_1^n(ax^i) + \epsilon$, $\gcd(i, 2^n - 1) = 1$). However, the definition of hyper-bent functions was given by Gong and Golomb [15] by a property of the extend Hadamard transform of Boolean functions. Hyper-bent functions as special bent functions with strong properties are hard to characterize and many related problems are open. Much research give the precise characterization of hyper-bent functions in certain forms.

The complete classification of bent and hyper-bent functions is not yet achieved. The monomial bent functions in the form $\mathrm{Tr}_1^n(ax^s)$ are considered in [2,21]. Leander [21] described the necessary conditions for s such that $\mathrm{Tr}_1^n(ax^s)$ is a bent function. In particular, when $s = r(2^m - 1)$ and $(r, 2^m + 1) = 1$, the monomial functions $\mathrm{Tr}_1^n(ax^s)$ (i.e., the Dillon functions) were extensively studied in [7,10,21]. A class of quadratic functions over $\mathbb{F}_{2^n}$ in polynomial form $\sum_{i=1}^{\frac{n}{2}-1} a_i \mathrm{Tr}_1^n(x^{1+2^i}) + a_{\frac{n}{2}} \mathrm{Tr}_1^{\frac{n}{2}}(x^{\frac{n}{2}+1})$ $(a_i \in \mathbb{F}_2)$ was described and studied in [9,17–19,23,32]. Dobbertin et al. [13] constructed a class of binomial bent functions of the form $\mathrm{Tr}_1^n(a_1 x^{s_1} + a_2 x^{s_2}), (a_1, a_2) \in (\mathbb{F}_{2^n}^*)^2$ with Niho power functions. Garlet and Mesanager [6] studied the duals of the Niho bent functions in [13]. In [25,26,29], Mesnager considered the binomial functions of the form $\mathrm{Tr}_1^n(ax^{r(2^m-1)}) + \mathrm{Tr}_1^2(bx^{\frac{2^n-1}{3}})$, where $a \in \mathbb{F}_{2^n}^*$ and $b \in \mathbb{F}_4^*$. Then he gave the link between the bentness property of such functions and Kloosterman sums. Leander and Kholosha [22] generalized one of the constructions provided by Dobbertin et al. [13] and presented a new primary construction of bent functions consisting of a linear combination of 2^r Niho exponents. Carlet et al. [5] computed the dual of the Niho bent function with 2^r exponents found by Leander and Kholosha [22] and showed that this new bent function is

not of the Niho type. Charpin and Gong [7] presented a characterization of bentness of Boolean functions over $\mathbb{F}_{2^n}$ of the form $\sum_{r\in R} \mathrm{Tr}_1^n(a_r x^{r(2^m-1)})$, where R is a subset of the set of representatives of the cyclotomic cosets modulo 2^m+1 of maximal size n. These functions include the well-known monomial functions with the Dillon exponent as a special case. Then they described the bentness of these functions with the Dickson polynomials. Mesnager et al. [27, 28] generalized the results of Charpin and Gong [7] and considered the bentness of Boolean functions over $\mathbb{F}_{2^n}$ of the form $\sum_{r\in R} \mathrm{Tr}_1^n(a_r x^{r(2^m-1)}) + \mathrm{Tr}_1^2(bx^{\frac{2^n-1}{3}})$, where $n=2m$, $a_r \in \mathbb{F}_{2^m}$ and $b \in \mathbb{F}_4$. Further, they presented the link between the bentness of such functions and some exponential sums (involving Dickson polynomials).

In this paper, we consider a class of Boolean functions defined over $\mathbb{F}_{2^n}$ by the form: $f_{a,b}^{(r)} := \mathrm{Tr}_1^n(ax^{r(2^m-1)}) + \mathrm{Tr}_1^4(bx^{\frac{2^n-1}{5}})$, where $n=2m$, $m \equiv 2 \pmod 4$, $a \in \mathbb{F}_{2^m}$ and $b \in \mathbb{F}_{16}$. When $r=1$, this class of Boolean functions is studied in [1]. Generally, it is elusive to give a characterization of bentness and hyper-bentness of Boolean functions. When $r \equiv 0 \pmod 5$, the hyper-bentness of $f_{a,b}^{(r)}$ is characterized in this paper. When $r \not\equiv 0 \pmod 5$ and $(b+1)(b^4+b+1=0)=0$, this paper presents the hyper-bentness of $f_{a,b}^{(r)}$ by the factorization of x^5+x+a^{-1} and Kloosterman sums. For $a \in \mathbb{F}_{2^{\frac{m}{2}}}$, we give all the hyper-bent functions $f_{a,b}^{(r)}$.

The rest of paper is organized as follows. In Section 2, we give some notations and recall some basic knowledge for this paper. In Section 3, we study the hyper-bentness of the Boolean functions $f_{a,b}^{(r)}$ for two cases (1) $(b+1)(b^4+b+1)=0$; (2) $a \in \mathbb{F}_{2^{\frac{m}{2}}}$. Finally, Section 4 makes a conclusion.

2 Preliminaries

2.1 Boolean functions

Let n be a positive integer. $\mathbb{F}_2^n$ is a n-dimensional vector space defined over finite field $\mathbb{F}_2$. Take two vectors $x=(x_1,\cdots,x_n)$ and $y=(y_1,\cdots,x_n)$ in $\mathbb{F}_2^n$. Their dot product is defined by

$$\langle x,y\rangle := \sum_{i=1}^{n} x_i y_i.$$

$\mathbb{F}_{2^n}$ is a finite field with 2^n elements and $\mathbb{F}_{2^n}^*$ is the multiplicative group of $\mathbb{F}_{2^n}$. Let $\mathbb{F}_{2^k}$ be a subfield of $\mathbb{F}_{2^n}$. The trace function from $\mathbb{F}_{2^n}$ to $\mathbb{F}_{2^k}$, denoted by Tr_k^n, is a map defined as

$$\mathrm{Tr}_k^n(x) := x + x^{2^k} + x^{2^{2k}} + \cdots + x^{2^{n-k}}.$$

When $k=1$, Tr_1^n is called the absolute trace. The trace function Tr_k^n satisfies the following properties.

$$\mathrm{Tr}_k^n(ax+by)=a\mathrm{Tr}_k^n(x)+b\mathrm{Tr}_k^n(y), \quad a,b\in\mathbb{F}_{2^k}, x,y\in\mathbb{F}_{2^n}.$$
$$\mathrm{Tr}_k^n(x^{2^k})=\mathrm{Tr}_k^n(x), \quad x\in\mathbb{F}_{2^n}.$$

When $\mathbb{F}_{2^k}\subseteq\mathbb{F}_{2^r}\subseteq\mathbb{F}_{2^n}$, the trace function Tr_k^n satisfies the following transitivity property.

$$\mathrm{Tr}_k^n(x)=\mathrm{Tr}_k^r(\mathrm{Tr}_r^n(x)), \quad x\in\mathbb{F}_{2^n}.$$

A Boolean function over $\mathbb{F}_2^n$ or $\mathbb{F}_{2^n}$ is an $\mathbb{F}_2$-valued function. The absolute trace function is a useful tool in constructing Boolean functions over $\mathbb{F}_{2^n}$. From the absolute trace function, a dot product over $\mathbb{F}_{2^n}$ is defined by

$$\langle x,y\rangle:=\mathrm{Tr}_1^n(xy), \quad x,y\in\mathbb{F}_{2^n}.$$

A Boolean function over $\mathbb{F}_{2^n}$ is often represented by the algebraic normal form (ANF):

$$f(x_1,\cdots,x_n)=\sum_{I\subseteq\{1,\cdots,n\}} a_I(\prod_{i\in I}x_i), \quad a_I\in\mathbb{F}_2.$$

When $I=\emptyset$, let $\prod_{i\in I}=1$. The terms $\prod_{i\in I}x_i$ are called monomials. The algebraic degree of a Boolean function f is the globe degree of its ANF, that is, $\deg(f):=\max\{\#(I)|a_I\neq 0\}$, where $\#(I)$ is the order of I and $\#(\emptyset)=0$.

Another representation of a Boolean function is of the form

$$f(x)=\sum_{j=0}^{2^n-1}a_jx^j.$$

In order to make f a Boolean function, we should require $a_0,a_{2^n-1}\in\mathbb{F}_2$ and $a_{2j}=a_j^2$, where $2j$ is taken modulo 2^n-1. This makes that f can be represented by a trace expansion of the form

$$f(x)=\sum_{j\in\Gamma_n}\mathrm{Tr}_1^{o(j)}(a_jx^j)+\epsilon(1+x^{2^n-1})$$

called its polynomial form, where

- Γ_n is the set of integers obtained by choosing one element in each cyclotomic class of 2 module 2^n-1 (j is often chosen as the smallest element in its cyclotomic class, called the coset leader of the class);
- $o(j)$ is the size of the cyclotomic coset of 2 modulo 2^n-1 containing j;
- $a_j\in\mathbb{F}_{2^{o(j)}}$;
- $\epsilon=wt(f) \pmod 2$, where $\mathrm{wt}(f):=\#\{x\in\mathbb{F}_{2^n}|f(x)=1\}$.

Let $\mathrm{wt}_2(j)$ be the number of 1's in the binary expansion of j. Then

$$\deg(f)=\begin{cases}n, & \epsilon=1\\ \max\{\mathrm{wt}_2(j)|a_j\neq 0\}, & \epsilon=0.\end{cases}$$

2.2 Bent and hyper-bent functions

The "sign" function of a Boolean function f is defined by

$$\chi(f) := (-1)^f.$$

When f is a Boolean function over $\mathbb{F}_2^n$, the Walsh Hadamard transform of f is the discrete Fourier transform of $\chi(f)$, whose value at $w \in \mathbb{F}_2^n$ is defined by

$$\widehat{\chi}_f(w) := \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x)+\langle w,x \rangle}.$$

When f is a Boolean function over $\mathbb{F}_{2^n}$, the Walsh Hadamard transform of f is defined by

$$\widehat{\chi}_f(w) := \sum_{x \in \mathbb{F}_{2^n}} (-1)^{f(x)+\mathrm{Tr}_1^n(wx)},$$

where $w \in \mathbb{F}_{2^n}$. Then we can define the bent functions.

Definition 1 A Boolean function $f : \mathbb{F}_{2^n} \to \mathbb{F}_2$ is called a bent function, if $\widehat{\chi}_f(w) = \pm 2^{\frac{n}{2}}$ $(\forall w \in \mathbb{F}_{2^n})$.

If f is a bent function, n must be even. Further, $\deg(f) \leq \frac{n}{2}$ [3]. Hyper-bent functions are an important subclass of bent functions. The definition of hyper-bent functions is given below.

Definition 2 A bent function $f : \mathbb{F}_{2^n} \to \mathbb{F}_2$ is called a hyper-bent function, if, for any i satisfying $(i, 2^n - 1) = 1$, $f(x^i)$ is also a bent function.

[4] and [31] proved that if f is a hyper-bent function, then $\deg(f) = \frac{n}{2}$. For a bent function f, $\mathrm{wt}(f)$ is even. Then $\epsilon = 0$, that is,

$$f(x) = \sum_{j \in \Gamma_n} \mathrm{Tr}_1^{o(j)}(a_j x^j).$$

If a Boolean function f is defined over $\mathbb{F}_{2^{\frac{n}{2}}} \times \mathbb{F}_{2^{\frac{n}{2}}}$, then we have a class of bent functions[10,24].

Definition 3 The Maiorana-McFarland class $\mathcal{M}$ is the set of all the Boolean functions f defined on $\mathbb{F}_{2^{\frac{n}{2}}} \times \mathbb{F}_{2^{\frac{n}{2}}}$ of the form $f(x, y) = \langle x, \pi(y) \rangle + g(y)$, where $x, y \in \mathbb{F}_{2^{\frac{n}{2}}}$, π is a permutation of $\mathbb{F}_{2^{\frac{n}{2}}}$ and $g(x)$ is a Boolean function over $\mathbb{F}_{2^{\frac{n}{2}}}$.

For Boolean functions over $\mathbb{F}_{2^{\frac{n}{2}}} \times \mathbb{F}_{2^{\frac{n}{2}}}$, we have a class of hyper-bent functions $\mathcal{PS}_{ap}$ [4].

Definition 4 Let $n = 2m$, the $\mathcal{PS}_{ap}$ class is the set of all the Boolean functions of the form $f(x, y) = g(\frac{x}{y})$, where $x, y \in \mathbb{F}_{2^m}$ and g is a balanced Boolean functions (i.e., $\mathrm{wt}(f) = 2^{m-1}$) and $g(0) = 0$. When $y = 0$, let $\frac{x}{y} = xy^{2^n-2} = 0$.

Each Boolean function f in $\mathcal{PS}_{ap}$ satisfies $f(\beta z) = f(z)$ and $f(0) = 0$, where $\beta \in \mathbb{F}_m^*$ and $z \in \mathbb{F}_m \times \mathbb{F}_m$. Youssef and Gong [31] studied these functions over $\mathbb{F}_{2^n}$ and gave the following property.

Proposition 1 *Let $n = 2m$, α be a primitive element in $\mathbb{F}_{2^n}$ and f be a Boolean function over $\mathbb{F}_{2^n}$ such that $f(\alpha^{2^m+1}x) = f(x)(\forall x \in \mathbb{F}_{2^n})$ and $f(0) = 0$, then f is a hyper-bent function if and only if the weight of $(f(1), f(\alpha), f(\alpha^2), \cdots, f(\alpha^{2^m}))$ is 2^{m-1}.*

Further, [4] proved the following result.

Proposition 2 *Let f be a Boolean function defined in Proposition 1. If $f(1) = 0$, then f is in $\mathcal{PS}_{ap}$. If $f(1) = 1$, then there exists a Boolean function g in $\mathcal{PS}_{ap}$ and $\delta \in \mathbb{F}_{2^n}^*$ satisfying $f(x) = g(\delta x)$.*

Let $\mathcal{PS}_{ap}^{\#}$ be the set of hyper-bent functions in the form of $g(\delta x)$, where $g(x) \in \mathcal{PS}_{ap}$, $\delta \in \mathbb{F}_{2^n}^*$ and $g(\delta) = 1$. Charpin and Gong expressed Proposition 2 in a different version below.

Proposition 3 *Let $n = 2m$, α be a primitive element of $\mathbb{F}_{2^n}$ and f be a Boolean function over $\mathbb{F}_{2^n}$ satisfying $f(\alpha^{2^{m+1}}x) = f(x)$ $(\forall x \in \mathbb{F}_{2^n})$ and $f(0) = 0$. Let ξ be a primitive $2^m + 1$-th root in $\mathbb{F}_{2^n}^*$. Then f is a hyper-bent function if and only if the cardinality of the set $\{i | f(\xi^i) = 1, 0 \le i \le 2^m\}$ is 2^{m-1}.*

In fact, Dillon [10] introduced the Partial Spreads class $\mathcal{PS}^-$, which is a bigger class of bent functions than $\mathcal{PS}_{ap}$ and $\mathcal{PS}_{ap}^{\#}$.

Theorem 1 *Let $E_i (i = 1, 2, \cdots, N)$ be N subspaces in $\mathbb{F}_{2^n}$ of dimension m such that $E_i \cap E_j = \{0\}$ for all $i, j \in \{1, \cdots, N\}$ with $i \neq j$. Let f be a Boolean function over $\mathbb{F}_{2^n}$. If the support of f is given by $supp(f) = \bigcup_{i=1}^{N} E_i^*$, where $E_i^* = E_i \backslash \{0\}$, then f is a bent function if and only if $N = 2^{m-1}$.*

The set of all the functions in Theorem 1 is defined by $\mathcal{PS}^-$.

2.3 Kloosterman sums and Weil sums

The Kloosterman sums on $\mathbb{F}_{2^n}$ are:

$$K_m(a) := \sum_{x \in \mathbb{F}_{2^m}} \chi(\mathrm{Tr}_1^m(ax + \frac{1}{x})), \quad a \in \mathbb{F}_{2^m}.$$

Some properties of Kloosterman sums are given by the following proposition[16,20].

Proposition 4 *Let $a \in \mathbb{F}_{2^m}$. Then $K_m(a) \in [1 - 2^{(m+2)/2}, 1 + 2^{(m+2)/2}]$ and $4 \mid K_m(a)$.*

Quintic Weil sums on $\mathbb{F}_{2^m}$ are:

$$Q_m(a) := \sum_{x \in \mathbb{F}_{2^m}} \chi(\mathrm{Tr}_1^m(a(x^5 + x^3 + x))), \quad a \in \mathbb{F}_{2^m}.$$

To determine the value of $Q_m(a)$, we should consider the factorization of the polynomial $P(x) = x^5 + x + a^{-1}$. We write that $P(x) = (n_1)^{r_1}(n_2)^{r_2} \cdots (n_t)^{r_t}$

to indicate that r_i of the irreducible factors of $P(x)$ have degree n_i. When $P(x) = x^5 + x + a^{-1}$ is irreducible over $\mathbb{F}_{2^m}$, the value of $Q_m(a)$ is related to the parity of the quadratic form $\mathfrak{q}(x) = \mathrm{Tr}_1^m(x(ax^4 + ax^2 + a^2x))$. $\mathfrak{q}(x)$ is the quadratic form associated to the simplectic form:

$$< x, y >_{\mathfrak{q}} := \mathrm{Tr}_1^m(x(ay^4 + ay^2 + a^2y) + y(ax^4 + ax^2 + a^2x)),$$

which is non-degenerate. Then there exists a normal simplectic basis e_1, e_{m_1+1}, $\cdots$, e_{m_1}, e_{2m_1} $(2m_1 = m)$. If $i \not\equiv j(\mod m_1)$, $< e_i, e_j >_{\mathfrak{q}} = 0$. For any i $(1 \le i \le m_1)$, $< e_i, e_{m_1+i} >_{\mathfrak{q}} = 1$. If $\#\{i|\mathfrak{q}(e_i) = \mathfrak{q}(e_{m_1+i}) = 1, 1 \le i \le m_1\}$ is even, then the quadratic form $\mathfrak{q}(x)$ is called an even quadratic form and $Q_m(a) = 2^{m_1}$. If $\#\{i|\mathfrak{q}(e_i) = \mathfrak{q}(e_{m_1+i}) = 1, 1 \le i \le m_1\}$ is odd, then the quadratic form $\mathfrak{q}(x)$ is called a odd quadratic form and $Q_m(a) = -2^{m_1}$.

3 A generalization of the class of hyper-bent functions in binomial forms

In this section, we will discuss the hyper-bentness of $f_{a,b}^{(r)}(x)$. We introduce some notations on character sums in [1]. Let $\xi = \alpha^{2^m-1}$, then $U =< \xi >$. Let $V =< \xi^5 >$. Since $5|(2^m + 1)$, V is the subgroup of U and $\#V = \frac{2^m+1}{5}$. Let $\beta = \alpha^{\frac{2^m-1}{5}}$.

For any $i \in \mathbb{F}_{2^m}$ and an integer i, we define

$$S_i = \sum_{v \in V} \mathrm{Tr}(a\xi^{i(2^m-1)}v) = \sum_{v \in V} \mathrm{Tr}(a\xi^{3i}v).$$

From the definition of S_i, $S_i = S_j$ when $i \equiv j \pmod 5$. Further, $S_i = S_{-i}$.

3.1 The hyper-bentness of Boolean functions $f_{a,b}^{(5)}(x)$

In this subsection, we consider the hyper-bentness of $f_{a,b}^{(r)}(x)$ with $r = 5$ of the form

$$f_{a,b}^{(5)}(x) := \mathrm{Tr}_1^n(ax^{5(2^m-1)}) + \mathrm{Tr}_1^4(bx^{\frac{2^n-1}{5}}), \tag{1}$$

where $n = 2m$, $m \equiv 2 \pmod 4$, $a \in \mathbb{F}_{2^m}$ and $b \in \mathbb{F}_{16}$.

Since $m \equiv 2 \pmod 4$, $2^m + 1 \equiv 0 \pmod 5$. For any $y \in \mathbb{F}_{2^m}$, $y^{2^m-1} = 1$. Then

$$f_{a,b}^{(5)}(\alpha^{2^m+1}x) = f_{a,b}^{(5)}(x), \quad x \in \mathbb{F}_{2^n},$$

where α is a primitive element of $\mathbb{F}_{2^n}$. Further, $f_{a,b}^{(5)}(0) = 0$. Then, from Proposition 3, we have the following proposition on the hyper-bentness of $f_{a,b}^{(5)}(x)$.

Proposition 5 *Let $f_{a,b}^{(5)}$ be the Boolean function defined by (1), where $a \in \mathbb{F}_{2^m}$ and $b \in \mathbb{F}_{16}$. Define the following character sum*

$$\Lambda_5(a,b) := \sum_{u \in U} \chi(f_{a,b}^{(5)}(u)) \tag{2}$$

where U is the group of all the 2^m+1-th root of unity in $\mathbb{F}_{2^n}$, that is, $U = \{x \in \mathbb{F}_{2^n} | x^{2^m+1} = 1\}$. Then $f_{a,b}^{(5)}$ is a hyper-bent function if and only if $\Lambda_5(a,b) = 1$. Further, the hyper-bent function $f_{a,b}^{(5)}$ lies in $\mathcal{PS}_{ap}$ if and only if $\mathrm{Tr}_1^4(b) = 0$.

Proof Similar to the proof of Proposition 9 in [1], this proposition follows.

Proposition 6 *Let $n = 2m$ and $m \equiv \pm 2, \pm 6 \pmod{20}$, If $b \in \{0\} \bigcup \{\beta^i | i = 0,1,2,3,4\}$, then the Boolean function $f_{a,b}^{(5)}$ in (1) is not a hyper-bent function. Further, if $b \in \mathbb{F}_{16}^* \backslash \{\beta^i | 0 \le i \le 4\}$, $f_{a,b}^{(5)}$ is a hyper-bent function if and only if*

$$\sum_{v \in V} \chi(\mathrm{Tr}_1^n(av)) = 1.$$

Proof From (2),

$$\begin{aligned}
\Lambda_5(a,b) &= \sum_{u \in U} \chi(f_{a,b}^{(5)}(u)) \\
&= \sum_{u \in U} \chi(\mathrm{Tr}_1^n(au^{5(2^m-1)}) + \mathrm{Tr}_1^4(bu^{\frac{2^n-1}{5}})) \\
&= \sum_{u \in U} \chi(\mathrm{Tr}_1^n(au^{5(2^m-1)}))\chi(\mathrm{Tr}_1^4(bu^{\frac{2^n-1}{5}})).
\end{aligned}$$

Note that $U =< \xi >$, $V =< \xi^5 >$ and

$$U = \xi^0 V \bigcup \xi^1 V \bigcup \xi^2 V \bigcup \xi^3 V \bigcup \xi^4 V. \tag{3}$$

Then,

$$\begin{aligned}
\Lambda_5(a,b) &= \sum_{i=0}^{4} \sum_{v \in V} \chi(\mathrm{Tr}_1^4(b(\xi^i v)^{\frac{2^n-1}{5}}))\chi(\mathrm{Tr}_1^n(a(\xi^i v)^{5(2^m-1)})) \\
&= \sum_{i=0}^{4} \sum_{v \in V} \chi(\mathrm{Tr}_1^4(b(\xi^i v)^{\frac{2^n-1}{5}}))\chi(\mathrm{Tr}_1^n(a(\xi^{5i})^{2^m-1} v^{5(2^m-1)})) \qquad (4)
\end{aligned}$$

Since $(\xi^{5i})^{2^m-1} \in V$ and $m \equiv \pm 2, \pm 6 \pmod{20}$, $(5(2^m-1), \#V) = (5, \frac{2^m+1}{5}) = 1$. Then $v \longmapsto (\xi^{5i})^{2^m-1} v^{5(2^m-1)}$ is a permutation of V. Hence,

$$\begin{aligned}\Lambda_5(a,b) =& \sum_{i=0}^{4} \sum_{v\in V} \chi(\mathrm{Tr}_1^4(b(\xi^i v)^{\frac{2^n-1}{5}}))\chi(\mathrm{Tr}_1^n(av)) \\ =& \sum_{i=0}^{4} \sum_{v\in V} \chi(\mathrm{Tr}_1^4(b\xi^{i\frac{2^n-1}{5}}))\chi(\mathrm{Tr}_1^n(av)) \\ =& (\sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(b\xi^{i\frac{2^n-1}{5}})))(\sum_{v\in V} \chi(\mathrm{Tr}_1^n(av))).\end{aligned}$$

Since $\xi^{\frac{2^n-1}{5}} = (\alpha^{2^m-1})^{\frac{(2^m-1)(2^m+1)}{5}} = \beta^{2^m-1} = \beta^{2^m+1-2} = \beta^3$, then

$$\begin{aligned}\Lambda_5(a,b) =& (\sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(b\beta^{3i}))(\sum_{v\in V} \chi(\mathrm{Tr}_1^n(av))) \\ =& (\sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(b\beta^{i}))(\sum_{v\in V} \chi(\mathrm{Tr}_1^n(av))). \qquad (5)\end{aligned}$$

From (5), when $b = 0$, $\Lambda_5(a,0) = 5 \sum\limits_{v\in V} \chi(\mathrm{Tr}_1^n(av))$. Hence, $\Lambda_5(a,0) \neq 1$. From Proposition 5, $f_{a,0}^{(5)}$ is not a hyper-bent function.

When $b \neq 0$, b can be represented by $b = \omega\beta^j$, where $\omega^3 = 1$ and $0 \leq j \leq 4$. Then

$$\sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(b\beta^i)) = \sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(\omega\beta^{i+j})) = \sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(\omega\beta^i))). \qquad (6)$$

Since $\omega^3 = 1$ and $\omega^4 = \omega$,

$$\mathrm{Tr}_1^4(\omega\beta^i) = \mathrm{Tr}_1^4(\omega^4\beta^{4i}) = \mathrm{Tr}_1^4(\omega\beta^{4i}).$$

In particular, we take $i = 1, 2$. Then

$$\mathrm{Tr}_1^4(\omega\beta) = \mathrm{Tr}_1^4(\omega\beta^4), \qquad (7)$$

$$\mathrm{Tr}_1^4(\omega\beta^2) = \mathrm{Tr}_1^4(\omega\beta^3). \qquad (8)$$

If $\omega = 1$, $\sum\limits_{i=0}^{4} \chi(\mathrm{Tr}_1^4(b\beta^i) = \sum\limits_{i=0}^{4} \chi(\mathrm{Tr}_1^4(\beta^i))$. Since β satisfies $\beta^4+\beta^3+\beta^2+\beta+1 = 0$, $\mathrm{Tr}_1^4(\beta^i) = 1$. Then $\sum\limits_{i=0}^{4} \chi(\mathrm{Tr}_1^4(b\beta^i) = -3$. Therefore,

$$\Lambda_5(a,b) = -3 \sum_{v\in V} \chi(\mathrm{Tr}_1^n(av)), b = \beta^j, 0 \leq j \leq 4.$$

From Propsition 5, $f^{(5)}_{a,\beta^j}$ is not a hyper-bent function. When $\omega \neq 1$, we have

$$\mathrm{Tr}_1^4(\omega\beta)+\mathrm{Tr}_1^4(\omega\beta^2) = \mathrm{Tr}_1^4(\omega(\beta+\beta^2)) = \omega(\beta+\beta^2+\beta^3+\beta^4)+\omega^2(\beta+\beta^2+\beta^3+\beta^4) = 1.$$

Then $\chi(\mathrm{Tr}_1^4(\omega\beta)) + \chi(\mathrm{Tr}_1^4(\omega\beta^2)) = 0$. Similarly, $\chi(\mathrm{Tr}_1^4(\omega\beta^3)) + \chi(\mathrm{Tr}_1^4(\omega\beta^4)) = 0$. Therefore,

$$\Lambda_5(a,b) = \sum_{v\in V} \chi(\mathrm{Tr}_1^n(av)), b = \omega\beta^j, 0 \leq j \leq 4, \omega^3 = 1, \omega \neq 1.$$

From Proposition 5, the second part of this proposition follows.

In Proposition 6, we consider the hyper-bentness of the Boolean function $f^{(5)}_{a,b}$ for $m \equiv \pm 2, \pm 6 \pmod{20}$. The proposition below discusses the hyper-bentness of $f^{(5)}_{a,b}$ for $m \equiv 10 \pmod{20}$.

Proposition 7 *Let $n = 2m$, $m \equiv 10 \pmod{20}$, $a \in \mathbb{F}_{2^m}$, $b \in \mathbb{F}_{16}$. then the Boolean function $f^{(5)}_{a,b}$ in (1) is not a hyper-bent function.*

Proof Note that

$$\Lambda_5(a,b) = \sum_{i=0}^{4}\sum_{v\in V} \chi(\mathrm{Tr}_1^4(b\xi^{i\frac{2^n-1}{5}}))\chi(\mathrm{Tr}_1^n(a(\xi^{5i})^{2^m-1}v^{5(2^m-1)})).$$

Since $m \equiv 10 \pmod{20}$, $25|(2^m+1)$ and $(5(2^m-1), \frac{2^m+1}{5}) = 5$. Then $v \longmapsto v^{5(2^m-1)}$ is 5 to 1 from V to $V^5 := \{v^5 | v \in V\}$. Therefore,

$$\Lambda_5(a,b) = 5\sum_{i=0}^{4}\sum_{v\in V^5} \chi(\mathrm{Tr}_1^4(b\xi^{i\frac{2^n-1}{5}}))\chi(\mathrm{Tr}_1^n(a(\xi^{5i})^{2^m-1}v)).$$

Hence, $5|\Lambda_5(a,b)$ and $\Lambda_5(a,b)$ is not equal to 1, From Proposition 5, $f^{(5)}_{a,b}$ is not a hyper-bent function.

From Proposition 6,

$$\sum_{v\in V} \chi(\mathrm{Tr}_1^n(av)) = \sum_{v\in V} \chi(\mathrm{Tr}_1^n(av^{2^m-1})).$$

Note that $\sum\limits_{v\in V} \chi(\mathrm{Tr}_1^n(av)) = S_0$ in [1]. From Proposition 15 in [1],

$$\sum_{v\in V} \chi(\mathrm{Tr}_1^n(av)) = \frac{1}{5}[1 - K_m(a) + 2Q_m(a)]. \tag{9}$$

Further, from Proposition 16 and 18 in [1], we have the following results.

Proposition 8 *Let $n = 2m$, $m \equiv \pm 2, \pm 6 \pmod{20}$, $m \geq 6$ and $b \in \mathbb{F}_{16}^* \backslash \{\beta^i | 0 \leq i \leq 4\}$, then $f^{(5)}_{a,b}$ is a hyper-bent function if and only if one of the assertions (1) and (2) holds.*

(1) $Q_m(a) = 0$, $K_m(a) = -4$.

(2) $Q_m(a) = 2^{m_1}$, $K_m(a) = 2 \cdot 2^{m_1} - 4$.

From Theorem 3 in [1], we have the following theorem.

Theorem 2 *Let $n = 2m$, $m \equiv \pm 2, \pm 6 \pmod{20}$, $m \geq 6$ and $b \in \mathbb{F}_{16}^* \backslash \{\beta^i | 0 \leq i \leq 4\}$, then $f_{a,b}^{(5)}$ is a hyper-bent function if and only if one of the following assertions* (1) *and* (2) *holds.*

(1) *$p(x) = x^5 + x + a^{-1}$ over $\mathbb{F}_{2^m}$ is* (1)(2)^2 *and $K_m(a) = -4$.*

(2) *$p(x) = x^5 + x + a^{-1}$ is irreducible over $\mathbb{F}_{2^m}$. The quadratic form $\mathfrak{q}(x) = \mathrm{Tr}_1^m(x(ax^4 + ax^2 + a^2x))$ over $\mathbb{F}_{2^m}$ is even. $K_m(a) = 2 \cdot 2^{m_1} - 4$.*

3.2 The hyper-bentness of $f_{a,b}^{(r)}(x)$

In the rest of the paper, we consider the Boolean function

$$f_{a,b}^{(r)}(x) := \mathrm{Tr}_1^n(ax^{r(2^m-1)}) + \mathrm{Tr}_1^4(bx^{\frac{2^n-1}{5}}), \tag{10}$$

where $n = 2m$, $m \equiv 2 \pmod 4$, $a \in \mathbb{F}_{2^m}$ and $b \in \mathbb{F}_{16}$. Then we define the character sum

$$\Lambda_r(a,b) := \sum_{u \in U} \chi(f_{a,b}^{(r)}(u)). \tag{11}$$

Similarly, $f_{a,b}^{(r)}(x)$ is a hyper-bent function if and only if $\Lambda_r(a,b) = 1$.

Theorem 3 *Let $n = 2m$, $m \equiv 2 \pmod 4$, $a \in \mathbb{F}_{2^m}$ and $b \in \mathbb{F}_{16}$. If $(r, \frac{2^m+1}{5}) > 1$, then $f_{a,b}^{(r)}$ is not a hyper-bent function. Further, if $(r, \frac{2^m+1}{5}) = 1$, then*

(1) *If $r \equiv 0 \pmod 5$, then $f_{a,b}^{(r)}$ and $f_{a,b}^{(5)}$ have the same hyper-bentness.*

(2) *If $r \equiv \pm 1 \pmod 5$, then $f_{a,b}^{(r)}$ and $f_{a,b}^{(1)}$ have the same hyper-bentness.*

(3) *If $r \equiv \pm 2 \pmod 5$, then $f_{a,b}^{(r)}$ and $f_{a,b}^{(2)}$ have the same hyper-bentness.*

Proof Note that

$$\begin{aligned}\Lambda_r(a,b) &= \sum_{i=0}^{4} \sum_{v \in V} \chi(\mathrm{Tr}_1^4(b(\xi^i v)^{\frac{2^n-1}{5}}))\chi(\mathrm{Tr}_1^n(a(\xi^i v)^{r(2^m-1)})) \\ &= \sum_{i=0}^{4} \sum_{v \in V} \chi(\mathrm{Tr}_1^4(b\xi^{i\frac{2^n-1}{5}}))\chi(\mathrm{Tr}_1^n(a\xi^{ri(2^m-1)} v^{r(2^m-1)})).\end{aligned}$$

Let $d := (r(2^m-1), \#V) = (r, \frac{2^m+1}{5})$, then

$$\Lambda_r(a,b) = d \sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(b\xi^{i\frac{2^n-1}{5}})) \sum_{v \in V^d} \chi(\mathrm{Tr}_1^n(a\xi^{ri(2^m-1)} v^{r(2^m-1)})), \tag{12}$$

where $V^d := \{v^d | v \in V\}$. If $d = (r, \frac{2^m+1}{5}) > 1$, $d | \Lambda_r(a,b)$ and $\Lambda_r(a,b) \neq 1$. Hence, $f_{a,b}^{(r)}$ is not a hyper-bent function.

When $d=(r,\frac{2^m+1}{5})=1$,

$$\Lambda_r(a,b)=\sum_{i=0}^{4}\chi(\mathrm{Tr}_1^4(b\xi^{i\frac{2^n-1}{5}}))\sum_{v\in V}\chi(\mathrm{Tr}_1^n(a\xi^{ri(2^m-1)}v)). \tag{13}$$

If $r\equiv 0 \pmod 5$, from $\xi^{\frac{2^n-1}{5}}=\beta^3$, we have

$$\begin{aligned}\Lambda_r(a,b)&=\sum_{i=0}^{4}\chi(\mathrm{Tr}_1^4(b\beta^{3i}))\sum_{v\in V}\chi(\mathrm{Tr}_1^n(a\xi^{ri(2^m-1)}v))\\&=\sum_{i=0}^{4}\chi(\mathrm{Tr}_1^4(b\beta^{3i}))\sum_{v\in V}\chi(\mathrm{Tr}_1^n(av))\\&=\sum_{i=0}^{4}\chi(\mathrm{Tr}_1^4(b\beta^{i}))\sum_{v\in V}\chi(\mathrm{Tr}_1^n(av)).\end{aligned}$$

Then $\Lambda_r(a,b)=\Lambda_5(a,b)$. Therefore, $f_{a,b}^{(r)}$ and $f_{a,b}^{(5)}$ have the same hyper-bentness.

If $r\equiv 1 \pmod 5$, then

$$\Lambda_r(a,b)=\sum_{i=0}^{4}\chi(\mathrm{Tr}_1^4(b\xi^{i\frac{2^n-1}{5}}))\sum_{v\in V}\chi(\mathrm{Tr}_1^n(a\xi^{i(2^m-1)}v)).$$

From Proposition 10 in [1], $\Lambda_r(a,b)=\Lambda_1(a,b)$. Hence, $f_{a,b}^{(r)}$ and $f_{a,b}^{(1)}$ have the same hyper-bentness.

If $r\equiv 2 \pmod 5$, then

$$\begin{aligned}\Lambda_r(a,b)&=\sum_{i=0}^{4}\chi(\mathrm{Tr}_1^4(b\xi^{i\frac{2^n-1}{5}}))\sum_{v\in V}\chi(\mathrm{Tr}_1^n(a\xi^{2i(2^m-1)}v))\\&=\sum_{i=0}^{4}\chi(\mathrm{Tr}_1^4(b\beta^{3i}))S_{2i}\\&=\sum_{i=0}^{4}\chi(\mathrm{Tr}_1^4(b\beta^{9i}))S_{6i}\\&=\sum_{i=0}^{4}\chi(\mathrm{Tr}_1^4(b\beta^{4i}))S_{i}.\end{aligned}$$

From Lemma 1 in [1], then

$$\Lambda_r(a,b)=\chi(\mathrm{Tr}_1^4(b))S_0+(\chi(\mathrm{Tr}_1^4(b\beta))+\chi(\mathrm{Tr}_1^4(b\beta^4)))S_1+(\chi(\mathrm{Tr}_1^4(b\beta^2))+\chi(\mathrm{Tr}_1^4(b\beta^3)))S_2. \tag{14}$$

Hence, $\Lambda_r(a,b)=\Lambda_2(a,b)$. $f_{a,b}^{(r)}$ and $f_{a,b}^{(2)}$ have the same hyper-bentness.

If $r \equiv 3 \pmod 5$,

$$\begin{aligned}\Lambda_r(a,b) &= \sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(b\xi^{i\frac{2^n-1}{5}})) \sum_{v\in V} \chi(\mathrm{Tr}_1^n(a\xi^{3i(2^m-1)}v)) \\ &= \sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(b\beta^{3i}))S_{3i} \\ &= \sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(b\beta^{i}))S_{i}.\end{aligned}$$

From Lemma 1 in [1],

$$\Lambda_r(a,b) = \chi(\mathrm{Tr}_1^4(b))S_0 + (\chi(\mathrm{Tr}_1^4(b\beta)) + \chi(\mathrm{Tr}_1^4(b\beta^4)))S_1 + (\chi(\mathrm{Tr}_1^4(b\beta^2)) + \chi(\mathrm{Tr}_1^4(b\beta^3)))S_2. \tag{15}$$

Hence, $\Lambda_r(a,b) = \Lambda_3(a,b)$. From (14) and (15),

$$\Lambda_2(a,b) = \Lambda_3(a,b).$$

$f_{a,b}^{(r)}$ and $f_{a,b}^{(2)}$ have the same hyper-bentness.

Similarly, if $r \equiv 4 \pmod 5$,

$$\Lambda_r(a,b) = \Lambda_4(a,b) = \Lambda_1(a,b).$$

$f_{a,b}^{(r)}$ and $f_{a,b}^{(1)}$ have the same hyper-bentness.

Above all, this theorem follows.

From Theorem 3, to characterize the hyper-bentness of $f_{a,b}^{(r)}$, we just consider the hyper-bentness of $f_{a,b}^{(1)}$, $f_{a,b}^{(2)}$ and $f_{a,b}^{(5)}$. The hyper-bentness of $f_{a,b}^{(1)}$ is considered in [1]. And the hyper-bentness of $f_{a,b}^{(5)}$ is discussed before. Next, we just study the hyper-bentness of $f_{a,b}^{(2)}$.

When $b = 0$, the hyper-bentness of $f_{a,0}^{(2)}$ is given in [2]. Then we just consider the case $b \neq 0$. We first give properties of $\Lambda_2(a,b)$ in the following proposition.

Proposition 9 *Let $a \in \mathbb{F}_{2^m}$ and $b \in \mathbb{F}_{16}^*$, then*

(1) *If $b = 1$, then $\Lambda_2(a,b) = S_0 - 2(S_1 + S_2) = 2S_0 - \Lambda_2(a,0)$.*

(2) *If $b \in \{\beta+\beta^2, \beta+\beta^3, \beta^2+\beta^4, \beta^3+\beta^4\}$, that is, b is a primitive element satisfying $\mathrm{Tr}_1^4(b) = 0$, then $\Lambda_2(a,b) = S_0$.*

(3) *If $b = \beta$ or β^4, then $\Lambda_2(a,b) = -S_0 - 2S_2$.*

(4) *If $b = \beta^2$ or β^3, then $\Lambda_2(a,b) = -S_0 - 2S_1$.*

(5) *If $b = 1+\beta$ or $1+\beta^4$, then $\Lambda_2(a,b) = -S_0 + 2S_2$.*

(6) *If $b = 1+\beta^2$ or $1+\beta^3$, then $\Lambda_2(a,b) = -S_0 + 2S_1$.*

(7) *If $b = \beta+\beta^4$, then $\Lambda_2(a,b) = S_0 + 2S_2 - 2S_1$.*

(8) *If $b = \beta^2+\beta^3$, then $\Lambda_2(a,b) = S_0 - 2S_2 + 2S_1$.*

Proof From (14) and the similar proof of Proposition 13 in [1], this proposition follows.

Corollary 1 *Let $a \in \mathbb{F}_{2^m}$ and $b \in \mathbb{F}_{16}^*$, then*

(1) *$f_{a,b}^{(2)}$ and $f_{a,b^2}^{(1)}$ have the same hyper-bentness.*

(2) *If b satisfies $(b+1)(b^4+b+1)=0$, then $f_{a,b}^{(2)}$ and $f_{a,b}^{(1)}$ have the same hyper-bentness.*

Proof (1) From Proposition 13 in [1] and Proposition 9,

$$\Lambda_2(a,b^2) = \Lambda_1(a,b).$$

Hence, $f_{a,b}^{(2)}$ and $f_{a,b^2}^{(1)}$ have the same hyper-bentness.

(2) Similarly, if b satisfies $(b+1)(b^4+b+1)=0$,

$$\Lambda_2(a,b) = \Lambda_1(a,b).$$

Hence, $f_{a,b}^{(2)}$ and $f_{a,b}^{(1)}$ have the same hyper-bentness.

From the above discussion, we have the following result on $f_{a,b}^{(r)}$.

Proposition 10 *Let $a \in \mathbb{F}_{2^m}$ and $(r, \frac{2^m+1}{5}) = 1$, then*

(1) *If $\frac{1}{5}[1 - K_m(a) + 2Q_m(a)] = 1$, then the following Boolean functions*

(a) *$f_{a,b}^{(r)}$, $b \in \mathbb{F}_{16}^* \backslash \{\beta^i | i = 0,1,2,3,4\}$, $r \equiv 0 \pmod 5$.*

(b) *$f_{a,b}^{(r)}$, $r \not\equiv 0 \pmod 5$, $b^4+b+1=0$.*

are hyper-bent functions.

(2) *If $-\frac{1}{5}[3(1-K_m(a))-4Q_m(a)] = 1$, then the Boolean function $f_{a,1}^{(r)}$ ($r \not\equiv 0 \pmod 5$) is a hyper-bent function.*

In fact, the converse proposition still holds.

Proof From Proposition 16 in [1] and Theorem 3, (9) and Proposition 6, this proposition follows.

We generalize Theorem 3 in [1] and get the following theorem.

Theorem 4 *Let $n = 2m$, $m = 2m_1$, $m_1 \equiv 1 \pmod 2$, $m_1 \geq 3$ and $(r, \frac{2^m+1}{5}) = 1$, If one of two assertions (1) and (2) holds,*

(1) *$p(x) = x^5 + x + a^{-1}$ over $\mathbb{F}_{2^m}$ is $(1)(2)^2$ and $K_m(a) = -4$.*

(2) *$p(x) = x^5 + x + a^{-1}$ is irreducible over $\mathbb{F}_{2^m}$. The quadratic form $\mathfrak{q}(x) = \mathrm{Tr}_1^m(x(ax^4 + ax^2 + a^2x))$ over $\mathbb{F}_{2^m}$ is even. $K_m(a) = 2 \cdot 2^{m_1} - 4$.*

Then the Boolean functions

(a) *$f_{a,b}^{(r)}$, $b \in \mathbb{F}_{16}^* \backslash \{\beta^i | i = 0,1,2,3,4\}$, $r \equiv 0 \pmod 5$.*

(b) *$f_{a,b}^{(r)}$, $r \not\equiv 0 \pmod 5$, $b^4+b+1=0$.*

are hyper-bent functions

In fact, the converse theorem still holds.

Proof From Proposition 16 and Theorem 3 in [1] and Proposition 10, this theorem follows.

Similar to Theorem 2 in [1], we have the following result.

Theorem 5 *Let $n = 2m$, $m = 2m_1$, $m_1 \equiv 1 \pmod 2$, $m_1 \geq 3$, $(r, \frac{2^m+1}{5}) = 1$ and $r \not\equiv 0 \pmod 5$, then $f_{a,1}^{(r)}$ is a hyper-bent function if and only if the following assertions holds.*

(1) *$p(x) = x^5 + x + a^{-1}$ is irreducible over $\mathbb{F}_{2^m}$.*

(2) *The quadratic form $\mathfrak{q}(x) = \mathrm{Tr}_1^m(x(ax^4 + ax^2 + a^2x))$ over $\mathbb{F}_{2^m}$ is even.*

(3) *$K_m(a) = \frac{4}{3}(2 - 2^{m_1})$.*

In fact, the converse theorem still holds.

Proof From Proposition 16 and Theorem 2 in [1] and Proposition 10, this theorem follows.

If $a \in \mathbb{F}_{2^{\frac{m}{2}}}$, we have the hyper-bentness of $f_{a,b}^{(r)}$ in the theorem below.

Theorem 6 *Let $n = 2m$, $m = 2m_1$, $m_1 \equiv 1 \pmod 2$ and $m_1 \geq 3$. If $n \neq 12, 28$, any Boolean function in*

$$\{f_{a,b}^{(r)} | a \in \mathbb{F}_{2^{\frac{m}{2}}}, b \in \mathbb{F}_{16}\} \tag{16}$$

is not a hyper-bent function. Further, if $n = 12$, all the hyper-bent functions in (16) are

$$\mathrm{Tr}_1^{12}(ax^{r(2^6-1)}) + \mathrm{Tr}_1^4(bx^{\frac{2^{12}-1}{5}}),$$

where $r \not\equiv 0 \pmod 5$, $(r, \frac{2^m+1}{5}) = 1$, $(a+1)(a^3+a^2+1) = 0$ and $b = \beta^i, i = 1, 2, 3, 4$. If $n = 28$, all the hyper-bent functions in (16) are

$$\mathrm{Tr}_1^{28}(ax^{r(2^{14}-1)}) + \mathrm{Tr}_1^4(bx^{\frac{2^{28}-1}{5}}),$$

where $r \not\equiv 0 \pmod 5$, $(r, \frac{2^m+1}{5}) = 1$, $(a+1)(a^7+a^6+a^5+a^4+a^3+a^2+1) = 0$ and $b = \beta^i, i = 1, 2, 3, 4$.

Proof Note that $a \in \mathbb{F}_{2^{\frac{m}{2}}}$. From Theorem 3, if $f_{a,b}^{(r)}$ is a hyper-bent function, $(r, \frac{2^m+1}{5}) = 1$.

Suppose $(r, \frac{2^m+1}{5}) = 1$. we first prove that $f_{a,0}^{(r)}$ is not a hyper-bent function when $r \equiv 0 \pmod 5$. From Theorem 3, $f_{a,b}^{(r)}$ is a hyper-bent function if and only if $f_{a,b}^{(5)}$ is a hyper-bent function. If $b = 0$,

$$\Lambda_5(a, 0) = \sum_{u \in U} \chi(\mathrm{Tr}_1^n(au^{5(2^m-1)})) = 5 \sum_{v \in V} \chi(\mathrm{Tr}_1^n(av^{2^m-1})).$$

Hence, $5 | \Lambda_5(a, 0)$ and $\Lambda_5(a, 0) \neq 1$. Therefore, $f_{a,0}^{(5)}$ is not a hyper-bent function. Then $f_{a,0}^{(r)}$ is not a hyper-bent function.

When $b \neq 0$, from Theorem 4, $f_{a,b}^{(r)}$ is a hyper-bent function if and only if $f_{a,b'}^{(1)}$ $(b'^4 + b' + 1 = 0)$ is a hyper-bent function. From Theorem 5 in [1], $f_{a,b'}^{(1)}$ $(b'^4 + b' + 1 = 0)$ is not a hyper-bent function. Hence, $f_{a,b}^{(r)}$ is not a hyper-bent function when $r \equiv 0 \pmod 5$.

Then we discuss the case $r \equiv \pm 1 \pmod 5$ and $(r, \frac{2^m+1}{5}) = 1$. From Theorem 3, $f_{a,b}^{(r)}$ is a hyper-bent function if and only if $f_{a,b}^{(1)}$ is a hyper-bent function. From Theorem 5 in [1], there are only two cases. The first case is $n = 12$, where a and b satisfy

$$(a+1)(a^3+a^2+1) = 0, b = \beta^i, i = 1, 2, 3, 4.$$

The second case is $n = 28$, where a and b satisfy

$$(a+1)(a^7+a^6+a^5+a^4+a^3+a^2+1) = 0, b = \beta^i, i = 1, 2, 3, 4.$$

When $r \equiv \pm 2 \pmod 5$ and $(r, \frac{2^m+1}{5}) = 1$, we have similar results.
Above all, this theorem follows.

4 Conclusion

This paper considers the hyper-bentness of the Boolean functions $f_{a,b}^{(r)}$ of the form $f_{a,b}^{(r)} := \mathrm{Tr}_1^n(ax^{r(2^m-1)}) + \mathrm{Tr}_1^4(bx^{\frac{2^n-1}{5}})$, where $n = 2m$, $m = 2 \pmod 4$, $a \in \mathbb{F}_{2^m}$ and $b \in \mathbb{F}_{16}$. When $r \equiv 0 \pmod 5$, we give the characterization of hyper-bentness of $f_{a,b}^{(r)}$. If $r \not\equiv 0 \pmod 5$ and $b = 1$ or b is a primitive element in $\mathbb{F}_{16}$ such that $\mathrm{Tr}_1^4(b) = 0$, the hyper-bentness of $f_{a,b}^{(r)}$ can be characterized by Kloosterman sums and the factorization of $x^5 + x + a^{-1}$. If $a \in \mathbb{F}_{2^{\frac{m}{2}}}$, with the results of [1], we prove that $f_{a,b}^{(r)}$ is not a hyper-bent function unless $n = 12$ or $n = 28$. Further, we give all the hyper-bent functions for $n = 12$ or $n = 28$.

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