Identity-Based (Lossy) Trapdoor Functions and Applications

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Abstract

We provide the first constructions of identity-based (injective) trapdoor functions. Furthermore, they are lossy. Constructions are given both with pairings (DLIN) and lattices (LWE). Our lossy identity-based trapdoor functions provide an automatic way to realize, in the identity-based setting, many functionalities previously known only in the public-key setting. In particular we obtain the first deterministic and efficiently searchable IBE schemes and the first hedged IBE schemes, which achieve best possible security in the face of bad randomness. Underlying our constructs is a new definition, of *partial* lossiness, that may be of broader interest.

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1 Introduction

A trapdoor function F specifies, for each public key pk, an injective, *deterministic* map F_{pk} that can be inverted given an associated secret key (trapdoor). The most basic measure of security is one-wayness. The canonical example is RSA [50].

Suppose there is an algorithm that generates a "fake" public key pk^* such that F_{pk^*} is no longer injective but has image much smaller than its domain and, moreover, given a public key, you can't tell whether it is real or fake. Peikert and Waters [48] call such a TDF lossy. Intuitively, F_{pk} is close to a function F_{pk^*} that provides information-theoretic security. Lossiness implies one-wayness [48].

Lossy TDFs have quickly proven to be a powerful tool. Applications include IND-CCA [48], deterministic [15], hedged [7] and selective-opening secure public-key encryption [9]. Lossy TDFs can be constructed from DDH [48], QR [33], DLIN [33], DBDH [23], LWE [48] and HPS (hash proof systems) [38]. RSA was shown in [42] to be lossy under the Φ-hiding assumption of [25], leading to the first proof of security of RSA-OAEP [12] without random oracles.

Lossy TDFs and their benefits belong, so far, to the realm of public-key cryptography. The purpose of this paper is to bring them to identity-based cryptography, defining and constructing identity-based TDFs (IB-TDFs), both one-way and lossy. We see this as having two motivations, one more theoretical, the other more applied, yet admittedly both foundational, as we discuss before moving further.

Theoretical angle. Trapdoor functions are the primitive that began public key cryptography [30, 50]. Public-key encryption was built from TDFs. (Via hardcore bits.) Lossy TDFs enabled the first DDH and lattice (LWE) based TDFs [48].

It is striking that identity-based cryptography developed entirely differently. The first realizations of IBE [20, 29, 53] directly used randomization and were neither underlain by, nor gave rise to, any IB-TDFs.

We ask whether this asymmetry between the public-key and identity-based worlds (TDFs in one but not the other) is inherent. This seems to us a basic question about the nature of identity-based cryptography that is worth asking and answering.

Application angle. Is there anything here but idle curiosity? IBE has already been achieved *without* IB-TDFs, so why go backwards to define and construct the latter? The answer is that *losssy* IB-TDFs enable new applications that we do not know how to get in other ways.

Stepping back, identity-based cryptography [54] offers several advantages over its public-key counterpart. Key management is simplified because an entity's identity functions as their public key. Key revocation issues that plague PKI can be handled in alternative ways, for example by using $\mathsf{identity}$+date as the key under which to encrypt to $\mathsf{identity}$ [20]. There is thus good motivation to go beyond basics like IBE [20, 29, 53, 16, 17, 56, 34] and identity-based signatures [10, 31] to provide identity-based counterparts of other public-key primitives.

Furthermore we would like to do this in a systematic rather than ad hoc way, leading us to seek tools that enable the transfer of multiple functionalities in relatively blackbox ways. The applications of lossiness in the public-key realm suggest that lossy IBTDFs will be such a tool also in the identity-based realm. As evidence we apply them to achieve identity-based deterministic encryption and identity-based hedged encryption. The first, the counterpart of deterministic public-key encryption [6, 15], allows efficiently searchable identity-based encryption of database entries while maintaining the maximal possible privacy, bringing the key-management benefits of the identity-based setting to this application. The second, counterpart of hedged symmetric and public-key encryption [51, 7], makes IBE as resistant as possible in the face of low-quality randomness, which is important given the widespread deployment of IBE and the real danger of bad-randomness based attacks evidenced by the ones on the Sony Playstation and Debian Linux. We hope that our framework will facilitate further such transfers.

We clarify that the solutions we obtain are not practical but they show that the security goals can be achieved in principle, which was not at all clear prior to our work. Allowed random oracles, we can give solutions that are much more efficient and even practical.

CONTRIBUTIONS IN BRIEF. We define IB-TDFs and two associated security notions, one-wayness and lossiness, showing that the second implies the first.

The first wave of IBE schemes was from pairings [20, 53, 16, 17, 56, 55] but another is now emerging from lattices [34, 28, 2, 3]. We aim accordingly to reach our ends with either route and do so successfully. We provide lossy IB-TDFs from a standard pairings assumption, namely the Decision Linear (DLIN) assumption of [18]. We also provide IB-TDFs based on Learning with Errors (LWE) [49], whose hardness follows from the worst-case hardness of certain lattice-related problems [49, 47]. (The same assumption underlies lattice-based IBE [34, 28, 2, 3] and public-key lossy TDFs [48].) None of these results relies on random oracles.

Existing work brought us closer to the door with lattices, where one-way IB-TDFs can be built by combining ideas from [34, 28, 2]. Based on techniques from [47, 43] we show how to make them lossy. With pairings, however it was unclear how to even get a one-way IB-TDF, let alone one that is lossy. We adapt the matrix-based framework of [48] so that by populating matrix entries with ciphertexts of a very special kind of *anonymous* IBE scheme it becomes possible to implicitly specify per-identity matrices defining the function. No existing anonymous IBE has the properties we need but we build one that does based on methods of [22]. Our results with pairings are stronger because the lossy branches are universal hash functions which is important for applications.

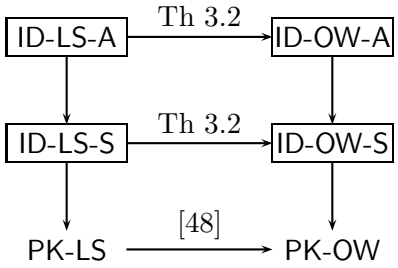
Public-key lossy TDFs exist aplenty and IBE schemes do as well. It is natural to think one could easily combine them to get IB-TDFs. We have found no simple way to do this. Ultimately we do draw from both sources for techniques but our approaches are intrusive. Let us now look at our contributions in more detail.

NEW PRIMITIVES AND DEFINITIONS. Public parameters *pars* and an associated master secret key having been chosen, an IB-TDF F associates to any identity a map $F_{pars,id}$, again injective and deterministic, inversion being possible given a secret key derivable from id via the master secret key. One-wayness means F_{pars,id^*} is hard to invert on random inputs for an adversary-specified challenge identity id^*. Importantly, as in IBE, this must hold even when the adversary may obtain, via a key-derivation oracle, a decryption key for any non-challenge identity of its choice [20]. This key-derivation capability contributes significantly to the difficulty of realizing the primitive. As with IBE, security may be selective (the adversary must specify id^* before seeing *pars*) [27] or adaptive (no such restriction) [20].

The most direct analog of the definition of lossiness from the public-key setting would ask that there be a way to generate "fake" parameters $pars^*$, indistinguishable from the real ones, such that F_{pars^*,id^*} is lossy (has image smaller than domain). In the selective setting, the fake parameter generation algorithm Pg^* can take id^* as input, making the goal achievable at least in principle, but in the adaptive setting it is impossible to achieve, since, with id^* not known in advance, Pg^* is forced to make $F_{pars^*,id}$ lossy for all id, something the adversary can immediately detect using its key-derivation oracle.

We ask whether there is an adaptation of the definition of lossiness that is achievable in the adaptive case while sufficing for applications. Our answer is a definition of *δ-lossiness*, a metric of partial lossiness parameterized by the probability δ that F_{pars^*,id^*} is lossy. The definition is unusual, involving an adversary advantage that is the difference, not of two probabilities as is common in cryptographic metrics, but of two differently weighted ones. We will achieve selective lossiness with degree $\delta = 1$, but in the adaptive case the best possible is degree $1/\operatorname{poly}$ with the polynomial depending on the number of key-derivation queries of the adversary, and this what we will achieve. We show that lossiness with degree δ implies one-wayness, in both the selective and adaptive settings, as long as δ is at least $1/\operatorname{poly}$.

In summary, in the identity-based setting (ID) there are two notions of security, one-wayness (OW) and lossiness (LS), each of which could be selective (S) or adaptive (A), giving rise to four kinds of IB-TDFs. The left side of Figure 1 shows how they relate to each other and to the two kinds of TDFs —OW and LS— in the public-key setting (PK). The un-annotated implications are trivial, ID-LS-A $\rightarrow$ ID-LS-S meaning that δ-lossiness of the first type implies δ-lossiness of the other for all δ. It is not however via this implication that we achieve ID-LS-S, for, as the table shows, we achieve it with degree higher than ID-LS-A.



Primitive	δ	Achieved under
ID-LS-A	$1/\text{poly}$	DLIN, LWE
ID-LS-S	1	DLIN, LWE

Figure 1: Types of TDFs based on setting (PK=Public-key, ID=identity-based), security (OW=one-way, LS=loss) and whether the latter is selective (S) or adaptive (A). An arrow A $\rightarrow$ B in the diagram on the left means that TDF of type B is implied by (can be constructed from) TDF of type A. Boxed TDFs are the ones we define and construct. The table on the right shows the δ for which we prove δ-lossiness and the assumptions used. In both the S and A settings the δ we achieve is best possible and suffices for applications.

Closer Look. One's first attempt may be to build an IB-TDF from an IBE scheme. In the random oracle (RO) model, this can be done by a method of [8], namely specify the coins for the IBE scheme by hashing the message with the RO. It is entirely unclear how to turn this into a standard model construct and it is also unclear how to make it lossy.

To build ID-TDFs from lattices we consider starting from the public-key TDF of [48] which is already lossy and trying to make it identity-based but it is unclear how to do this. However, Gentry, Peikert and Vaikuntanathan (GPV) [34] showed that the function $g_{\mathbf{A}} : \mathbb{Z}_q^n \times B_\alpha^m \rightarrow \mathbb{Z}_q^n$ defined by $g_{\mathbf{A}}(\mathbf{x}, \mathbf{e}) = \mathbf{A}^T \cdot \mathbf{x} + \mathbf{e}$ is a TDF for appropriate choices of the domain and parameters, where matrix $\mathbf{A}$ is the public key. We make this function identity-based using basis delegation methods of Cash, Hofheinz, Kiltz and Peikert [28] or, for greater efficiency, those of Agrawal, Boneh and Boyen [2]. Finally, we obtain a lossy IB-TDF by showing that this construct is already lossy.

With pairings we face greater difficulties, for there is no immediate way to get an IB-TDF that is even one-way, let alone lossy. We aim for the latter, there being no obviously simpler way to get the former. In the selective case we need to ensure that the function is lossy on the challenge identity id^* yet injective on others, this setup being indistinguishable from the one where the function is always injective. Whereas the matrix diagonals in the construction of [48] consisted of ElGamal ciphertexts, in ours they are ciphertexts for identity id^* under an anonymous IBE scheme, the salient property being that the "anonymity" property should hide whether the underlying ciphertext is to id^* or is a random group element. Existing anonymous IBE schemes, in particular that of Boyen and Waters (BW) [22], are not conducive and we create a new one. A side benefit is a new anonymous IBE scheme with ciphertexts and private keys having one less group element than BW but still proven secure under DLIN.

A method of Boneh and Boyen [16] can be applied to turn selective into adaptive security but the reduction incurs a factor that is equal to the size of the identity space and thus ultimately exponential in the security parameter, so that adaptive security according to the standard asymptotic convention would not have been achieved. To achieve it, we want to be able to "program" the public parameters so that they will be lossy on about a $1/Q$ fraction of "random-ish" identities, where Q is the number of key-derivation queries made by the attacker. Ideally, with probability around $1/Q$ all of (a successful) attacker's queries will land outside the lossy identity-space, but the challenge identity will land inside it so that we achieve δ-lossiness with δ around $1/Q$.

This sounds similar to the approach of Waters [56] for achieving adaptively secure IBE but there are some important distinctions, most notably that the technique of Waters is information-theoretic while ours is of necessity computational, relying on the DLIN assumption. In the reduction used by Waters the partitioning of the identities into two classes was based solely on the reduction algorithm's internal view of the public parameters; the parameters themselves were distributed independently of this partitioning and thus the adversary view was the same as in a normal setup. In contrast, the partitioning in our

scheme will actually directly affect the parameters and how the system behaves. This is why we must rely on a computational assumption to show that the partitioning in undetectable. A key novel feature of our construction is the introduction of a system that will produce lossy public parameters for about a $1/Q$ fraction of the identities.

Applications. Deterministic PKE is a TDF providing the best possible privacy subject to being deterministic, a notion called PRIV that is much stronger than one-wayness [6]. An application is encryption of database records in a way that permits logarithmic-time search, improving upon the linear-time search of PEKS [19]. Boldyreva, Fehr and O'Neill [15] show that lossy TDFs whose lossy branch is a universal hash (called universal lossy TDFs) achieve (via the LHL [14, 37]) PRIV-security for message sequences which are blocksources, meaning each message has some min-entropy even given the previous ones, which remains the best result without ROs. Deterministic IBE and the resulting efficiently-searchable IBE are attractive due to the key-management benefits. We can achieve them because our DLIN-based lossy IB-TDFs are also universal lossy. (This is not true, so far, for our LWE based IB-TDFs.)

To provide IND-CPA security in practice, IBE relies crucially on the availability of fresh, high-quality randomness. This is fine in theory but in practice RNGs (random number generators) fail due to poor entropy gathering or bugs, leading to prominent security breaches [35, 36, 24, 46, 45, 1, 57, 32]. Expecting systems to do a better job is unrealistic. Hedged encryption [7] takes poor randomness as a fact of life and aims to deliver best possible security in the face of it, providing privacy as long as the message together with the "randomness" have some min-entropy. Hedged PKE was achieved in [7] by combining IND-CPA PKE with universal lossy TDFs. We can adapt this to IBE and combine existing (randomized) IBE schemes with our DLIN-based universal lossy IB-TDFs to achieved hedged IBE. This is attractive given the widespread use of IBE in practice and the real danger of randomness failures.

Related Work. A number of papers have studied security notions of trapdoor functions beyond traditional one-wayness. Besides lossiness [48] there is Rosen and Segev's notion of correlated-product security [52], and Canetti and Dakdouk's extractable trapdoor functions [26]. The notion of adaptive one-wayness for tag-based trapdoor functions from Kiltz, Mohassel and O'Neill [41] can be seen as the special case of our selective IB-TDF in which the adversary is denied key-derivation queries. Security in the face of these queries was one of the main difficulties we faced in realizing IB-TDFs.

Organization. We define IB-TDFs, one-wayness and δ-lossiness in Section 2. We also define extended IB-TDFs, an abstraction that will allow us to unify and shorten the analyses for the selective and adaptive security cases. In Appendix 3 we show that δ-lossiness implies one-wayness as long as δ is at least $1/\operatorname{poly}$. This allows us to focus on achieving δ-lossiness.

We have put our pairing-based lossy IB-TDFs in the body (Section 4) because the materiel is more broadly accessible than lattices, we face greater challenges in this area, and we get universal lossy IB-TDFs as needed by applications. Proofs of the crucial Lemmas 4.1 and 4.2 are however in appendices 4.7 and 4.5, respectively. The lattice-based IB-TDFs are in Appendix 5. In Appendix B we sketch how to apply δ-lossy IB-TDFs to achieve deterministic and hedged IBE.

2 Definitions

Notation and conventions. If $\mathbf{x}$ is a vector then $|\mathbf{x}|$ denotes the number of its coordiates and $\mathbf{x}[i]$ denotes its i-th coordinate. Coordinates may be numbered $1, \ldots, |\mathbf{x}|$ or $0, \ldots, |\mathbf{x}| - 1$ as convenient. A string x is identified with a vector over $\{0,1\}$ so that $|x|$ denotes its length and $x[i]$ its i-th bit. The empty string is denoted ε. If S is a set then $|S|$ denotes its size, S^a denotes the set of a-vectors over S, $S^{a\times b}$ denotes the set of a by b matrices with entries in S, and so on. The (i,j)-th entry of a 2 dimensional matrix $\mathbf{M}$ is denoted $\mathbf{M}[i,j]$ and the (i,j,k)-th entry of a 3 dimensional matrix $\mathbf{M}$ is denoted $\mathbf{M}[i,j,k]$. If $\mathbf{M}$ is a n by μ matrix then $\mathbf{M}[j,\cdot]$ denotes the vector $(\mathbf{M}[j,1], \ldots, \mathbf{M}[j,\mu])$. If $a = (a_1, \ldots, a_n)$ then $(a_1, \ldots, a_n) \leftarrow a$ means we parse a as shown. Unless otherwise indicated, an algorithm may be randomized. By $y \xleftarrow{\$} A(x_1, x_2, \ldots)$ we denote the operation of running A on inputs $x_1, x_2, \ldots$ and fresh

proc Initialize(id) // $\mathrm{OW}_{\mathsf{F}}, \mathrm{Real}_{\mathsf{F}}$
$(pars, msk) \xleftarrow{\$} \mathsf{F.Pg}$; $IS \leftarrow \emptyset$; $id^* \leftarrow id$
Return $pars$

proc GetDK(id) // $\mathrm{OW}_{\mathsf{F}}, \mathrm{Real}_{\mathsf{F}}$
$IS \leftarrow IS \cup \{id\}$
$dk \leftarrow \mathsf{F.Kg}(pars, msk, id)$
Return dk

proc Ch(id) // OW_{F}
$id^* \leftarrow id$; $x \xleftarrow{\$} \mathsf{InSp}$
$y \leftarrow \mathsf{F.Ev}(pars, id^*, x)$
Return y

proc Finalize(x') // OW_{F}
Return $((x' = x)$ and $(id^* \notin IS))$

proc Initialize(id) // $\mathrm{Lossy}_{\mathsf{F},\mathsf{LF},\ell}$
$(pars, msk) \xleftarrow{\$} \mathsf{LF.Pg}(id)$; $IS \leftarrow \emptyset$; $id^* \leftarrow id$
Return $pars$

proc GetDK(id) // $\mathrm{Lossy}_{\mathsf{F},\mathsf{LF},\ell}$
$IS \leftarrow IS \cup \{id\}$
$dk \leftarrow \mathsf{LF.Kg}(pars, msk, id)$
Return dk

proc Ch(id) // $\mathrm{Real}_{\mathsf{F}}, \mathrm{Lossy}_{\mathsf{F},\mathsf{LF},\ell}$
$id^* \leftarrow id$

proc Finalize(d') // $\mathrm{Real}_{\mathsf{F}}$
Return $((d' = 1)$ and $(id^* \notin IS))$

proc Finalize(d') // $\mathrm{Lossy}_{\mathsf{F},\mathsf{LF},\ell}$
Return $((d' = 1)$ and $(id^* \notin IS)$ and $(\lambda(\mathsf{F.Ev}(pars, id^*, \cdot)) \geq \ell))$

Figure 2: Games defining one-wayness and δ-lossiness of IBTDF F with associated sibling LF.

coins and letting y denote the output. We denote by $[A(x_1, x_2, \ldots)]$ the set of all possible outputs of A on inputs $x_1, x_2, \ldots$. The (Kronecker) delta function Δ is defined by $\Delta(a, b) = 1$ if $a = b$ and 0 otherwise. If a, b are equal-length vectors of reals then $\langle a, b\rangle = a[1]b[1] + \cdots + a[|a|]b[|b|]$ denotes their inner product.

Games. A game —look at Figure 2 for an example— has an **Initialize** procedure, procedures to respond to adversary oracle queries, and a **Finalize** procedure. To execute a game G is executed with an adversary A means to run the adversary and answer its oracle queries by the corresponding procedures of G. The adversary must make exactly one query to **Initialize**, this being its first oracle query. (This means the adversary can give **Initialize** an input, an extension of the usual convention [13].) It must make exactly one query to **Finalize**, this being its last oracle query. The reply to this query, denoted G^A, is called the output of the game, and we let "G^A" denote the event that this game output takes value true. Boolean flags are assumed initialized to false.

IBTDFs. An *identity-based trapdoor function* (IBTDF) is a tuple $\mathsf{F} = (\mathsf{F.Pg}, \mathsf{F.Kg}, \mathsf{F.Ev}, \mathsf{F.Ev}^{-1})$ of algorithms with associated input space InSp and identity space IDSp. The parameter generation algorithm $\mathsf{F.Pg}$ takes no input and returns common parameters $pars$ and a master secret key msk. On input $pars, msk, id$, the key generation algorithm $\mathsf{F.Kg}$ produces a decryption key dk for identity id. For any $pars$ and $id \in \mathsf{IDSp}$, the *deterministic* evaluation algorithm $\mathsf{F.Ev}$ defines a function $\mathsf{F.Ev}(pars, id, \cdot)$ with domain InSp. We require *correct inversion*: For any $pars$, any $id \in \mathsf{IDSp}$ and any $dk \in [\mathsf{F.Kg}(pars, id)]$, the deterministic inversion algorithm $\mathsf{F.Ev}^{-1}$ defines a function that is the inverse of $\mathsf{F.Ev}(pars, id, \cdot)$, meaning $\mathsf{F.Ev}^{-1}(pars, id, dk, \mathsf{F.Ev}(pars, id, x)) = x$ for all $x \in \mathsf{InSp}$.

E-IBTDF. To unify and shorten the selective and adaptive cases of our analyses it is useful to define and specify a more general primitive. An extended IBTDF (E-IBTDF) $\mathsf{E} = (\mathsf{E.Pg}, \mathsf{E.Kg}, \mathsf{E.Ev}, \mathsf{E.Ev}^{-1})$ consists of four algorithms that are just like the ones for an IBTDF except that $\mathsf{F.Pg}$ takes an additional *auxiliary* input from an auxiliary input space AxSp. Fixing a particular auxiliary input $aux \in \mathsf{AxSp}$ for $\mathsf{F.Pg}$ results in an IBTDF scheme that we denote $\mathsf{E}(aux)$ and call the IBTDF induced by aux. Not all these induced schemes need, however, satisfy the correct inversion requirement. If the one induced by aux does, we say that aux grants invertibility. Looking ahead we will build an E-IBTDF and then obtain our IBTDF as the one induced by a particular auxiliary input, the other induced schemes being the basis of the siblings and being used in the proof.

One-wayness. One-wayness of IBTDF $\mathsf{F} = (\mathsf{F.Pg}, \mathsf{F.Kg}, \mathsf{F.Ev}, \mathsf{F.Ev}^{-1})$ is defined via game OW_{F} of Figure 2. The adversary is allowed only one query to its challenge oracle **Ch**. The advantage of such an

adversary I is $\mathbf{Adv}^{\mathrm{ow}}_{\mathsf{F}}(I) = \Pr\left[\,\mathrm{OW}^{I}_{\mathsf{F}}\,\right]$.

SELECTIVE VERSUS ADAPTIVE ID. We are interested in both these variants for all the notions we consider. To avoid a proliferation of similar definitions, we capture the variants instead via different adversary classes relative to the same game. To exemplify, consider game OW_{F} of Figure 2. Say that an adversary A is *selective-id* if the identity id in its queries to **Initialize** and **Ch** is always the same, and say it is *adaptive-id* if this is not necessarily true. Selective-id security for one-wayness is thus captured by restricting attention to selective-id adversaries and full (adaptive-id) security by allowing adaptive-id adversaries. Now, adopt the same definitions of selective and adaptive adversaries relative to *any* game that provides procedures called **Initialize** and **Ch**, regardless of how these procedures operate. In this way, other notions we will introduce, including partial lossiness defined via games also in Figure 2, will automatically have selective-id and adaptive-id security versions.

PARTIAL LOSSINESS. We first provide the formal definitions and later explain them and their relation to standard definitions. If f is a function with domain a (non-empty) set $\mathrm{Dom}(f)$ then its image is $\mathrm{Im}(f) = \{\, f(x) \,:\, x \in \mathrm{Dom}(f) \,\}$. We define the *lossiness* $\lambda(f)$ of f via

$$\lambda(f) = \lg \frac{|\mathrm{Dom}(f)|}{|\mathrm{Im}(f)|} \qquad \text{or equivalently} \qquad |\mathrm{Im}(f)| = |\mathrm{Dom}(f)| \cdot 2^{-\lambda(f)} \ .$$

We say that f is ℓ-lossy if $\lambda(f) \geq \ell$. Let IBTDF $\mathsf{F} = (\mathsf{F.Pg}, \mathsf{F.Kg}, \mathsf{F.Ev}, \mathsf{F.Ev}^{-1})$ be an IBTDF with associated input space InSp and identity space IDSp. A *sibling* for F is an E-IBTDF $\mathsf{LF} = (\mathsf{LF.Pg}, \mathsf{LF.Kg}, \mathsf{F.Ev}, \mathsf{F.Ev}^{-1})$ whose evaluation and inversion algorithms, as the notation indicates, are those of F and whose auxiliary input space is IDSp. Algorithm $\mathsf{LF.Pg}$ will use this input in the selective-id case and ignore it in the adaptive-id case. Consider games $\mathrm{Real}_{\mathsf{F}}$ and $\mathrm{Lossy}_{\mathsf{F},\mathsf{LF},\ell}$ of Figure 2. The first uses the real parameter and key-generation algorithms while the second uses the sibling ones. A los-adversary A is allowed just one **Ch** query, and the games do no more than record the challenge identity id^*. The advantage of the adversary is *not*, as usual, the difference in the probabilities that the games return true, but is instead parameterized by a probability $\delta \in [0,1]$ and defined via

$$\mathbf{Adv}^{\delta\text{-los}}_{\mathsf{F},\mathsf{LF},\ell}(A) = \delta \cdot \Pr\left[\,\mathrm{Real}^{A}_{\mathsf{F}}\,\right] - \Pr\left[\,\mathrm{Lossy}^{A}_{\mathsf{F},\mathsf{LF},\ell}\,\right] \ . \tag{1}$$

DISCUSSION. The PW [48] notion of lossy TDFs in the public-key setting asks for an alternative "sibling" key-generation algorithm, producing a public key but no secret key, such that two conditions hold. The first, which is combinatorial, asks that the functions defined by sibling keys are lossy. The second, which is computational, asks that real and sibling keys are indistinguishable. The first change for the IB setting is that one needs an alternative parameter generation algorithm which produces not only *pars* but a master secret key *msk*, and an alternative key-generation algorithm that, based on *msk*, can issue decryption keys to users. Now we would like to ask that the function $\mathsf{F.Ev}(\mathit{pars}, id^*, \cdot)$ be lossy on the challenge identity id^* when *pars* is generated via $\mathsf{LF.Pg}$, but, in the adaptive-id case, we do not know id^* in advance. Thus the requirement is made via the games.

We would like to define the advantage normally, meaning with $\delta = 1$, but the resulting notion is not achievable in the adaptive-id case. (This can be shown via attack.) With the relaxation, a low (close to zero) advantage means that the probability that the adversary finds a lossy identity id^* and then outputs 1 is less than the probability that it merely outputs 1 by a factor not much less than δ. Roughly, it means that a δ fraction of identities are lossy. The advantage represents the computational loss while δ represents a necessary information-theortic loss.

IBE. Recall that an IBE scheme $\mathsf{IBE} = (\mathsf{IBE.Pg}, \mathsf{IBE.Kg}, \mathsf{IBE.Enc}, \mathsf{IBE.Dec})$ is a tuple of algorithms with associated message space InSp and identity space IDSp. The parameter generation algorithm $\mathsf{IBE.Pg}$ takes no input and returns common parameters *pars* and a master secret key *msk*. On input $\mathit{pars}, \mathit{msk}, id$, the key generation algorithm $\mathsf{IBE.Kg}$ produces a decryption key dk for identity id. On input pars, $id \in \mathsf{IDSp}$ and a message $M \in \mathsf{InSp}$ the encryption algorithm $\mathsf{IBE.Enc}$ returns a ciphertext. The decryption algorithm $\mathsf{IBE.Dec}$ is deterministic. The scheme has decryption error ϵ if

$\Pr[\mathsf{IBE.Dec}(pars, id, dk, \mathsf{IBE.Enc}(pars, id, M)) \neq M] \leq \epsilon$ for all $pars$, all $id \in \mathsf{IDSp}$, all $dk \in [\mathsf{F.Kg}(pars, id)]$ and all $M \in \mathsf{InSp}$. We say that IBE is deterministic if $\mathsf{IBE.Enc}$ is deterministic. A deterministic IBE scheme is identical to an IBTDF.

3 Implications of Partial Lossiness

Theorem 3.2 shows that partial lossiness implies one-wayness. We discuss other applications in Appendix B. We first need a simple lemma.

Lemma 3.1 *Let f be a function with non-empty domain $\mathrm{Dom}(f)$. Then for any adversary A*

$$\Pr[A(y) = x \,:\, x \xleftarrow{\$} \mathrm{Dom}(f)\,;\, y \leftarrow f(x)] \;\leq\; 2^{-\lambda(f)}\,. \quad \blacksquare$$

Proof of Lemma 3.1: For $y \in \mathrm{Im}(f)$ let $f^{-1}(y)$ be the set of all $x \in \mathrm{Dom}(f)$ such that $f(x) = y$. The probability in question is

$$\sum_{y \in \mathrm{Im}(f)} \Pr[\, A(y) = x \mid f(x) = y \,] \cdot \Pr[\, f(x) = y \,] \;\leq\; \sum_{y \in \mathrm{Im}(f)} \frac{1}{|f^{-1}(y)|} \cdot \frac{|f^{-1}(y)|}{|\mathrm{Dom}(f)|} \;=\; \frac{|\mathrm{Im}(f)|}{|\mathrm{Dom}(f)|} \;=\; 2^{-\lambda(f)}$$

where the probability is over x chosen at random from $\mathrm{Dom}(f)$ and the coins of A if any. (Since A is unbounded, it can be assumed wlog to be deterministic.) $\blacksquare$

Theorem 3.2 [δ-lossiness implies one-wayness] *Let $\mathsf{F} = (\mathsf{F.Pg}, \mathsf{F.Kg}, \mathsf{F.Ev}, \mathsf{F.Ev}^{-1})$ be a IBTDF with associated input space InSp. Let $\mathsf{LF} = (\mathsf{LF.Pg}, \mathsf{LF.Kg}, \mathsf{F.Ev}, \mathsf{F.Ev}^{-1})$ be a lossy sibling for F. Let $\delta > 0$ and let $\ell \geq 0$. Then for any ow-adversary I there is a los-adversary A such that*

$$\mathbf{Adv}^{\mathrm{ow}}_{\mathsf{F}}(I) \leq \frac{\mathbf{Adv}^{\delta\text{-los}}_{\mathsf{F},\mathsf{LF},\ell}(A) + 2^{-\ell}}{\delta}\,. \tag{2}$$

The running time of A is that if I plus the time for a computation of $\mathsf{F.Ev}$. If I is a selective adversary then so is A. $\blacksquare$

In asymptotic terms, the theorem says that δ-lossiness implies one-wayness as long as δ^{-1} is bounded above by a polynomial in the security parameter and ℓ is super-logarithmic. This means δ need only be non-negligible. The last sentence of the theorem, saying that if I is selective then so is A, is important because it says that the theorem covers both the selective and adaptive security cases, meaning selective δ-lossiness implies selective one-wayness and adaptive δ-lossiness implies adaptive one-wayness.

Proof of Theorem 3.2: Adversary A runs I. When I makes query $\mathbf{Initialize}(id)$, adversary A does the same, obtaining $pars$ and returning this to I. Adversary A answers I's queries to its $\mathbf{GetDK}$ oracle via its own oracle of the same name. When I makes its (single) $\mathbf{Ch}$ query id^*, adversary A also makes query $\mathbf{Ch}(id^*)$. Additionally, it picks x at random from InSp and returns $y = \mathsf{F.Ev}(pars, id^*, x)$ to I. The latter eventually halts with output x'. Adversary A returns 1 if $x' = x$ and 0 otherwise. By design we clearly have $\Pr\left[\,\mathrm{Real}^A_{\mathsf{F}}\,\right] = \mathbf{Adv}^{\mathrm{ow}}_{\mathsf{F}}(I)$. But game $\mathrm{Lossy}_{\mathsf{F},\mathsf{LF},\ell}$ returns true only if $\mathsf{F.Ev}(pars, id^*, \cdot)$ is ℓ-lossy, in which case the probability that $x = x'$ is small by Lemma 3.1. In detail, assuming wlog that I never queries id^* to $\mathbf{GetDK}$, we have

$$\begin{aligned}
\Pr\left[\,\mathrm{Lossy}^A_{\mathsf{F},\mathsf{LF},\ell}\,\right] &= \Pr\left[\, x = x' \mid \lambda(\mathsf{F.Ev}(pars, id^*, \cdot)) \geq \ell \,\right] \cdot \Pr[\, \lambda(\mathsf{F.Ev}(pars, id^*, \cdot)) \geq \ell \,] \\
&\leq \Pr\left[\, x = x' \mid \lambda(\mathsf{F.Ev}(pars, id^*, \cdot)) \geq \ell \,\right] \;\leq\; 2^{-\ell}\,,
\end{aligned}$$

the last inequality by Lemma 3.1 applied to the function $f = \mathsf{F.Ev}(pars, id^*, \cdot)$. From Equation (1) we have

$$\mathbf{Adv}^{\delta\text{-los}}_{\mathsf{F},\mathsf{LF},\ell}(A) \;=\; \delta \cdot \Pr\left[\,\mathrm{Real}^A_{\mathsf{F}}\,\right] - \Pr\left[\,\mathrm{Lossy}^A_{\mathsf{F},\mathsf{LF},\ell}\,\right] \;\geq\; \delta \cdot \mathbf{Adv}^{\mathrm{ow}}_{\mathsf{F}}(I) - 2^{-\ell}\,.$$

Equation (2) follows. $\blacksquare$ In Section B we discuss the application to deterministic and hedged IBE.

4 IB-TDFs from pairings

In Appendix 3 we show that δ-lossiness implies one-wayness in both the selective and adaptive cases. We now show how to achieve δ-lossiness using pairings.

SETUP. Throughout we fix a bilinear map $\mathbf{e}$: $\mathbb{G} \times \mathbb{G} \to \mathbb{G}_T$ where $\mathbb{G}, \mathbb{G}_T$ are groups of prime order p. By $\mathbf{1}, \mathbf{1}_T$ we denote the identity elements of $\mathbb{G}, \mathbb{G}_T$, respectively. By $\mathbb{G}^* = \mathbb{G} - \{\mathbf{1}\}$ we denote the set of generators of $\mathbb{G}$. The advantage of a dlin-adversary B is

$$\mathbf{Adv}^{\mathrm{dlin}}(B) \;=\; 2\Pr[\mathrm{DLIN}^B] - 1 \;,$$

where game DLIN is as follows. The **Initialize** procedure picks $g, \hat{g}$ at random from $\mathbb{G}^*$, s at random from $\mathbb{Z}_p^*$, $\hat{s}$ at random from $\mathbb{Z}_p$ and X at random from $\mathbb{G}$. It picks a random bit b. If $b = 1$ it lets $T \leftarrow X^{s+\hat{s}}$ and otherwise picks T at random from $\mathbb{G}$. It returns $(g, \hat{g}, g^s, \hat{g}^{\hat{s}}, X, T)$ to the adversary B. The adversary outputs a bit b' and **Finalize**, given b' returns true if $b = B'$ and false otherwise. For integer $\mu \geq 1$, vectors $\mathbf{U} \in \mathbb{G}^{\mu+1}$ and $\mathbf{y} \in \mathbb{Z}_p^{\mu+1}$, and vector $id \in \mathbb{Z}_p^{\mu}$ we let

$$\overline{id} = (1, id[1], \ldots, id[\mu]) \in \mathbb{Z}_p^{\mu+1} \qquad \text{and} \qquad \mathcal{H}(\mathbf{U}, id) = \textstyle\prod_{k=0}^{\mu} \mathbf{U}[k]^{\overline{id}[k]} \;.$$

$\mathcal{H}$ is the BB hash function [16] when $\mu = 1$, and the Waters' one [22] when $\mathsf{IDSp} = \{0,1\}^{\mu}$ and an $id \in \mathsf{IDSp}$ is viewed as an μ-vector over $\mathbb{Z}_p^{\mu}$. We also let

$$f(\mathbf{y}, id) = \textstyle\sum_{k=0}^{\mu} \mathbf{y}[k]\overline{id}[k] \qquad \text{and} \qquad \overline{f}(\mathbf{y}, id) = f(\mathbf{y}, id) \bmod p \;.$$

4.1 Overview

In the Peikert-Waters [48] design, the matrix entries are ciphertexts of an underlying homomorphic encryption scheme, and the function output is a vector of ciphertexts of the same scheme. We begin by presenting an IBE scheme, that we call the basic IBE scheme, such that the function outputs of our eventual IB-TDF will be a vector of ciphertexts of this IBE scheme. Towards building the IB-TDF, the first difficulty we run into in setting up the matrix is that ciphertexts depend on the identity and we cannot have a different matrix for every identity. Thus, our approach is more intrusive. We will have many matrices which contain certain "atoms" from which, given an identity, one can reconstruct ciphertexts of the IBE scheme. The result of this intrusive approach is that security of the IB-TDF relies on more than security of the base IBE scheme. Our ciphertext pseudorandomness lemma (Lemma 4.1) shows something stronger, namely that even the atoms from which the ciphertexts are created look random under DLIN. This will be used to establish Lemma 4.2, which moves from the real to the lossy setup. The heart of the argument is the proofs of the lemmas, which are in the appendices.

We introduce and use a general framework that allows us to treat both the selective-id and adaptive-id cases in as unified a way as possible. We will first specify an E-IBTDF. The selective-id and adaptive-id IB-TDFs are obtained via different auxiliary inputs. Furthermore, the siblings used to prove lossiness also emanate from this E-IBTDF. With this approach, the main lemmas become usable in both the selective-id and adaptive-id cases with only minor adjustments for the latter due to artifical aborts. At the cost of some complexity, this approach eventually saves us from repeating similar arguments and significantly compacts the proof.

4.2 Our basic IBE scheme

We associate to any integer $\mu \geq 1$ and any identity space $\mathsf{IDSp} \subseteq \mathbb{Z}_p^{\mu}$ an IBE scheme $\mathsf{IBE}[\mu, \mathsf{IDSp}]$ that has message space $\{0,1\}$ and algorithms as follows:

1. Parameters: Algorithm $\mathsf{IBE}[\mu, \mathsf{IDSp}].\mathsf{Pg}$ lets $g \xleftarrow{\$} \mathbb{G}^*$; $t \xleftarrow{\$} \mathbb{Z}_p^*$; $\hat{g} \leftarrow g^t$. It then lets $H, \hat{H} \xleftarrow{\$} \mathbb{G}$; $\mathbf{U}, \hat{\mathbf{U}} \xleftarrow{\$} \mathbb{G}^{\mu+1}$. It returns $pars = (g, \hat{g}, H, \hat{H}, \mathbf{U}, \hat{\mathbf{U}})$ as the public parameters and $msk = t$ as the master secret key.

2. Key generation: Given parameters $(g, \hat{g}, H, \hat{H}, \mathbf{U}, \hat{\mathbf{U}})$, master secret t and identity $id \in \mathsf{IDSp}$, algorithm $\mathsf{IBE}[\mu, \mathsf{IDSp}].\mathsf{Kg}$ returns decryption key (D_1, D_2, D_3, D_4) computed by letting $r, \hat{r} \xleftarrow{\$} \mathbb{Z}_p$ and setting

$$D_1 \leftarrow \mathcal{H}(\mathbf{U}, id)^{tr} \cdot H^{t\hat{r}} \; ; \; D_2 \leftarrow \mathcal{H}(\hat{\mathbf{U}}, id)^{r} \cdot \hat{H}^{\hat{r}} \; ; \; D_3 \leftarrow g^{-tr} \; ; \; D_4 \leftarrow g^{-t\hat{r}} \; .$$

3. Encryption: Given parameters $(g, \hat{g}, H, \hat{H}, \mathbf{U}, \hat{\mathbf{U}})$, identity $id \in \mathsf{IDSp}$ and message $M \in \{0,1\}$, algorithm $\mathsf{IBE}[\mu, \mathsf{IDSp}].\mathsf{Enc}$ returns ciphertext (C_1, C_2, C_3, C_4) computed as follows. If $M = 0$ then it lets $s, \hat{s} \xleftarrow{\$} \mathbb{Z}_p$ and $C_1 \leftarrow g^s$; $C_2 \leftarrow \hat{g}^{\hat{s}}$; $C_3 \leftarrow \mathcal{H}(\mathbf{U}, id)^s \cdot \mathcal{H}(\hat{\mathbf{U}}, id)^{\hat{s}}$; $C_4 \leftarrow H^s \hat{H}^{\hat{s}}$. If $M = 1$ it lets $C_1, C_2, C_3, C_4 \xleftarrow{\$} \mathbb{G}$.

4. Decryption: Given parameters $(g, \hat{g}, H, \hat{H}, \mathbf{U}, \hat{\mathbf{U}})$, identity $id \in \mathsf{IDSp}$, decryption key (D_1, D_2, D_4, D_4) for id and ciphertext (C_1, C_2, C_3, C_4), algorithm $\mathsf{IBE}[\mu, \mathsf{IDSp}].\mathsf{Dec}$ returns 0 if $\mathbf{e}(C_1, D_1)\mathbf{e}(C_2, D_2)$ $\mathbf{e}(C_3, D_3)\mathbf{e}(C_4, D_4) = \mathbf{1}_T$ and 1 otherwise.

This scheme has non-zero decryption error (at most $2/p$) yet our IBTDF will have zero inversion error. This scheme turns out to be IND-CPA+ANON-CPA although we will not need this in what follows. Instead we will have to consider a distinguishing game related to this IBE scheme and our IBTDF. In Appendix A we give a (more natural) variant of $\mathsf{IBE}[\mu, \mathsf{IDSp}]$ that is more efficient and has exponential message space (instead of just bits). The improved IBE scheme can still be proved IND-CPA+ANON-CPA but it cannot be used for our purpose of building IB-TDFs.

4.3 Our E-IBTDF and IB-TDF

Our E-IBTDF $\overline{\mathsf{E}}[n, \mu, \mathsf{IDSp}]$ is associated to any integers $n, \mu \geq 1$ and any identity space $\mathsf{IDSp} \subseteq \mathbb{Z}_p^{\mu}$. It has message space $\{0,1\}^n$ and auxiliary input space $\mathbb{Z}_p^{\mu+1}$, and the algorithms are as follows:

1. Parameters: Given auxiliary input $\mathbf{y}$, algorithm $\overline{\mathsf{E}}[n, \mu, \mathsf{IDSp}].\mathsf{Pg}$ lets $g \xleftarrow{\$} \mathbb{G}^*$; $t \xleftarrow{\$} \mathbb{Z}_p^*$; $\hat{g} \leftarrow g^t$; $U \xleftarrow{\$} \mathbb{G}^*$. It then lets $\mathbf{H}, \hat{\mathbf{H}} \xleftarrow{\$} \mathbb{G}^n$; $\mathbf{V}, \hat{\mathbf{V}} \xleftarrow{\$} \mathbb{G}^{n \times (\mu+1)}$ and $\mathbf{s} \xleftarrow{\$} (\mathbb{Z}_p^*)^n$; $\hat{\mathbf{s}} \xleftarrow{\$} \mathbb{Z}_p^n$. It returns $pars = (g, \hat{g}, \mathbf{G}, \hat{\mathbf{G}}, \mathbf{J}, \mathbf{W}, \mathbf{H}, \hat{\mathbf{H}}, \mathbf{V}, \hat{\mathbf{V}}, U)$ as the public parameters and $msk = t$ as the master secret key where for $1 \leq i, j \leq n$ and $0 \leq k \leq \mu$

$$\mathbf{G}[i] \leftarrow g^{\mathbf{s}[i]} \; ; \; \hat{\mathbf{G}}[i] \leftarrow \hat{g}^{\hat{\mathbf{s}}[i]} \; ; \; \mathbf{J}[i,j] \leftarrow \mathbf{H}[j]^{\mathbf{s}[i]} \hat{\mathbf{H}}[j]^{\hat{\mathbf{s}}[i]} \; ; \; \mathbf{W}[i,j,k] \leftarrow \mathbf{V}[j,k]^{\mathbf{s}[i]} \hat{\mathbf{V}}[j,k]^{\hat{\mathbf{s}}[i]} U^{\mathbf{s}[i]\mathbf{y}[k]\Delta(i,j)} \; ,$$

where we recall that $\Delta(i,j) = 1$ if $i = j$ and 0 otherwise is the Kronecker Delta function.

2. Key generation: Given parameters $(g, \hat{g}, \mathbf{G}, \hat{\mathbf{G}}, \mathbf{J}, \mathbf{W}, \mathbf{H}, \hat{\mathbf{H}}, \mathbf{V}, \hat{\mathbf{V}}, U)$, master secret t and identity $id \in \mathsf{IDSp}$, algorithm $\overline{\mathsf{E}}[n, \mu, \mathsf{IDSp}].\mathsf{Kg}$ returns decryption key $(\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3, \mathbf{D}_4)$ where $\mathbf{r} \xleftarrow{\$} (\mathbb{Z}_p^*)^n$; $\hat{\mathbf{r}} \xleftarrow{\$} \mathbb{Z}_p^n$ and for $1 \leq i \leq n$

$$\mathbf{D}_1[i] \leftarrow \mathcal{H}(\mathbf{V}, id)^{t\mathbf{r}[i]} \cdot \mathbf{H}[i]^{t\hat{\mathbf{r}}[i]} \; ; \; \mathbf{D}_2[i] \leftarrow \mathcal{H}(\hat{\mathbf{V}}, id)^{\mathbf{r}[i]} \cdot \hat{H}[i]^{\hat{\mathbf{r}}[i]} \; ; \; \mathbf{D}_3[i] \leftarrow g^{-t\mathbf{r}[i]} \; ; \; \mathbf{D}_4[i] \leftarrow g^{-t\hat{\mathbf{r}}[i]} \; .$$

3. Evaluate: Given parameters $(g, \hat{g}, \mathbf{G}, \hat{\mathbf{G}}, \mathbf{J}, \mathbf{W}, \mathbf{H}, \hat{\mathbf{H}}, \mathbf{V}, \hat{\mathbf{V}}, U)$, identity $id \in \mathsf{IDSp}$ and input $x \in \{0,1\}^n$, algorithm $\overline{\mathsf{E}}[n, \mu, \mathsf{IDSp}].\mathsf{Ev}$ returns output $(C_1, C_2, \mathbf{C}_3, \mathbf{C}_4)$ where for $1 \leq j \leq n$

$$C_1 \leftarrow \prod_{i=1}^n \mathbf{G}[i]^{x[i]} \; ; \; C_2 \leftarrow \prod_{i=1}^n \hat{\mathbf{G}}[i]^{x[i]} \; ; \; \mathbf{C}_3[j] \leftarrow \prod_{i=1}^n \prod_{k=0}^{\mu} \mathbf{W}[i,j,k]^{x[i]\overline{id}[k]} \; ; \; \mathbf{C}_4[j] \leftarrow \prod_{i=1}^n \mathbf{J}[i,j]^{x[i]}$$

4. Invert: Given parameters $(g, \hat{g}, \mathbf{G}, \hat{\mathbf{G}}, \mathbf{J}, \mathbf{W}, \mathbf{H}, \hat{\mathbf{H}}, \mathbf{V}, \hat{\mathbf{V}}, U)$, identity $id \in \mathsf{IDSp}$, decryption key $(\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3, \mathbf{D}_4)$ for id and output (ciphertext) $(C_1, C_2, \mathbf{C}_3, \mathbf{C}_4)$, algorithm $\overline{\mathsf{E}}[n, \mu, \mathsf{IDSp}].\mathsf{Ev}^{-1}$ returns $x \in \{0,1\}^n$ where for $1 \leq j \leq n$ it sets $x[j] = 0$ if $\mathbf{e}(C_1, \mathbf{D}_1[j])\mathbf{e}(C_2, \mathbf{D}_2[j])\mathbf{e}(\mathbf{C}_3[j], \mathbf{D}_3[j])\mathbf{e}(\mathbf{C}_4[j], \mathbf{D}_4[j]) = \mathbf{1}_T$ and 1 otherwise.

INVERTIBILITY. We observe that if parameters $(g, \hat{g}, \mathbf{G}, \hat{\mathbf{G}}, \mathbf{J}, \mathbf{W}, \mathbf{H}, \hat{\mathbf{H}}, \mathbf{V}, \hat{\mathbf{V}}, U)$ were generated with auxiliary input $\mathbf{y}$ and $(C_1, C_2, \mathbf{C}_3, \mathbf{C}_4) = \overline{\mathsf{E}}[n, \mu, \mathsf{IDSp}].\mathsf{Ev}((g, \hat{g}, \mathbf{G}, \hat{\mathbf{G}}, \mathbf{J}, \mathbf{W}), id, x)$ then for $1 \le j \le n$

$$
\begin{aligned}
C_1 &= \textstyle\prod_{i=1}^{n} g^{\mathbf{s}[i]x[i]} = g^{\langle \mathbf{s}, x\rangle} && (3)\\
C_2 &= \textstyle\prod_{i=1}^{n} \hat{g}^{\hat{\mathbf{s}}[i]x[i]} = \hat{g}^{\langle \hat{\mathbf{s}}, x\rangle} && (4)\\
\mathbf{C}_3[j] &= \textstyle\prod_{i=1}^{n}\prod_{k=0}^{\mu} \mathbf{V}[j,k]^{\mathbf{s}[i]x[i]\overline{id}[k]}\hat{\mathbf{V}}[j,k]^{\hat{\mathbf{s}}[i]x[i]\overline{id}[k]} U^{\mathbf{s}[i]x[i]\mathbf{y}[k]\overline{id}[k]\Delta(i,j)}\\
&= \textstyle\prod_{i=1}^{n} \mathcal{H}(\mathbf{V}[j,\cdot], id)^{\mathbf{s}[i]x[i]}\mathcal{H}(\hat{\mathbf{V}}[j,\cdot], id)^{\hat{\mathbf{s}}[i]x[i]} U^{\mathbf{s}[i]x[i]f(\mathbf{y},id)\Delta(i,j)}\\
&= \mathcal{H}(\mathbf{V}[j,\cdot], id)^{\langle \mathbf{s}, x\rangle}\mathcal{H}(\hat{\mathbf{V}}[j,\cdot], id)^{\langle \hat{\mathbf{s}}, x\rangle} U^{\mathbf{s}[j]x[j]f(\mathbf{y},id)} && (5)\\
\mathbf{C}_4[j] &= \textstyle\prod_{i=1}^{n} \mathbf{H}[j]^{\mathbf{s}[i]x[i]}\hat{\mathbf{H}}[j]^{\hat{\mathbf{s}}[i]x[i]} = \mathbf{H}[j]^{\langle \mathbf{s}, x\rangle}\hat{\mathbf{H}}[j]^{\langle \hat{\mathbf{s}}, x\rangle}\ . && (6)
\end{aligned}
$$

Thus if $x[j] = 0$ then $(C_1, C_2, \mathbf{C}_3[j], \mathbf{C}_4[j])$ is an encryption, under our base IBE scheme, of the message 0, with coins $\langle \mathbf{s}, x\rangle \bmod p, \langle \hat{\mathbf{s}}, x\rangle \bmod p$, parameters $(g, \hat{g}, \mathbf{H}[j], \hat{\mathbf{H}}[j], \mathbf{V}[j,\cdot], \hat{\mathbf{V}}[j,\cdot])$ and identity id. The inversion algorithm will thus correctly recover $x[j] = 0$. On the other hand suppose $x[j] = 1$. Then $\mathbf{e}(C_1, \mathbf{D}_1[j])\mathbf{e}(C_2, \mathbf{D}_2[j])\mathbf{e}(\mathbf{C}_3[j], \mathbf{D}_3[j])\mathbf{e}(\mathbf{C}_4[j], \mathbf{D}_4[j]) = \mathbf{e}(U^{\mathbf{s}[j]x[j]f(\mathbf{y},id)}, \mathbf{D}_3[j])$. Now suppose $f(\mathbf{y}, id) \bmod p \neq 0$. Then $U^{\mathbf{s}[j]x[j]f(\mathbf{y},id)} \neq \mathbf{1}$ because we chose $\mathbf{s}[j]$ to be non-zero modulo p and $\mathbf{D}_3[j] \neq \mathbf{1}$ because we chose $\mathbf{r}[j]$ to be non-zero modulo p. So the result of the pairing is never $\mathbf{1}_T$, meaning the inversion algorithm will again correctly recover $x[j] = 1$. We have established that auxiliary input $\mathbf{y}$ grants invertibility, meaning induced IBTDF $\overline{\mathsf{E}}[n, \mu, \mathsf{IDSp}](\mathbf{y})$ satisfies the correct inversion condition, if $f(\mathbf{y}, id) \bmod p \neq 0$ for all $id \in \mathsf{IDSp}$.

OUR IBTDF. We associate to any integers $n, \mu \ge 1$ and any identity space $\mathsf{IDSp} \subseteq \mathbb{Z}_p^{\mu}$ the IBTDF scheme induced by our E-IBTDF $\overline{\mathsf{E}}[n, \mu, \mathsf{IDSp}]$ via auxiliary input $\mathbf{y} = (1, 0, \ldots, 0) \in \mathbb{Z}_p^{\mu+1}$, and denote this IBTDF scheme by $\overline{\mathsf{F}}[n, \mu, \mathsf{IDSp}]$. This IBTDF satisfies the correct inversion requirement because $f(\mathbf{y}, id) = \overline{id}[0] = 1 \not\equiv 0 \pmod p$ for all id. We will show that this IBTDF is selective-id secure when $\mu = 1$ and $\mathsf{IDSp} = \mathbb{Z}_p$, and adaptive-id secure when $\mathsf{IDSp} = \{0,1\}^{\mu}$. In the first case, it is fully lossy (i.e. 1-lossy) and in the second it is δ-lossy for appropriate δ. First we prove two technical lemmas that we will use in both cases.

4.4 Ciphertext pseudorandomness lemma

Consider games ReC, RaC of Figure 3 associated to some choice of $\mathsf{IDSp} \subseteq \mathbb{Z}_p^{\mu}$. The adversary provides the **Initialize** procedure with an auxiliary input $\mathbf{y} \in \mathbb{Z}_p^{\mu+1}$. Parameters are generated as per our base IBE scheme with the addition of U. The decryption key for id is computed as per our base IBE scheme except that the games refuse to provide it when $f(\mathbf{y}, id) = 0$. The challenge oracle, however, does not return ciphertexts of our IBE scheme. In game ReC, it returns group elements that resemble diagonal entries of the matrices in the parameters of our E-IBTDF, and in game RaC it returns random group elements. Notice that the challenge oracle does not take an identity as input. (Indeed, it has no input.) As usual it must be invoked exactly once. The following lemma says the games are indistinguishable under DLIN. The proof is in Appendix 4.7.

Lemma 4.1 *Let $\mu \ge 1$ be an integer and $\mathsf{IDSp} \subseteq \mathbb{Z}_p^{\mu}$. Let P be an adversary. Then there is an adversary B such that*

$$\Pr\left[\,\mathrm{ReC}^P\,\right] - \Pr\left[\,\mathrm{RaC}^P\,\right] \;\le\; (\mu + 2) \cdot \mathbf{Adv}^{\mathrm{dlin}}(B)\ . \qquad (7)$$

The running time of B is that of P plus some overhead.

4.5 Proof of Lemma 4.2

Consider the games of Figure 4. Game $\mathrm{RL}_{l,b}$ makes the diagonal entries of $\mathbf{W}$ (namely all the $\mu + 1$ entries with $i = j$) random for $i \le l$ and otherwise makes them using $\mathbf{y}_b$. Game $\mathrm{RL}_{0,1}$ is the same as

proc Initialize($\mathbf{y}$) // ReC, RaC
$(pars, msk) \xleftarrow{\$} \mathsf{IBE}[\mu, \mathsf{IDSp}].\mathsf{Pg}$
$(g, \hat{g}, H, \hat{H}, \mathbf{U}, \hat{\mathbf{U}}) \leftarrow pars$
$U \xleftarrow{\$} \mathbb{G}^*$
Return $(g, \hat{g}, H, \hat{H}, \mathbf{U}, \hat{\mathbf{U}}, U)$

proc GetDK(id) // ReC, RaC
If $f(\mathbf{y}, id) = 0$ then $dk \leftarrow \perp$
Else $dk \leftarrow \mathsf{IBE}[\mu, \mathsf{IDSp}].\mathsf{Kg}(pars, msk, id)$
Return dk

proc Ch() // ReC
$s \xleftarrow{\$} \mathbb{Z}_p^*$; $\hat{s} \xleftarrow{\$} \mathbb{Z}_p$; $G \leftarrow g^s$; $\hat{G} \leftarrow \hat{g}^{\hat{s}}$; $S \leftarrow H^s \hat{H}^{\hat{s}}$
For $k = 0, \ldots, \mu$ do $\mathbf{Z}[k] \leftarrow (U^{\mathbf{y}[k]} \mathbf{U}[k])^s \hat{\mathbf{U}}[k]^{\hat{s}}$
Return $(G, \hat{G}, S, \mathbf{Z})$

proc Ch() // RaC
$G, \hat{G}, S \xleftarrow{\$} \mathbb{G}$; $\mathbf{Z} \xleftarrow{\$} \mathbb{G}^\mu$
Return $(G, \hat{G}, S, \mathbf{Z})$

proc Finalize(d') // ReC, RaC
Return $(d' = 1)$

Figure 3: Games ReC ("Real Ciphertexts") and RaC ("Random Ciphertexts") associated to $\mathsf{IDSp} \subseteq \mathbb{Z}_p^\mu$.

proc Initialize(id)
$\mathbf{y}_0 \xleftarrow{\$} \mathsf{Aux}(id)$; $\mathbf{y}_1 \leftarrow (1, 0, \ldots, 0)$; $\text{WIN} \leftarrow \mathsf{true}$
$g \xleftarrow{\$} \mathbb{G}^*$; $t \xleftarrow{\$} \mathbb{Z}_p^*$; $\hat{g} \leftarrow g^t$; $U \xleftarrow{\$} \mathbb{G}^*$
$\mathbf{H}, \hat{\mathbf{H}} \xleftarrow{\$} \mathbb{G}^n$; $\mathbf{V}, \hat{\mathbf{V}} \xleftarrow{\$} \mathbb{G}^{n \times (\mu+1)}$; $\mathbf{s} \xleftarrow{\$} (\mathbb{Z}_p^*)^n$; $\hat{\mathbf{s}} \xleftarrow{\$} \mathbb{Z}_p^n$
For $i = 1, \ldots, n$ do
  $\mathbf{G}[i] \leftarrow g^{\mathbf{s}[i]}$; $\hat{\mathbf{G}}[i] \leftarrow \hat{g}^{\hat{\mathbf{s}}[i]}$
  For $j = 1, \ldots, n$ do
    $\mathbf{J}[i, j] \leftarrow \mathbf{H}[j]^{\mathbf{s}[i]} \hat{\mathbf{H}}[j]^{\hat{\mathbf{s}}[i]}$
    For $k = 0, \ldots, \mu$ do
      If ($i = j$ and $i \le l$) then $\mathbf{W}[i, j, k] \xleftarrow{\$} \mathbb{G}$
      Else $\mathbf{W}[i, j, k] \leftarrow \mathbf{V}[j, k]^{\mathbf{s}[i]} \hat{\mathbf{V}}[j, k]^{\hat{\mathbf{s}}[i]} U^{\mathbf{s}[i] \mathbf{y}_b[k] \Delta(i,j)}$
$pars \leftarrow (g, \hat{g}, \mathbf{G}, \hat{\mathbf{G}}, \mathbf{J}, \mathbf{W}, \mathbf{H}, \hat{\mathbf{H}}, \mathbf{V}, \hat{\mathbf{V}}, U)$; $msk \leftarrow t$
$IS \leftarrow \emptyset$; $id^* \leftarrow id$
Return $pars$

proc GetDK(id)
$IS \leftarrow IS \cup \{id\}$
If $f(\mathbf{y}_0, id) = 0$ then $\text{WIN} \leftarrow \mathsf{false}$; $dk \leftarrow \perp$
Else $dk \leftarrow \overline{\mathsf{E}}[n, \mu, \mathsf{IDSp}].\mathsf{Kg}(pars, msk, id)$
Return dk

proc Ch(id)
$id^* \leftarrow id$
If $f(\mathbf{y}_0, id) \neq 0$ then $\text{WIN} \leftarrow \mathsf{false}$

proc Finalize(d')
Return $((d' = 1)$ and $(id^* \notin IS)$ and WIN$)$

Figure 4: Games $\text{RL}_{l,b}$ ($0 \le l \le n$ and $b \in \{0, 1\}$) associated to $n, \mu, \mathsf{IDSp}, \mathsf{Aux}$ for proof of Lemma 4.2.

game RL_0 and game $\text{RL}_{0,0}$ is the same as game RL_n. Games $\text{RL}_{n,0}, \text{RL}_{n,1}$ are identical, both making all diagonal entries of $\mathbf{W}$ random. Thus we have

$$\Pr[\text{RL}_0^A] - \Pr[\text{RL}_n^A] \;=\; \big(\Pr[\text{RL}_{0,1}^A] - \Pr[\text{RL}_{n,1}^A]\big) + \big(\Pr[\text{RL}_{n,0}^A] - \Pr[\text{RL}_{0,0}^A]\big) \;.$$

We will design adversaries P_0, P_1 so that

$$\Pr[\text{ReC}^{P_0}] - \Pr[\text{RaC}^{P_0}] \;=\; \frac{1}{n} \cdot \big(\Pr[\text{RL}_{n,0}^A] - \Pr[\text{RL}_{0,0}^A]\big) \tag{8}$$

$$\Pr[\text{ReC}^{P_1}] - \Pr[\text{RaC}^{P_1}] \;=\; \frac{1}{n} \cdot \big(\Pr[\text{RL}_{0,1}^A] - \Pr[\text{RL}_{n,1}^A]\big) \;. \tag{9}$$

Adversary P picks $b \xleftarrow{\$} \{0, 1\}$ and runs P_b. This yields Equation (10). Now we present adversary P_b ($b \in \{0, 1\}$). It runs adversary A, responding to its oracle queries as follows.

When A makes query **Initialize**(id), adversary P_b begins with

$l \xleftarrow{\$} \{1, \ldots, n\}$; $\mathbf{y}_0 \xleftarrow{\$} \mathsf{Aux}(id)$; $\mathbf{y}_1 \leftarrow (1, 0, \ldots, 0)$; $\text{WIN}_A \leftarrow \mathsf{true}$; $IS_A \leftarrow \emptyset$
$(g, \hat{g}, H, \hat{H}, \mathbf{U}, \hat{\mathbf{U}}, U) \xleftarrow{\$}$ **Initialize**($\mathbf{y}_b$) ; $(G, \hat{G}, S, Z) \xleftarrow{\$}$ **Ch**().

Here P_b has called its own **Initialize** procedure with input $\mathbf{y}_b$ and then called its **Ch** procedure. Now it creates parameters $pars$ for A as follows:

$\mathbf{h}, \hat{\mathbf{h}} \xleftarrow{\$} \mathbb{Z}_p^n$; $\mathbf{v}, \hat{\mathbf{v}} \xleftarrow{\$} \mathbb{Z}_p^{n\times(\mu+1)}$; $\mathbf{s} \xleftarrow{\$} (\mathbb{Z}_p^*)^n$; $\hat{\mathbf{s}} \xleftarrow{\$} \mathbb{Z}_p^n$
For $i = 1, \ldots, n$ do
  If $(i = l)$ then $\mathbf{H}[i] \leftarrow H$; $\hat{\mathbf{H}}[i] \leftarrow \hat{H}$; $\mathbf{G}[i] \leftarrow G$; $\hat{\mathbf{G}}[i] \leftarrow \hat{G}$
  If $(i \neq l)$ then $\mathbf{H}[i] \leftarrow g^{\mathbf{h}[i]}$; $\hat{\mathbf{H}}[i] \leftarrow \hat{g}^{\hat{\mathbf{h}}[i]}$; $\mathbf{G}[i] \leftarrow g^{\mathbf{s}[i]}$; $\hat{\mathbf{G}}[i] \leftarrow \hat{g}^{\hat{\mathbf{s}}[i]}$
  For $k = 0, \ldots, \mu$ do
    If $(i = l)$ then $\mathbf{V}[i,k] \leftarrow \mathbf{U}[k]$; $\hat{\mathbf{V}}[i,k] \leftarrow \hat{\mathbf{U}}[k]$
    If $(i \neq l)$ then $\mathbf{V}[i,k] \leftarrow g^{\mathbf{v}[i,k]}$; $\hat{\mathbf{V}}[i,k] \leftarrow \hat{g}^{\hat{\mathbf{v}}[i,k]}$
For $i = 1, \ldots, n$ do
  For $j = 1, \ldots, n$ do
    If $(i = l$ and $j = i)$ then $\mathbf{J}[i,j] \leftarrow S$
    If $(i = l$ and $j \neq i)$ then $\mathbf{J}[i,j] \leftarrow G^{\mathbf{h}[j]}\hat{G}^{\hat{\mathbf{h}}[j]}$
    If $(i \neq l)$ then $\mathbf{J}[i,j] \leftarrow \mathbf{H}[j]^{\mathbf{s}[i]}\hat{\mathbf{H}}[j]^{\hat{\mathbf{s}}[i]}$
    For $k = 0, \ldots, \mu$ do
      If $(i = j$ and $i \leq l-1)$ then $\mathbf{W}[i,j,k] \xleftarrow{\$} \mathbb{G}$
      If $(i = j$ and $i = l)$ then $\mathbf{W}[i,j,k] \leftarrow \mathbf{Z}[k]$
      Else $\mathbf{W}[i,j,k] \leftarrow \mathbf{V}[j,k]^{\mathbf{s}[i]}\hat{\mathbf{V}}[j,k]^{\hat{\mathbf{s}}[i]}U^{\mathbf{s}[i]\mathbf{y}_b[k]\Delta(i,j)}$
$pars \leftarrow (g, \hat{g}, \mathbf{G}, \hat{\mathbf{G}}, \mathbf{J}, \mathbf{W}, \mathbf{H}, \hat{\mathbf{H}}, \mathbf{V}, \hat{\mathbf{V}}, U)$

It returns *pars* to A.

When adversary A makes query $\mathbf{GetDK}(id)$, adversary P_b proceeds as follows. In this code, **GetDK** is P_b's own oracle:

$IS_A \leftarrow IS_A \cup \{id\}$
If $f(\mathbf{y}_0, id) = 0$ then $\text{WIN}_A \leftarrow \mathsf{false}$; $dk \leftarrow \perp$
Else
  $(D_1, D_2, D_3, D_4) \xleftarrow{\$} \mathbf{GetDK}(id)$
  $\mathbf{r}' \xleftarrow{\$} (\mathbb{Z}_p^*)^n$; $\hat{\mathbf{r}}' \xleftarrow{\$} \mathbb{Z}_p^n$
  For $i = 1, \ldots, n$ do
    If $i = l$ then $(\mathbf{D}_1[i], \mathbf{D}_2[i], \mathbf{D}_3[i], \mathbf{D}_4[i]) \leftarrow D$
    Else
      $\mathbf{D}_1[i] \leftarrow \mathcal{H}(\mathbf{V}[i,\cdot], id)^{\mathbf{r}'[i]}\mathbf{H}[i]^{\hat{\mathbf{r}}'[i]}$; $\mathbf{D}_2[i] \leftarrow g^{f(\hat{\mathbf{v}}, id)\mathbf{r}'[i]}g^{\hat{\mathbf{h}}[i]\hat{\mathbf{r}}[i]}$
      $\mathbf{D}_3[i] \leftarrow g^{-\mathbf{r}'[i]}$; $\mathbf{D}_4[i] \leftarrow g^{-\hat{\mathbf{r}}'[i]}$
  $dk \leftarrow (\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3, \mathbf{D}_4)$

It returns dk to A. Notice that P's invocation of **GetDK** will never return $\perp$. In the case $b = 1$ this is true because $f(\mathbf{y}_1, \cdot) = 1 \neq 0$. In the case $b = 0$ it is true because the case $f(\mathbf{y}_0, id) = 0$ was excluded by the If statement. To justify the above simulation, define $\mathbf{r}, \hat{\mathbf{r}}$ by $\mathbf{r}[i] = \mathbf{r}'[i]/t$ and $\hat{\mathbf{r}}[i] = \hat{\mathbf{r}}'[i]/t$ for $i \neq l$ and $\mathbf{r}[l], \hat{\mathbf{r}}[l]$ as the randomness underlying (D_1, D_2, D_3, D_4). Then think of $\mathbf{r}, \hat{\mathbf{r}}$ as the randomness used by the real key generation algorithm. Here t is the secret key, so that $\hat{g} = g^t$.

When adversary A makes query $\mathbf{Ch}(id)$, adversary P_b proceeds as follows:

$id^* \leftarrow id$
If $f(\mathbf{y}_0, id) \neq 0$ then $\text{WIN}_A \leftarrow \mathsf{false}$.

Finally, A halts with output d'. Adversaries P_0, P_1 compute their output differently. Adversary P_1 returns 1 if

$$(d' = 1) \text{ and } id^* \notin IS_A \text{ and } \text{WIN}_A$$

and 0 otherwise. Adversary P_0 does the opposite, returning 0 if the above condition is true and 1

proc Initialize(id) // RL_0

$\mathbf{y}_0 \xleftarrow{\$} \mathsf{Aux}(id)$; $\mathbf{y}_1 \leftarrow (1, 0, \ldots, 0)$; $\textsc{Win} \leftarrow \mathsf{true}$
$(pars, msk) \xleftarrow{\$} \overline{\mathsf{E}}[n, \mu, \mathsf{IDSp}].\mathsf{Pg}(\mathbf{y}_1)$
$IS \leftarrow \emptyset$; $id^* \leftarrow id$
Return $pars$

proc Initialize(id) // RL_n

$\mathbf{y}_0 \xleftarrow{\$} \mathsf{Aux}(id)$; $\mathbf{y}_1 \leftarrow (1, 0, \ldots, 0)$; $\textsc{Win} \leftarrow \mathsf{true}$
$(pars, msk) \xleftarrow{\$} \overline{\mathsf{E}}[n, \mu, \mathsf{IDSp}].\mathsf{Pg}(\mathbf{y}_0)$
$IS \leftarrow \emptyset$; $id^* \leftarrow id$
Return $pars$

proc GetDK(id) // $\mathrm{RL}_0, \mathrm{RL}_n$

$IS \leftarrow IS \cup \{id\}$
If $f(\mathbf{y}_0, id) = 0$ then $\textsc{Win} \leftarrow \mathsf{false}$; $dk \leftarrow \perp$
Else $dk \leftarrow \overline{\mathsf{E}}[n, \mu, \mathsf{IDSp}].\mathsf{Kg}(pars, msk, id)$
Return dk

proc Ch(id) // $\mathrm{RL}_0, \mathrm{RL}_n$

$id^* \leftarrow id$
If $f(\mathbf{y}_0, id) \neq 0$ then $\textsc{Win} \leftarrow \mathsf{false}$

proc Finalize(d') // $\mathrm{RL}_0, \mathrm{RL}_n$

Return $((d' = 1)$ and $(id^* \notin IS)$ and $\textsc{Win})$

Figure 5: Games $\mathrm{RL}_0, \mathrm{RL}_n$ ("Real-to-Losssy") associated to $n, \mu, \mathsf{IDSp} \subseteq \mathbb{Z}_p^\mu$ and auxiliary input generator algorithm Aux.

otherwise. We obtain Equations (8), (9) as follows:

$$\begin{aligned}
\Pr[\mathrm{ReC}^{P_1}] - \Pr[\mathrm{RaC}^{P_1}] &= \frac{1}{n}\sum_{l=1}^{n} \Pr[\mathrm{RL}^A_{l-1,1}] - \Pr[\mathrm{RL}^A_{l,1}] \\
&= \Pr[\mathrm{RL}^A_{0,1}] - \Pr[\mathrm{RL}^A_{n,1}] \\
\Pr[\mathrm{ReC}^{P_0}] - \Pr[\mathrm{RaC}^{P_0}] &= \frac{1}{n}\sum_{l=1}^{n} (1 - \Pr[\mathrm{RL}^A_{l-1,0}]) - (1 - \Pr[\mathrm{RL}^A_{l,0}]) \\
&= \frac{1}{n}\sum_{l=1}^{n} \Pr[\mathrm{RL}^A_{l,0}] - \Pr[\mathrm{RL}^A_{l-1,0}] \\
&= \Pr[\mathrm{RL}^A_{n,0}] - \Pr[\mathrm{RL}^A_{0,0}] \; .
\end{aligned}$$

4.6 Real-to-lossy lemma

Consider games $\mathrm{RL}_0, \mathrm{RL}_n$ of Figure 5 associated to some choice of $n, \mu, \mathsf{IDSp} \subseteq \mathbb{Z}_p^\mu$ and auxiliary input generator Aux for $\overline{\mathsf{E}}[n, \mu, \mathsf{IDSp}]$. The latter is an algorithm that takes input an identity in IDSp and returns an auxiliary input in $\mathbb{Z}_p^{\mu+1}$. Game RL_0 obtains an auxiliary input $\mathbf{y}_0$ via Aux but generates parameters exactly as $\overline{\mathsf{E}}[n, \mu, \mathsf{IDSp}].\mathsf{Pg}$ with the real auxiliary input $\mathbf{y}_1$. The game will return true under the same condition as game Real but additionally requiring that $f(\mathbf{y}_0, id) \neq 0$ for all $\mathbf{GetDK}(id)$ queries and $f(\mathbf{y}_0, id) = 0$ for the $\mathbf{Ch}(id)$ query. Game RL_n generates parameters with the auxiliary input provided by Aux but is otherwise identical to game RL_0. The following lemma says it is hard to distinguish these games. We will apply this by defining Aux in such a way that its output $\mathbf{y}_0$ results in a lossy setup. The proof of the following is in Appendix 4.5.

Lemma 4.2 *Let $n, \mu \geq 1$ be integers and $\mathsf{IDSp} \subseteq \mathbb{Z}_p^\mu$. Let Aux be an auxiliary input generator for $\overline{\mathsf{E}}[n, \mu, \mathsf{IDSp}]$ and A an adversary. Then there is an adversary P such that*

$$\Pr[\mathrm{RL}^A_0] - \Pr[\mathrm{RL}^A_n] \;\leq\; 2n \cdot \left(\Pr\left[\,\mathrm{ReC}^P\,\right] - \Pr\left[\,\mathrm{RaC}^P\,\right]\right) \; . \tag{10}$$

The running time of P is that of A plus some overhead. If A is selective-id then so is P.

The last statement allows us to use the lemma in both the selective-id and adaptive-id cases.

$\underline{\textbf{proc Initialize}(\mathbf{y})}$ // $\mathrm{PC}, \mathrm{PC}_l$
$(pars, msk) \xleftarrow{\$} \mathsf{IBE}[\mu, \mathsf{IDSp}].\mathsf{Pg}$
$(g, \hat{g}, H, \hat{H}, \mathbf{U}, \hat{\mathbf{U}}) \leftarrow pars$
$U \xleftarrow{\$} \mathbb{G}^*$
Return $(g, \hat{g}, H, \hat{H}, \mathbf{U}, \hat{\mathbf{U}}, U)$

$\underline{\textbf{proc GetDK}(id)}$ // $\mathrm{PC}, \mathrm{PC}_l$
If $f(\mathbf{y}, id) = 0$ then $dk \leftarrow \perp$
Else $dk \leftarrow \mathsf{IBE}[\mu, \mathsf{IDSp}].\mathsf{Kg}(pars, msk, id)$
Return dk

$\underline{\textbf{proc Ch}()}$ // PC
$s \xleftarrow{\$} \mathbb{Z}_p^*$; $\hat{s} \xleftarrow{\$} \mathbb{Z}_p$; $G \leftarrow g^s$; $\hat{G} \leftarrow \hat{g}^{\hat{s}}$; $S \leftarrow H^s \hat{H}^{\hat{s}}$
For $k = 0, \ldots, \mu$ do $\mathbf{Z}[k] \leftarrow (U^{\mathbf{y}[k]} \mathbf{U}[k])^s \hat{\mathbf{U}}[k]^{\hat{s}}$
Return $(G, \hat{G}, S, \mathbf{Z})$

$\underline{\textbf{proc Ch}()}$ // PC_l
$s \xleftarrow{\$} \mathbb{Z}_p^*$; $\hat{s} \xleftarrow{\$} \mathbb{Z}_p$; $G \leftarrow g^s$; $\hat{G} \leftarrow \hat{g}^{\hat{s}}$; $S \xleftarrow{\$} \mathbb{G}$
For $k = 0, \ldots, l-1$ do $\mathbf{Z}[k] \xleftarrow{\$} \mathbb{G}$
For $k = l, \ldots, \mu$ do $\mathbf{Z}[k] \leftarrow (U^{\mathbf{y}[k]} \mathbf{U}[k])^s \hat{\mathbf{U}}[k]^{\hat{s}}$
Return $(G, \hat{G}, S, \mathbf{Z})$

$\underline{\textbf{proc Finalize}(d')}$ // $\mathrm{PC}, \mathrm{PC}_l$
Return $(d' = 1)$

Figure 6: Games $\mathrm{PC}, \mathrm{PC}_l$ $(0 \le l \le \mu + 1)$ associated to $\mathsf{IDSp} \subseteq \mathbb{Z}_p^{\mu+1}$ for the proof of Lemma 4.1.

4.7 Proof of Lemma 4.1

Consider the games of Figure 6. Game PC is the same as game ReC. Game PC_l $(0 \le l \le \mu + 1)$ makes S random and also makes the first $l - 1$ entries of $\mathbf{Z}$ random and the rest real. Thus $\mathrm{PC}_{\mu+1}$ is the same as RaC. We will design adversaries B_1, B_2 so that

$$\mathbf{Adv}^{\mathrm{dlin}}(B_1) = \Pr[\mathrm{PC}^P] - \Pr[\mathrm{PC}_0^P] \quad (11)$$

$$\mathbf{Adv}^{\mathrm{dlin}}(B_2) = \frac{1}{\mu+1}\left(\Pr[\mathrm{PC}_0^P] - \Pr[\mathrm{PC}_{\mu+1}^P]\right) \quad (12)$$

Adversary B will run B_1 with probability $1/(\mu+2)$ and B_2 with probability $(\mu+1)/(\mu+2)$. This yields Equation (7).

On input $(g, \hat{g}, g^s, \hat{g}^{\hat{s}}, H, T)$ where T is either $H^{s+\hat{s}}$ or random, adversary B_1 runs adversary P, responding to its oracle queries as follows. When P makes query $\mathbf{Initialize}(\mathbf{y})$, adversary B_1 lets

$\mathbf{u}, \hat{\mathbf{u}} \xleftarrow{\$} \mathbb{Z}_p^{\mu+1}$; $u, v \xleftarrow{\$} \mathbb{Z}_p$; $\hat{H} \leftarrow H \hat{g}^v$; $U \leftarrow \hat{g}^u$
For $k = 0, \ldots, \mu$ do $\mathbf{U}[k] \leftarrow U^{-\mathbf{y}[k]} g^{\mathbf{u}[k]}$; $\hat{\mathbf{U}}[k] \leftarrow \hat{g}^{\hat{\mathbf{u}}[k]}$

It returns $(g, \hat{g}, H, \hat{H}, \mathbf{U}, \hat{\mathbf{U}}, U)$ to P. When P makes its (single) $\mathbf{Ch}()$ query, adversary B_1 lets

$S \leftarrow T \hat{g}^{v\hat{s}}$
For $k = 0, \ldots, \mu$ do $\mathbf{Z}[k] \leftarrow g^{s\mathbf{u}[k]} \hat{g}^{\hat{s}\hat{\mathbf{u}}[k]}$

It returns $(g^s, \hat{g}^{\hat{s}}, S, \mathbf{Z})$ to P. Notice that for $0 \le k \le \mu$

$$\mathbf{Z}[k] = g^{s\mathbf{u}[k]} \hat{g}^{\hat{s}\hat{\mathbf{u}}[k]} = (U^{\mathbf{y}[k]-\mathbf{y}[k]} g^{\mathbf{u}[k]})^s \hat{g}^{\hat{s}\hat{\mathbf{u}}[k]} = (U^{\mathbf{y}[k]} \mathbf{U}[k])^s \hat{\mathbf{U}}[k]^{\hat{s}} .$$

Also if $T = H^{s+\hat{s}}$ then $S = T \hat{g}^{v\hat{s}} = H^s (H \hat{g}^v)^{\hat{s}} = H^s \hat{H}^{\hat{s}}$ as in PC while if T is random, so is S, as in PC_0. When P makes query $\mathbf{GetDK}(id)$, adversary B_1 does the following:

If $f(\mathbf{y}, id) = 0$ then $dk \leftarrow \perp$
Else
  $r', \hat{r}' \xleftarrow{\$} \mathbb{Z}_p$
  $D_1 \leftarrow g^{-f(\mathbf{y},id)ur'} g^{f(\mathbf{u},id)r'} H^{-f(\mathbf{u},id)\hat{r}'/f(\mathbf{y},id)}$; $D_2 \leftarrow g^{f(\hat{u},id)r'} H^{-f(\mathbf{u},id)\hat{r}'/f(\mathbf{y},id)} \hat{H}^{u\hat{r}'}$
  $D_3 \leftarrow H^{\hat{r}'/f(\mathbf{y},id)} g^{-r'}$; $D_4 \leftarrow \hat{g}^{-u\hat{r}'}$; $dk \leftarrow (D_1, D_2, D_3, D_4)$

It returns dk to P. We now show this key is properly distributed. Let h be such that $H = g^h$ and let

$$r \;=\; \frac{r'}{t} - \frac{h\hat{r}'}{tf(\mathbf{y},id)} \bmod p \quad \text{and} \quad \hat{r} \;=\; u\hat{r}' \bmod p\,.$$

Since $t, f(\mathbf{y},uid)$ are non-zero modulo p and $r', \hat{r}'$ are random, $r, \hat{r}$ are random as well. The following computes the correct secret key components with the above randomness and shows that they are the ones of the simulation:

$$\begin{aligned}
\mathcal{H}(\mathbf{U},id)^{tr} H^{t\hat{r}} &= \mathbf{U}[0]^{tr} \left(\textstyle\prod_{k=1}^{\mu} \mathbf{U}[k]^{id[k]tr}\right) H^{t\hat{r}} \\
&= U^{-\mathbf{y}[0]tr} g^{\mathbf{u}[0]tr} \left(\textstyle\prod_{k=1}^{\mu} U^{-\mathbf{y}[k]id[k]tr} g^{\mathbf{u}[k]id[k]tr}\right) H^{t\hat{r}} \\
&= U^{-f(\mathbf{y},id)tr} g^{f(\mathbf{u},id)tr} H^{t\hat{r}} \\
&= U^{-f(\mathbf{y},id)(r'-h\hat{r}'/f(\mathbf{y},id))} g^{f(\mathbf{u},id)(r'-h\hat{r}'/f(\mathbf{y},id))} H^{tu\hat{r}'} \\
&= \hat{g}^{-hu\hat{r}'} g^{-f(\mathbf{y},id)ur'} g^{f(\mathbf{u},id)r'} g^{-f(\mathbf{u},id)h\hat{r}'/f(\mathbf{y},id)} g^{htu\hat{r}'} \\
&= g^{-f(\mathbf{y},id)ur'} g^{f(\mathbf{u},id)r'} H^{-f(\mathbf{u},id)\hat{r}'/f(\mathbf{y},id)} \;=\; D_1 \\
\mathcal{H}(\hat{\mathbf{U}},id)^{r} \hat{H}^{\hat{r}} &= \hat{\mathbf{U}}[0]^{r} \left(\textstyle\prod_{k=1}^{\mu} \hat{\mathbf{U}}[k]^{id[k]r}\right) \hat{H}^{\hat{r}} \;=\; \hat{g}^{\hat{u}[0]r} \left(\textstyle\prod_{k=1}^{\mu} \hat{g}^{\hat{u}[k]id[k]r}\right) \hat{H}^{\hat{r}} \\
&= \hat{g}^{f(\hat{u},id)r} \hat{H}^{\hat{r}} \;=\; g^{f(\hat{u},id)tr} \hat{H}^{\hat{r}} \\
&= g^{f(\hat{u},id)(r'-h\hat{r}'/f(\mathbf{y},id))} \hat{H}^{u\hat{r}'} \;=\; g^{f(\hat{u},id)r'} H^{-f(\mathbf{u},id)\hat{r}'/f(\mathbf{y},id)} \hat{H}^{u\hat{r}'} \;=\; D_2 \\
g^{-tr} &= g^{h\hat{r}'/f(\mathbf{y},id)-r'} \;=\; H^{\hat{r}'/f(\mathbf{y},id)} g^{-r'} \;=\; D_3 \\
g^{-t\hat{r}} &= g^{-tu\hat{r}'} \;=\; \hat{g}^{-u\hat{r}'} \;=\; D_4\,.
\end{aligned}$$

Finally adversary P outputs d'. Adversary B_1 also outputs d', so we have Equation (11).

On input $(g, \hat{g}, g^s, \hat{g}^{\hat{s}}, \hat{U}, T)$ where T is either $\hat{U}^{s+\hat{s}}$ or random, adversary B_2 runs adversary P, responding to its oracle queries as follows. When P makes query $\mathbf{Initialize}(\mathbf{y})$, adversary B_1 lets

$l \xleftarrow{\$} \{0,\ldots,\mu\}$; $\mathbf{u}, \hat{\mathbf{u}} \xleftarrow{\$} \mathbb{Z}_p^{\mu+1}$; $u, h, \hat{h} \xleftarrow{\$} \mathbb{Z}_p$; $H \leftarrow \hat{g}^h$; $\hat{H} \leftarrow \hat{g}^{\hat{h}}$; $U \leftarrow g^u$
For $k = 0,\ldots,\mu$ do $\mathbf{U}[k] \leftarrow \hat{U}^{\Delta(l,k)} g^{\mathbf{u}[k]}$; $\hat{\mathbf{U}}[k] \leftarrow \hat{U}^{\Delta(l,k)} \hat{g}^{\hat{\mathbf{u}}[k]}$

It returns $(g, \hat{g}, H, \hat{H}, \mathbf{U}, \hat{\mathbf{U}}, U)$ to P. When P makes its (single) $\mathbf{Ch}()$ query, adversary B_2 lets

$S \xleftarrow{\$} \mathbb{G}$
For $k = 0,\ldots,l-1$ do $\mathbf{Z}[k] \xleftarrow{\$} \mathbb{G}$
For $k = l,\ldots,\mu$ do $\mathbf{Z}[k] \leftarrow (g^s)^{u\mathbf{y}[k]+\mathbf{u}[k]} (\hat{g}^{\hat{s}})^{\hat{\mathbf{u}}[k]} T^{\Delta(l,k)}$

It returns $(g^s, \hat{g}^{\hat{s}}, S, \mathbf{Z})$ to P. Notice that for $l+1 \le k \le \mu$

$$\mathbf{Z}[k] = (g^s)^{u\mathbf{y}[k]+\mathbf{u}[k]} (\hat{g}^{\hat{s}})^{\hat{\mathbf{u}}[k]} = U^{s\mathbf{y}[k]} \mathbf{U}[k]^s \hat{\mathbf{U}}[k]^{\hat{s}} = (U^{\mathbf{y}[k]} \mathbf{U}[k])^s \hat{\mathbf{U}}[k]^{\hat{s}}\,.$$

If $T = \hat{U}^{s+\hat{s}}$ then

$$\mathbf{Z}[l] = (g^s)^{u\mathbf{y}[l]+\mathbf{u}[l]} (\hat{g}^{\hat{s}})^{\hat{\mathbf{u}}[l]} T = U^{s\mathbf{y}[l]} (\hat{U}^{-1}\mathbf{U}[l])^s (\hat{U}^{-1}\hat{\mathbf{U}}[l])^{\hat{s}} \hat{U}^s \hat{U}^{\hat{s}} = (U^{\mathbf{y}[l]}\mathbf{U}[l])^s \hat{\mathbf{U}}[l]^{\hat{s}}$$

as in game PC_l. On the other hand if T is random then so is $\mathbf{Z}[l]$, as in game PC_{l+1}. When P makes query $\mathbf{GetDK}(id)$, adversary B_2 does the following:

If $f(\mathbf{y},id) = 0$ then $dk \leftarrow \perp$
Else
  $r, \hat{r}' \xleftarrow{\$} \mathbb{Z}_p$
  $D_1 \leftarrow \hat{g}^{f(\mathbf{u},id)r} \hat{g}^{h\hat{r}'}$; $D_2 \leftarrow g^{f(\mathbf{u},id)r} \hat{U}^{id[l]r} g^{\hat{h}\hat{r}'} \hat{U}^{-\hat{h}id[l]r/h}$
  $D_3 \leftarrow \hat{g}^{-r}$; $D_4 \leftarrow \hat{U}^{id[l]r/h} g^{-\hat{r}'}$; $dk \leftarrow (D_1, D_2, D_3, D_4)$

It returns dk to P. We now show this key is properly distributed. Let $\hat{u}$ be such that $\hat{U} = g^{\hat{u}}$ and let

$$\hat{r} \;=\; \frac{\hat{r}'}{t} - \frac{id[l]\hat{u}r}{th} \bmod p\ .$$

Since t is non-zero modulo p and $\hat{r}'$ is random, $\hat{r}$ is random as well. The following computes the correct secret key components with the above randomness and shows that they are the ones of the simulation:

$$\begin{aligned}
\mathcal{H}(\mathbf{U}, id)^{tr} H^{t\hat{r}} &= \mathbf{U}[0]^{tr} \left(\textstyle\prod_{k=1}^{\mu} \mathbf{U}[k]^{id[k]tr}\right) H^{t\hat{r}} \\
&= g^{\mathbf{u}[0]tr} \left(\textstyle\prod_{k=1}^{\mu} \hat{U}^{id[k]tr\Delta(l,k)} g^{\mathbf{u}[k]id[k]tr}\right) \hat{g}^{ht\hat{r}} \\
&= g^{f(\mathbf{u},id)tr} \hat{U}^{id[l]tr} \hat{g}^{ht\hat{r}} \;=\; g^{f(\mathbf{u},id)tr} \hat{U}^{id[l]tr} \hat{g}^{h(\hat{r}' - id[l]\hat{u}r/h)} \\
&= \hat{g}^{f(\mathbf{u},id)r} \hat{U}^{id[l]tr} \hat{g}^{h\hat{r}'} \hat{g}^{-id[l]\hat{u}r} \;=\; \hat{g}^{f(\mathbf{u},id)r} g^{id[l]\hat{u}rt} \hat{g}^{h\hat{r}'} \hat{g}^{-id[l]\hat{u}r} \\
&= \hat{g}^{f(\mathbf{u},id)r} \hat{g}^{h\hat{r}'} \;=\; D_1 \\
\mathcal{H}(\hat{\mathbf{U}}, id)^{r} \hat{H}^{\hat{r}} &= \hat{\mathbf{U}}[0]^{r} \left(\textstyle\prod_{k=1}^{\mu} \hat{\mathbf{U}}[k]^{id[k]r}\right) \hat{H}^{\hat{r}} \;=\; g^{\hat{\mathbf{u}}[0]r} \left(\textstyle\prod_{k=1}^{\mu} \hat{U}^{id[k]r\Delta(l,k)} g^{\hat{\mathbf{u}}[k]id[k]r}\right) \hat{g}^{\hat{h}\hat{r}} \\
&= g^{f(\hat{\mathbf{u}},id)r} \hat{U}^{id[l]r} g^{t\hat{h}\hat{r}} \;=\; g^{f(\hat{\mathbf{u}},id)r} \hat{U}^{id[l]r} g^{\hat{h}(\hat{r}' - id[l]\hat{u}r/h)} \\
&= g^{f(\hat{\mathbf{u}},id)r} \hat{U}^{id[l]r} g^{\hat{h}\hat{r}'} g^{-\hat{h}id[l]\hat{u}r/h} \;=\; g^{f(\mathbf{u},id)r} \hat{U}^{id[l]r} g^{\hat{h}\hat{r}'} \hat{U}^{-\hat{h}id[l]r/h} \;=\; D_2 \\
g^{-tr} &= \hat{g}^{-r} \;=\; D_3 \\
g^{-t\hat{r}} &= g^{\hat{u}rid[l]/h - \hat{r}'} \;=\; \hat{U}^{id[l]r/h} g^{-\hat{r}'} \;=\; D_4\ .
\end{aligned}$$

Finally adversary P outputs d'. Adversary B_2 also outputs d'. So

$$\begin{aligned}
\mathbf{Adv}^{\mathrm{dlin}}(B_2) &= \frac{1}{\mu+1} \sum_{l=0}^{\mu} \Pr[\mathrm{PC}_l^P] - \Pr[\mathrm{PC}_{l+1}^P] \\
&= \frac{1}{\mu+1} \Pr[\mathrm{PC}_0^P] - \Pr[\mathrm{PC}_{\mu+1}^P]
\end{aligned}$$

and we have Equation (12).

4.8 Selective-id security

We consider IBTDF $\overline{\mathsf{F}}[n, 1, \mathbb{Z}_p]$, the instance of our construction with $\mu = 1$ and $\mathsf{IDSp} = \mathbb{Z}_p$. We show that this IBTDF is selective-id δ-lossy for $\delta = 1$, meaning fully selective-id lossy, and hence selective-id one-way. To do this we define a sibling $\overline{\mathsf{LF}}[n, 1, \mathbb{Z}_p]$. It preserves the key-generation, evaluation and inversion algorithms of $\overline{\mathsf{F}}[n, 1, \mathbb{Z}_p]$ and alters parameter generation to

Algorithm $\overline{\mathsf{LF}}[n, 1, \mathbb{Z}_p].\mathsf{Pg}(id)$: $\quad \mathbf{y} \leftarrow (-id, 1)$; $(pars, msk) \xleftarrow{\$} \overline{\mathsf{E}}[n, 1, \mathbb{Z}_p].\mathsf{Pg}(\mathbf{y})$; Return $(pars, msk)$.

The following says that our IBTDF is 1-lossy under the DLIN assumption with lossiness $\ell = n - 2\lg(p)$.

Theorem 4.3 *Let $n > 2\lg(p)$ and let $\ell = n - 2\lg(p)$. Let $\mathsf{F} = \overline{\mathsf{F}}[n, 1, \mathbb{Z}_p]$ be the IBTDF associated by our construction to parameters n, $\mu = 1$ and $\mathsf{IDSp} = \mathbb{Z}_p$. Let $\mathsf{LF} = \overline{\mathsf{LF}}[n, 1, \mathbb{Z}_p]$ be the sibling associated to it as above. Let $\delta = 1$ and let be A a selective-id adversary. Then there is an adversary B such that*

$$\mathbf{Adv}^{\delta\text{-los}}_{\mathsf{F},\mathsf{LF},\ell}(A) \;\le\; 2n(\mu+2) \cdot \mathbf{Adv}^{\mathrm{dlin}}(B)\ . \tag{13}$$

The running time of B is that of A plus overhead.

Proof: On input id, let algorithm Aux return $(-id, 1)$. Let $\mathrm{RL}_0, \mathrm{RL}_n$ be the games of Figure 5 with $\mu = 1$, $\mathsf{IDSp} = \mathbb{Z}_p$ and this Aux. Then we claim

$$\Pr\left[\,\mathrm{Real}^A_{\mathsf{F}}\,\right] = \Pr\left[\,\mathrm{RL}^A_0\,\right] \quad \text{and} \quad \Pr\left[\,\mathrm{Lossy}^A_{\mathsf{F},\mathsf{LF},\ell}\,\right] = \Pr\left[\,\mathrm{RL}^A_n\,\right]\ . \tag{14}$$

To justify this let id^* be the identity queried by A to both **Initialize** and **Ch**. (These queries are the same because A is selective-id.) Then $\mathbf{y}_0 = (-id^*, 1)$ so $f(\mathbf{y}_0, id) = id - id^*$. This is 0 iff $id = id^*$. This means that the conjunct $(id^* \notin IS) \wedge \text{WIN}$ is always true. The claim of Equation (14) is now true because game RL_0 generates parameters with the real auxiliary input $\mathbf{y}_1 = (1, 0) \in \mathbb{Z}_p^2$ that, via $\overline{\mathsf{E}}[n, 1, \mathbb{Z}_p]$, defines F. However game RL_n generates parameters with auxiliary input $\mathbf{y}_0$. Since $f(\mathbf{y}_0, id^*) = 0$, the dependency of $\mathbf{C}_3[j]$ on $x[j]$ in Equation (5) vanishes when $id = id^*$. Examing equations (3), (4), (5), (6), we now see that with *pars* fixed, the values $\langle \mathbf{s}, x\rangle, \langle \hat{\mathbf{s}}, x\rangle$ determine the ciphertext $(C_1, C_2, \mathbf{C}_3, \mathbf{C}_4)$. Thus there are at most p^2 possible ciphertexts when $id = id^*$, and 2^n possible inputs. This means that $\lambda(\mathsf{F}.\mathsf{Ev}(pars, id^*, \cdot)) \geq n - \lg(p^2) = \ell$, which justifies the second claim of Equation (14). Recalling that $\delta = 1$, Equation (13) follows from Equation (1), Equation (14), Lemma 4.2 and Lemma 4.1. ▌

4.9 Adaptive-id Security

We consider IBTDF $\overline{\mathsf{F}}[n, \mu, \{0,1\}^\mu]$, the instance of our construction with $\mathsf{IDSp} = \{0,1\}^\mu \subset \mathbb{Z}_p^\mu$. We show that this IBTDF is adaptive-id δ-lossy for $\delta = (4(\mu+1)Q)^{-1}$ where Q is the number of key-derivation queries of the adversary. By Theorem 3.2 this means $\overline{\mathsf{F}}[n, \mu, \{0,1\}^\mu]$ is adaptive-id one-way. To do this we define a sibling $\overline{\mathsf{LF}}_Q[n, \mu, \{0,1\}^\mu]$. It preserves the key-generation, evaluation and inversion algorithms of $\overline{\mathsf{F}}[n, \mu, \{0,1\}^\mu]$ and alters parameter generation to $\overline{\mathsf{LF}}[n, \mu, \{0,1\}^\mu].\mathsf{Pg}(id)$ defined via

$$\mathbf{y} \leftarrow \mathsf{Aux}\,;\ (pars, msk) \xleftarrow{\$} \overline{\mathsf{E}}[n, \mu, \{0,1\}^\mu].\mathsf{Pg}(\mathbf{y})\,;\ \text{ Return } (pars, msk)\ .$$

where algorithm Aux is defined via

$$\begin{array}{l}\mathbf{y}'[0] \xleftarrow{\$} \{0, \ldots, 2Q-1\}\ ;\ \ell \xleftarrow{\$} \{0, \ldots, \mu+1\}\ ;\ \mathbf{y}[0] \leftarrow \mathbf{y}'[0] - 2\ell Q \\ \text{For } i = 1 \text{ to } \mu \text{ do } \mathbf{y}[i] \xleftarrow{\$} \{0, \ldots, 2Q-1\} \\ \text{Return } \mathbf{y} \in \mathbb{Z}_p^{\mu+1}\end{array}$$

The following says that our IBTDF is δ-lossy under the DLIN assumption with lossiness $\ell = n - 2\lg(p)$.

Theorem 4.4 *Let $n > 2\lg(p)$ and let $\ell = n - 2\lg(p)$. Let $\mathsf{F} = \overline{\mathsf{F}}[n, \mu, \{0,1\}^\mu]$ be the IBTDF associated by our construction to parameters n, μ and $\mathsf{IDSp} = \{0,1\}^\mu$. Let A be an adaptive-id adversary that makes a maximal number of $Q < p/(3m)$ queries and let $\delta = (4(\mu+1)Q)^{-1}$. Let $\mathsf{LF} = \overline{\mathsf{LF}}_Q[n, \mu, \{0,1\}^\mu]$ be the sibling associated to F, A as above. Then there is an adversary B such that*

$$\mathbf{Adv}^{\delta\text{-los}}_{\mathsf{F},\mathsf{LF},\ell}(A) \ \leq\ 2n(\mu+2) \cdot \mathbf{Adv}^{\text{dlin}}(B)\ . \tag{15}$$

The running time of B is that of A plus $O(\mu^2\rho^{-1}((\mu Q\rho)^{-1}))$ overhead, where $\rho = \frac{1}{2} \cdot \mathbf{Adv}^{\delta\text{-los}}_{\mathsf{F},\mathsf{LF},\ell}(A)$.

Proof: Our proof uses a simulation technique due to Waters [56]. We used a slightly improved analysis from [40]. Let Q be the number of queries made by A and let algorithm Aux be defined as above. Let RL_0, RL_n be the games of Figure 5 with $\mathsf{IDSp} = \{0,1\}^\mu$ and this Aux. Let $\text{E}(IS, id^*)$ denote the event that when **Finalize**d' is called in RL_0^A the flag $\text{WIN} \leftarrow \mathsf{false}$ is set and $id^* \notin IS$. (Note that $\eta(IS, id^*)$ only depends on IS, id^* since $\mathbf{y}_0$ is exclusively used to set $\text{WIN} \leftarrow \mathsf{false}$.) Let $\eta(IS, id^*)$ be the probability that $\text{E}(IS, id^*)$ happens. In [40, Lemma 6.2], it was shown (using purely combinatorial arguments) that $\lambda_{\text{low}} := \frac{1}{4(\mu+1)Q} \leq \eta(IS, id^*) \leq \frac{1}{2Q} := \lambda_{\text{up}}$. Since RL_0^A and $\text{Real}_{\mathsf{F}}^A$ are only different when $\text{E}(IS, id^*)$ happens, one would like to argue that $\lambda_{\text{low}} \cdot \Pr\left[\text{Real}_{\mathsf{F}}^A\right] = \Pr\left[\text{RL}_0^A\right]$ but this is not true since $\text{E}(IS, id^*)$ and $\text{Real}_{\mathsf{F}}^A$ may not be independent. To get rid of this unwanted dependence we consider a modification of RL_0 and RL_n which adds some artificial abort such that in total it always sets $\text{WIN} \leftarrow \mathsf{false}$ with probability around $1 - \lambda_{\text{low}}$, independent of the view of the adversary. (Since, given IS, id^*, the exact value of $\eta(IS, id^*)$ cannot be computed efficiently, it needs to be approximated using sampling.) Concretely, games $\hat{\text{RL}}_0$ and $\hat{\text{RL}}_n$ are defined as RL_0 and RL_n, respectively, the only difference being **Finalize** which is defined as follows.

proc Finalize(d') // $\hat{\mathrm{RL}}_0, \hat{\mathrm{RL}}_n$
Compute an approximation $\eta'(IS, id^*)$ of $\eta(IS, id^*)$
If $\eta'(IS, id^*) > \lambda_{\mathrm{low}}$ then set $\textsc{Win} \leftarrow \mathsf{false}$ with probability $1 - \lambda_{\mathrm{low}}/\eta'(IS, id^*)$
Return $((d' = 1)$ and $(id^* \notin IS)$ and $\textsc{Win})$

We refer to [40] on details how to compute the approximation $\eta'(IS, id^*)$. Using [40, Lemma 6.3], one can show that if we use $O(\mu^2 \rho^{-1}((\mu Q \rho)^{-1}))$ samples to compute approximation $\eta'(IS, id^*)$, then

$$\Pr\left[\mathrm{Real}_{\mathsf{F}}^{A}\right] - \lambda_{\mathrm{low}}^{-1} \cdot \Pr\left[\hat{\mathrm{RL}}_0^{A}\right] = \rho. \tag{16}$$

Setting $\rho = \frac{1}{2} \cdot \Pr\left[\mathrm{Real}_{\mathsf{F}}^{A}\right]$ we obtain

$$\delta \cdot \Pr\left[\mathrm{Real}_{\mathsf{F}}^{A}\right] = \Pr\left[\hat{\mathrm{RL}}_0^{A}\right], \tag{17}$$

where $\delta = \lambda_{\mathrm{low}}/2$ is as in the theorem statement. As in the proof of Theorem 4.3, we can show that

$$\Pr\left[\mathrm{Lossy}_{\mathsf{F},\mathsf{LF},\ell}^{A}\right] = \Pr\left[\hat{\mathrm{RL}}_n^{A}\right]. \tag{18}$$

Now Equation (15) follows from Equations (1), (17), (18), Lemma 4.2 and (a version incorporating the artificial abort of) Lemma 4.1. ∎

We remark that we could use the proof technique of [11] which avoids the artificial abort but this increases the value of δ, making it dependent on the adversary advantage. The proof technique of [39] could be used to strengthen δ in Theorem 4.4 to $O(\sqrt{m}Q)^{-1}$ which is close to the optimal value Q^{-1}.

5 IB-TDFs from Lattices

Here we give a construction of a lossy IB-TDF from lattices (specifically, the LWE assumption). We note that a one-way IB-TDF can already be derived by applying methods from [28, 2] to the LWE-based injective (not identity-based) trapdoor function from [34].

LWE is a particular type of average-case BDD/GapSVP problem. It has been recognized since [47, 43] that BDD/GapSVP induces a form of lossiness. So there is folklore that the GPV LWE-based TDF can be made to satisfy some meaningful notion of lossiness (specifically, for an appropriate input distribution, the output does not reveal the entire input statistically) by replacing its normally uniformly random key with an LWE (BDD/GapSVP) instance.

However a full construction and proof according to the standard notion of lossiness (which compares the domain and images sizes of the function) have not yet appeared in the literature and there are many quantitative issues to address.

In this section we construct an (ID-based) TDF that is lossy for a natural (uniform) input distribution. We favor simplicity of analysis at the expense of tight bounds, so our construction is highly unoptimized and should be seen mainly as a proof of feasibility. Much tighter constructions and bounds can certainly be achieved using more sophisticated machinery from the literature.

5.1 Background

A full-rank m-dimensional integer lattice $\Lambda \subseteq \mathbb{Z}^m$ is a discrete additive subgroup whose linear span is $\mathbb{R}^m$. Every lattice is generated as the $\mathbb{Z}$-linear combination of some *basis* of linearly independent vectors $\mathbf{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_m\} \subset \mathbb{Z}^m$, i.e., $\Lambda = \mathbf{B} \cdot \mathbb{Z}^m = \{\sum_{i=1}^{m} z_i \mathbf{b}_i : z_i \in \mathbb{Z}\}$.

In this work we deal exclusively with "q-ary" lattices, where for simplicity we always take $q = \mathrm{poly}(n)$ to be prime. For a matrix $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$, define the integer lattices

$$\Lambda^{\perp}(\mathbf{A}) = \{\mathbf{z} \in \mathbb{Z}^m : \mathbf{A}\mathbf{z} = 0 \bmod q\},$$
$$\Lambda(\mathbf{A}^t) = \{\mathbf{z} \in \mathbb{Z}^m : \exists\, \mathbf{s} \in \mathbb{Z}_q^n \text{ s.t. } \mathbf{z} = \mathbf{A}^t\mathbf{s} \bmod q\}$$

which both contain $q\mathbb{Z}^m$ as a sublattice.

We work with *discrete Gaussian* error distributions $D_{\mathbb{Z},s}$ over the integers (for $s > 0$), where the probability of each $x \in \mathbb{Z}$ is proportional to $\exp(-\pi x^2/s^2)$. Given s, this distribution can be sampled efficiently via rejection [34].

The (decisional) *learning with errors* (LWE) problem [49] in dimension n with error rate $\alpha \in (0,1)$, stated in matrix form, is: given an input $(\mathbf{A}, \mathbf{b})$ where $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$ (for any $m = \text{poly}(n)$) is uniformly random and $\mathbf{b} \in \mathbb{Z}_q^m$ is either of the form $\mathbf{b} = \mathbf{A}^t\mathbf{s} + \mathbf{e} \mod q$ for uniform $\mathbf{s} \in \mathbb{Z}_q^n$ and $\mathbf{e} \leftarrow D_{\mathbb{Z},\alpha q}^m$ or is uniformly random (and independent of $\mathbf{A}$), distinguish which is the case, with non-negligible advantage. Note that in the former case, $\mathbf{b}$ is essentially a random point of $\Lambda(\mathbf{A}^t)$, perturbed by some (discrete) Gaussian noise. It is known that when $\alpha q > 2\sqrt{n}$, this decision problem is at least as hard as approximating several problems on n-dimensional lattices in the *worst case* to within $\tilde{O}(n/\alpha)$ factors with a *quantum* computer [49], or on a *classical* computer for a subset of these problems [47].

For a real-valued matrix $\mathbf{M}$, we let $s_1(\mathbf{M})$ denote the largest singular value of $\mathbf{M}$, i.e., $\max_{\mathbf{y}} \|\mathbf{M}\mathbf{y}\|$ where $\mathbf{y}$ ranges over all unit vectors of appropriate dimension. A random variable X over $\mathbb{R}$ is said to be subgaussian with parameter s if $\Pr[|X| \geq t] \leq 2\exp(-\pi t^2/s^2)$ for all $t \geq 0$; a random variable X over $\mathbb{R}^n$ is subgaussian (of parameter s) if the marginal $\langle X, \mathbf{y}\rangle$ is subgaussian (of parameter s) for every unit vector $\mathbf{y} \in \mathbb{R}^n$. In particular, any bounded random variable is subgaussian, and it is known that a discrete Gaussian $D_{\Lambda,s}$ over any lattice Λ is subgaussian (with parameter s). We need the following standard fact from random matrix theory: an m-by-n matrix $\mathbf{M}$ whose entries are independent mean-zero subgaussian random variables with common parameter s has largest singular value $s_1(\mathbf{M}) = s \cdot O(\sqrt{m} + \sqrt{n})$, except with probability $2^{-\Omega(m+n)}$.

We need the following lemma showing how to generate a (nearly) uniformly random $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$ together with a 'trapdoor' in the form of a short basis. Such a construction is given in [4]. That work is focused on the standard parameter regime where $m = O(n \log q)$, and does not actually contain a theorem statement for the non-standard parameters $m = O(n)$ that we need. Fortunately, it follows by a straightforward adaption of the construction using a tradeoff between the base of the logarithm and the length of the trapdoor basis vectors. The concurrent work [44] contains a full (and simpler) proof of this fact.

Lemma 5.1 ([4, 44]) *Let n, q be positive integers, and let $b \geq 2$. For large enough $m = O(n \log_b q)$, there is an efficient randomized algorithm that outputs a matrix $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$ and a basis $\mathbf{T}$ of $\Lambda^\perp(\mathbf{A})$, such that $\mathbf{A}$ is negligibly far from uniform and $\|\tilde{\mathbf{T}}\| = O(b \cdot \sqrt{n \log_b q})$.*

The above lemma is usually invoked with $b = 2$, yielding a dimension $m = O(n \log q)$. Because our constructions need m to grow only linearly in n, we will instead use base $b = q^{1/C}$ for some constant C, which yields $\|\mathbf{T}\| = O(q^{1/C} \cdot \sqrt{n})$.

The following lemma from [34] (using the "nearest-plane" algorithm [5]) says that for appropriate parameters, the LWE one-way function has an inversion trapdoor.

Lemma 5.2 *Let $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$ be full-rank. Given $\mathbf{A}$ and any basis $\mathbf{T} \in \mathbb{Z}^{m \times m}$ of $\Lambda^\perp(\mathbf{A})$, one can efficiently recover $\mathbf{x} \in \mathbb{Z}_q^n$ and $\mathbf{e} \in \mathbb{Z}^m$ from $g_{\mathbf{A}}(\mathbf{x}, \mathbf{e}) = \mathbf{A}^T\mathbf{x} + \mathbf{e} \bmod q$, as long as $\|\mathbf{e}\| < q/(2\|\tilde{\mathbf{T}}\|)$.*

In our constructions, we will be working with vectors $\mathbf{e}$ whose entries are bounded in magnitude by some β. Using Lemma 5.1 we will have $m = O(n)$ and $\|\tilde{\mathbf{T}}\| = O(q^{1/C} \cdot \sqrt{n})$ for some constant C. So for correctness of inversion, it will suffice to take a small enough $\beta \leq q^{1-1/C} \cdot O(n^{-1})$.

We also need the following specialized basis-delegation algorithm, which combines the SampleRight algorithm from [2] with the basis-delegation algorithm of [28].

Lemma 5.3 *Let $q \geq 2$ and $m > n \geq 1$. Let $\mathbf{A}, \mathbf{B} \in \mathbb{Z}_q^{n \times m}$, $\mathbf{B}$ full rank, $\mathbf{T}$ be a basis of $\Lambda^\perp(\mathbf{B})$ with $\|\tilde{\mathbf{T}}\| \leq \tilde{L}$, let $\mathbf{R} \in \mathbb{Z}^{m \times m}$ with $s_1(\mathbf{R}) \leq R$. There exists an efficient randomized algorithm* SampleRight *that, given $\mathbf{T}$, $\mathbf{R}$, $\mathbf{A}$ and $\mathbf{B}$, computes a basis $\mathbf{S}$ of $\mathbf{F} = [\mathbf{A} \mid \mathbf{AR} + \mathbf{B}]$ with $\|\tilde{\mathbf{S}}\| \leq O(\sqrt{m} \cdot \tilde{L} \cdot R)$ whose distribution depends (up to negligible statistical distance) only on $\mathbf{F}$, $\tilde{L}$, and R.*

5.2 Our basic trapdoor function

Let $c_2 > c_1 > 1$ be positive integers to be determined later in the analysis, and let $m = c_2 n$, $\hat{n} = c_1 n$. Define $D_\beta = [0, \beta)$ for some positive integer β to be determined later. (We sometimes drop the subscript when it is clear from context.) The analysis also goes through unchanged with $D_\beta = [-\beta, \beta)$.

1. Parameters: Algorithm $\mathsf{LWE.Pg}$ uses the algorithm from Lemma 5.1 to generate a (nearly) uniform $\mathbf{A} \in \mathbb{Z}_q^{\hat{n} \times m}$, together with a basis $\mathbf{T}$ for lattice $\Lambda_q^\perp(\mathbf{A})$ such that $\|\tilde{\mathbf{T}}\| = O(q^{1/C}\sqrt{n})$ for some constant C. It returns $pars = \mathbf{A}$ as the public parameters and $msk = \mathbf{T}$ as the trapdoor.
2. Evaluate: Given parameters $\mathbf{A}$ and input $(\mathbf{x}, \mathbf{e}) \in D^{\hat{n}} \times D^m$, algorithm $\mathsf{LWE.Ev}$ returns output $\mathbf{c} = g_{\mathbf{A}}(\mathbf{x}, \mathbf{e}) = \mathbf{A}^T\mathbf{x} + \mathbf{e} \bmod q$.
3. Invert: Given parameter $\mathbf{A}$, trapdoor $\mathbf{T}$ and output (ciphertext) $\mathbf{c}$, algorithm $\mathsf{LWE.Ev}^{-1}$ returns $(\mathbf{x}, \mathbf{e})$ using the inversion algorithm from Lemma 5.2.

The next lemma shows that when $\mathbf{A}$ has a particular (non-uniform) structure, then the function $g_{\mathbf{A}} \colon D^{\hat{n}+m} \to \mathbb{Z}_q^m$ is lossy. We show how to instantiate all the parameters after the proof.

Lemma 5.4 *Let $m = c_2 n > c_1 n$. Suppose that $\mathbf{A} \in \mathbb{Z}_q^{\hat{n} \times m}$ is of the form $\mathbf{A}^T = \left[\bar{\mathbf{A}}^T \mid \bar{\mathbf{A}}^T\mathbf{S} + \mathbf{E} \bmod q\right]$ for some $\bar{\mathbf{A}} \in \mathbb{Z}_q^{n \times m}$, $\mathbf{S} \in \mathbb{Z}_q^{n \times (c_1 - 1)n}$, and some $\mathbf{E} \in \mathbb{Z}^{m \times (c_1-1)n}$. Then the number of possible values of the form $g_{\mathbf{A}}(\mathbf{x}, \mathbf{e}) = \mathbf{A}^T\mathbf{x} + \mathbf{e}$ for $(\mathbf{x}, \mathbf{e}) \in D_\beta^{\hat{n}} \times D_\beta^m$ is at most $q^n \cdot O(\beta \cdot (s_1(\mathbf{E}) + 1))^m$.*

In particular, for large enough $\beta = \Omega(s_1(\mathbf{E}))^{c_2/c_1} \cdot q^{1/c_1}$, the function $g_{\mathbf{A}} : D_\beta^{\hat{n}} \times D_\beta^m \to \mathbb{Z}_q^m$ is m-lossy.

Proof: Write $\mathbf{x} = (\bar{\mathbf{x}}, \hat{\mathbf{x}}) \in D^n \times D^{(c_1-1)n}$. Then

$$\mathbf{A}^T\mathbf{x} + \mathbf{e} = \bar{\mathbf{A}}^T(\bar{\mathbf{x}} + \mathbf{S}\hat{\mathbf{x}}) + (\mathbf{E}\hat{\mathbf{x}} + \mathbf{e}).$$

The number of possible values of $\bar{\mathbf{A}}^T(\bar{\mathbf{x}} + \mathbf{S}\hat{\mathbf{x}})$ is at most q^n.

Define $N_m(r)$ to be the number of integer points in an m-dimensional ball of radius r. For $r \geq \sqrt{m}$, from the volume of the ball and Stirling's approximation, we have $N_m(r) = O(r/\sqrt{m})^m$. By the triangle inequality, the number of possible values of $\mathbf{E}\hat{\mathbf{x}} + \mathbf{e}$ is

$$N_m(\beta \cdot (s_1(\mathbf{E})\sqrt{(c_1 - 1)n} + \sqrt{m})) \leq O(\beta \cdot (s_1(\mathbf{E}) + 1))^m.$$

For lossiness, observe that the base-2 logarithm of the domain size of $g_{\mathbf{A}}$ is

$$(m + \hat{n}) \lg(\beta) \geq m \lg \beta^{(1+c_1/c_2)}.$$

Whereas by the above, the base-2 logarithm of the image size of $g_{\mathbf{A}}$ is

$$n \lg q + m \lg O(\beta \cdot (s_1(\mathbf{E}) + 1)) \leq m \lg O(\beta \cdot (s_1(\mathbf{E}) + 1) \cdot q^{1/c_2}).$$

For β as in the lemma statement, the two quantities above differ by at least m, as desired. ∎

We now discuss the constraints on the parameters and show how they can be instantiated. To accommodate both the upper bound on β that suffices for invertibility (Lemma 5.2), and the lower bound on β that suffices for lossiness (Lemma 5.4), it is enough to have

$$\Omega(s_1(\mathbf{E}))^{c_2/c_1} \cdot q^{1/c_1} \quad \ll \quad \beta \quad \ll \quad q^{1-1/C} \cdot O(n^{-1}).$$

These constraints can be satisfied for large enough

$$q^{1-1/C-1/c_1} \quad \gg \quad \Omega(n) \cdot \Omega(s_1(\mathbf{E}))^{c_2/c_1}. \tag{19}$$

Now, C and c_1 are free constants of our choice, which determine the constant c_2 and the hidden constants in the above $\Omega(\cdot)$ notation via Lemmas 5.1 and 5.4. In summary, if we have some $\mathrm{poly}(n)$ upper bound on $s_1(\mathbf{E})$, then we can choose C, c_1 and sufficiently large $q = \mathrm{poly}(n)$ to satisfy both invertibility (for uniform $\mathbf{A}$) and lossiness (for structured $\mathbf{A}$).

Remark 5.5 As a concrete example, consider a matrix $\mathbf{A}$ having the form from Lemma 5.4, where $\mathbf{S}$ is uniformly random and the entries of $\mathbf{E}$ are chosen independently from $D_{\mathbb{Z},\alpha q}$, where $\alpha q = \Theta(\sqrt{n})$ so as to invoke known worst-case hardness results. Under the LWE assumption (in dimension n) with noise rate α, such an $\mathbf{A}$ is indistinguishable from uniform, and we can have $s_1(\mathbf{E}) = O(n)$ with overwhelming probability by subgaussianity of $D_{\mathbb{Z},\alpha q}$.

Remark 5.6 Our constructions of ID-based lossy TDFs below involve two small variations on the above example. First, the trapdoor $\mathbf{D}_{id}$ for an identity is delegated from a master trapdoor $\mathbf{T}$ using the algorithm from Lemma 5.3, so we will have $\|\tilde{\mathbf{D}}_{id}\| \leq \text{poly}(n) \cdot \|\tilde{\mathbf{T}}\|$. This only turns the $\Omega(n)$ term in Equation (19) into a larger polynomial. Second, the hidden $\mathbf{E}$ term in the structured matrix $\mathbf{A}$ will no longer be Gaussian itself, but will always be of the form $\mathbf{E} = \left[\begin{smallmatrix}\mathbf{I}\\ \mathbf{R}^t\end{smallmatrix}\right] \mathbf{E}'$ for some Gaussian $\mathbf{E}'$ (of parameter αq) and some $\mathbf{R}$ with $s_1(\mathbf{R}) = \text{poly}(n)$. Since $s_1(\mathbf{E}) \leq s_1(\mathbf{E}') \cdot (1 + s_1(\mathbf{R})) = \text{poly}(n)$, we can still instantiate all the parameters so that $q, 1/\alpha = \text{poly}(n)$.

5.3 Our id-based lossy trapdoor function

SETUP. For constants $c_2 > c_1$ let $m = c_2 n$, $\hat{n} = c_1 n$. For integer $\mu \geq 1$, let $\mathsf{C} : \mathsf{IDSp} \to (\mathbb{Z}_q^{\hat{n}\times\hat{n}})^\mu$ be an injective encoding which we call suitable if $\mathsf{C}(id) \in \mathbb{Z}_q^{\hat{n}\times\hat{n}} \times \{\mathbf{0}_{\hat{n}\times\hat{n}}, \pm\mathbf{1}_{\hat{n}\times\hat{n}}\}^{\mu-1}$, for all $id \in \mathsf{IDSp}$. For matrices $\mathbf{U} = \mathbf{U}[0], \ldots, \mathbf{U}[\mu] \in (\mathbb{Z}_q^{\hat{n}\times m})^{\mu+1}$ and $\mathbf{y} \in (\mathbb{Z}_q^{\hat{n}\times\hat{n}})^{\mu+1}$, and vector $id \in \mathsf{IDSp}$ we let

$$\overline{id} = (\mathbf{1}_{\hat{n}\times\hat{n}}, \mathsf{C}(id)) \in (\mathbb{Z}_q^{\hat{n}\times\hat{n}})^{\mu+1} \qquad \text{and} \qquad \mathcal{H}(\mathbf{U}, id) = \textstyle\sum_{k=0}^{\mu} \overline{id}[k]\mathbf{U}[k] \ .$$

We also let

$$f(\mathbf{y}, id) = \textstyle\sum_{k=0}^{\mu} \overline{id}[k]\mathbf{y}[k] \in \mathbb{Z}_q^{\hat{n}\times\hat{n}}$$

Define the following parameters which are motivated by the discussion in Remark 5.6.

$$q^{1-1/C-1/c_1} \quad \gg \quad \Omega(n^2\mu) \cdot \Omega(s)^{c_2/c_1}, \tag{20}$$

where $s = O(n^{3/2}\mu)$ is a bound on $s_1(\mathbf{E})$. This enables us to chose β such that

$$\Omega(s)^{c_2/c_1} \cdot q^{1/c_1} \quad \ll \quad \beta \quad \ll \quad q^{1-1/C} \cdot O(\mu^{-1}n^{-2}). \tag{21}$$

We further define

$$\tilde{L} = q^{1/C} \cdot \sqrt{n}, \quad \tilde{L}_1 := \tilde{L} \cdot (\mu+1) \cdot m, \quad R = \sqrt{2m}(\mu+1), \quad \text{and} \quad \alpha \geq 2\sqrt{n}/q \tag{22}$$

OUR E-IBTDF. Our E-IBTDF $\overline{\mathsf{L}}[\mu, \mathsf{IDSp}, \mathsf{C}]$ is associated to any integer $\mu \geq 1$, any identity space IDSp and any suitable injective encoding C. It has domain $\mathsf{InSp} = D_\beta^{\hat{n}} \times D_\beta^{2m}$ and auxiliary input space $(\mathbb{Z}_q^{\hat{n}\times\hat{n}})^{\mu+1}$, and is given by the following algorithms.

1. Parameters: Given auxiliary input $\mathbf{y}$, algorithm $\overline{\mathsf{L}}[\mu, \mathsf{IDSp}, \mathsf{C}].\mathsf{Pg}$ lets $(\mathbf{B}, \mathbf{T}) \xleftarrow{\$} \mathsf{LWE.Pg}$ and $\mathbf{A} \in \mathbb{Z}_q^{\hat{n}\times m}$ as in Remark 5.5, with Gaussian parameter α. It then lets $\mathbf{R} = (\mathbf{R}[0], \ldots, \mathbf{R}[\mu]) \xleftarrow{\$} (\{-1, 1\}^{m\times m})^{\mu+1}$, and $\mathbf{U} = (\mathbf{U}[0], \ldots, \mathbf{U}[\mu]) \in (\mathbb{Z}_q^{\hat{n}\times m})^{\mu+1}$, where

$$\mathbf{U}[i] \leftarrow \begin{cases} \mathbf{A} \cdot \mathbf{R}[i] + \mathbf{y}[i] \cdot \mathbf{B} & : \quad \overline{id}[i] \in \{\mathbf{0}_{\hat{n}\times\hat{n}}, \pm\mathbf{1}_{\hat{n}\times\hat{n}}\} \\ \mathbf{y}[i] \cdot \mathbf{B} & : \quad \text{otherwise} \end{cases}$$

It returns $pars = (\mathbf{A}, \mathbf{U})$ as the public parameters and $msk = (\mathbf{T}, \mathbf{B}, \mathbf{R}, \mathbf{y})$ as the master secret key. We define $\mathbf{F}(\mathbf{A}, \mathbf{U}, id) = [\mathbf{A} \mid \mathcal{H}(\mathbf{U}, id)]$. Since C is suitable, we have

$$\mathbf{F}(\mathbf{A}, \mathbf{U}, id) = [\mathbf{A} \mid \mathbf{A} \cdot R(\mathbf{R}, id) + f(\mathbf{y}, id) \cdot \mathbf{B}], \tag{23}$$

where $R(\mathbf{R}, id) = \sum_{k:\overline{id}[k]=\mathbf{1}_{\hat{n}\times\hat{n}}} \mathbf{R}[k] - \sum_{k:\overline{id}[k]=-\mathbf{1}_{\hat{n}\times\hat{n}}} \mathbf{R}[k]$. Note that, with high probability, $\|R(\mathbf{R}, id)\| \leq R$ as defined in Equation (22).

proc Initialize(id) // RL_0
$\mathbf{y}_0 \xleftarrow{\$} \mathsf{Aux}_0(id)$; $\mathbf{y}_1 \xleftarrow{\$} \mathsf{Aux}_1(id)$; $\textsc{Win} \leftarrow \mathsf{true}$
$(pars, msk) \xleftarrow{\$} \overline{\mathsf{L}}[\mu, \mathsf{IDSp}, \mathsf{C}].\mathsf{Pg}(\mathbf{y}_1)$
$IS \leftarrow \emptyset$; $id^* \leftarrow id$
Return $pars$

proc Initialize(id) // RL_1
$\mathbf{y}_0 \xleftarrow{\$} \mathsf{Aux}(id)$; $\mathbf{y}_1 \xleftarrow{\$} \mathsf{Aux}_1(id)$; $\textsc{Win} \leftarrow \mathsf{true}$
$(pars, msk) \xleftarrow{\$} \overline{\mathsf{L}}[\mu, \mathsf{IDSp}, \mathsf{C}].\mathsf{Pg}(\mathbf{y}_0)$
$IS \leftarrow \emptyset$; $id^* \leftarrow id$
Return $pars$

proc GetDK(id) // $\mathrm{RL}_0, \mathrm{RL}_1$
$IS \leftarrow IS \cup \{id\}$
If $det(f(\mathbf{y}_0, id)) = 0$ or $\det(f(\mathbf{y}_1, id)) = 0$
then $\textsc{Win} \leftarrow \mathsf{false}$; $dk \leftarrow \perp$
Else $dk \leftarrow \overline{\mathsf{L}}[\mu, \mathsf{IDSp}, \mathsf{C}].\mathsf{Kg}(pars, msk, id)$
Return dk

proc Ch(id) // $\mathrm{RL}_0, \mathrm{RL}_1$
$id^* \leftarrow id$
If $f(\mathbf{y}_0, id) \neq \mathbf{0}_{n\times n}$ or $f(\mathbf{y}_1, id) \neq \mathbf{0}_{n\times n}$ then $\textsc{Win} \leftarrow \mathsf{false}$

proc Finalize(d') // $\mathrm{RL}_0, \mathrm{RL}_1$
Return $((d' = 1)$ and $(id^* \notin IS)$ and $\textsc{Win})$

Figure 7: Games $\mathrm{RL}_0, \mathrm{RL}_1$ ("Real-to-Losssy") associated to n, μ, IDSp and auxiliary input generator algorithms Aux_0 and Aux_1.

2. Key generation: Given parameters $(\mathbf{A}, \mathbf{U})$, master secret $(\mathbf{T}, \mathbf{B}, \mathbf{R}, \mathbf{y})$ and identity $id \in \mathsf{IDSp}$, algorithm $\overline{\mathsf{L}}[\mu, \mathsf{IDSp}, \mathsf{C}].\mathsf{Kg}$ proceeds as follows. By the structure of $\mathbf{F}(\mathbf{A}, \mathbf{U}, id)$ from Equation (23), it can invoke algorithm $\mathsf{SampleRight}$ from Lemma 5.3 that returns decryption key $\mathbf{D}_{id}$, which is a basis of $\Lambda^{\perp}(\mathbf{F}(\mathbf{A}, \mathbf{U}, id))$ with a distribution that only depends on $\mathbf{F}(\mathbf{A}, \mathbf{U}, id))$, R and $\tilde{L}$, and satisfies $\|\tilde{\mathbf{D}}_{id}\| \leq \tilde{L}R\sqrt{m} = \tilde{L}_1$. This can be done as long as $f(\mathbf{y}, id)$ is a full-rank matrix.
3. Evaluate: Given parameters $(\mathbf{A}, \mathbf{U})$, identity $id \in \mathsf{IDSp}$ and input $(\mathbf{x}, \mathbf{e}) \in D_{\beta}^{\hat{n}} \times D_{\beta}^{2m}$, algorithm $\overline{\mathsf{L}}[\mu, \mathsf{IDSp}, \mathsf{C}].\mathsf{Ev}$ returns output $\mathbf{c} \leftarrow \mathsf{LWE.Ev}(\mathbf{F}(\mathbf{A}, \mathbf{U}, id), \mathbf{x}, \mathbf{e}) = \mathbf{F}(\mathbf{A}, \mathbf{U}, id)^{\mathsf{T}} \cdot \mathbf{x} + \mathbf{e}$.
4. Invert: Given parameters $(\mathbf{A}, \mathbf{U})$, identity $id \in \mathsf{IDSp}$, decryption key $\mathbf{D}_{id}$ for id and output (ciphertext) $\mathbf{c}$, algorithm $\overline{\mathsf{L}}[\mu, \mathsf{IDSp}, \mathsf{C}].\mathsf{Ev}^{-1}$ returns $\mathsf{LWE.Ev}^{-1}(\mathbf{F}(\mathbf{A}, \mathbf{U}, id), \mathbf{D}_{id})$.

Invertibility. Auxiliary input $\mathbf{y}$ grants invertibility, meaning induced IBTDF $\overline{\mathsf{L}}[\mu, \mathsf{IDSp}, \mathsf{C}](\mathbf{y})$ satisfies the correct inversion condition, if $f(\mathbf{y}, id)$ is a full-rank matrix for all $id \in \mathsf{IDSp}$. In that case we have $\|\mathbf{D}_{id}\| \leq \tilde{L}_1$ and $\overline{\mathsf{L}}[\mu, \mathsf{IDSp}, \mathsf{C}].\mathsf{Ev}^{-1}$ is correct by Lemma 5.2 and the choice of β in Equation (21).

5.4 Real-to-lossy lemma

Consider games $\mathrm{RL}_0, \mathrm{RL}_1$ which are defined as in Figure 7. The following lemma says it is hard to distinguish these games. We will apply this by defining Aux_0 in such a way that its output $\mathbf{y}_0$ results in a lossy setup.

Lemma 5.7 *Let $n, \mu \geq 1$ be integers and IDSp. Let Aux_0 and Aux_1 be auxiliary input generators for $\overline{\mathsf{L}}[\mu, \mathsf{IDSp}, \mathsf{C}]$ and A an adversary. Then there is an adversary P such that*

$$\Pr[\mathrm{RL}_0^A] - \Pr[\mathrm{RL}_1^A] \;\leq\; 2 \cdot \mathbf{Adv}^{\mathrm{lwe}}_{n,\alpha}(B) + negl(n) \; . \tag{24}$$

The running time of P is that of A plus some overhead. If A is selective-id then so is P.

The last statement allows us to use the lemma in both the selective-id and adaptive-id cases.

Proof: We define games R_0 and R_1 to be the same as RL_0 and RL_1, respectively, with the difference that in $\overline{\mathsf{L}}[\mu, \mathsf{IDSp}, \mathsf{C}].\mathsf{Pg}$ the distribution of $\mathbf{A}$ is changed to uniform random over $\mathbb{Z}_q^{\hat{n}\times m}$. By Remark 5.5 we have that

$$\Pr[\mathrm{R}_0^A] - \Pr[\mathrm{RL}_0^A] \;\leq\; \mathbf{Adv}^{\mathrm{lwe}}_{n,\alpha}(B), \qquad \Pr[\mathrm{R}_1^A] - \Pr[\mathrm{RL}_1^A] \;\leq\; \mathbf{Adv}^{\mathrm{lwe}}_{n,\alpha}(B) \; . \tag{25}$$

We claim that in R_0 and R_1 the values $\mathbf{y}_0$ and $\mathbf{y}_1$ are statistically hidden from A's view. Because $\mathbf{A}$ is uniform and therefore of full rank (with high probability), matrices $\mathbf{R}[i]$ act as strong extractors, i.e.,

$\mathbf{A}, \mathbf{AR}[0], \ldots, \mathbf{AR}[\mu]$ is statistically close to a uniform string of the same size. (This was formally proved in [2].) Hence for all i with $\overline{id}[i] \in \{\mathbf{0}_{\hat{n}\times\hat{n}}, \pm\mathbf{1}_{\hat{n}\times\hat{n}}\}$, $\mathbf{U}[i]$ statistically hides $\mathbf{y}[i]$. Because C is suitable there exists at most one index i^* with $\overline{id}[i^*] \notin \{\mathbf{0}_{\hat{n}\times\hat{n}}, \pm\mathbf{1}_{\hat{n}\times\hat{n}}\}$. For that index $\mathbf{U}[i^*] = \mathbf{y}[i^*] \cdot \mathbf{B}$ (for uniform $\mathbf{B}$) statistically hides $\mathbf{y}[i^*]$. Since the execution of the remaining game is independent of whether $\mathbf{y}$ comes from Aux_0 or Aux_1, we obtain

$$\Pr[\mathrm{R}_0^A] - \Pr[\mathrm{R}_1^A] \;\le\; negl(n)\;. \tag{26}$$

which concludes the proof. ∎

5.5 Selective-id Security

We consider IBTDF $\overline{\mathsf{L}}[\mu, \mathbb{Z}_q^{\hat{n}} \setminus \{\mathbf{0}\}, \mathsf{C}_{\mathrm{FRD}}]$, the instance of our construction with $\mathsf{IDSp} = \mathbb{Z}_q^{\hat{n}} \setminus \{\mathbf{0}\}$, auxiliary input $\mathbf{y} = (-\mathsf{C}_{\mathrm{FRD}}(\mathbf{0}), \mathbf{1}_{\hat{n}\times\hat{n}}) \in (\mathbb{Z}_q^{\hat{n}\times\hat{n}})^2$ and $\mathsf{C}_{\mathrm{FRD}}$, where $\mathsf{C}_{\mathrm{FRD}} : \mathbb{Z}_q^{\hat{n}} \to \mathbb{Z}_q^{\hat{n}\times\hat{n}}$ is a full-rank difference encoding as constructed in [2]. (I.e., for each $\mathbf{x} \neq \mathbf{x}'$, matrix $\mathsf{C}_{\mathrm{FRD}}(\mathbf{x}) - \mathsf{C}_{\mathrm{FRD}}(\mathbf{x}')$ is of full rank.)

Note that our scheme satisfies the correct inversion requirement because $f(\mathbf{y}, id) = \mathsf{C}_{\mathrm{FRD}}(id) - \mathsf{C}_{\mathrm{FRD}}(\mathbf{0})$ is of full rank for all $id \in \mathsf{IDSp} = \mathbb{Z}_q^{\hat{n}} \setminus \{\mathbf{0}\}$. We show that this IBTDF is selective-id δ-lossy for $\delta = 1$, meaning fully selective-id lossy, and hence selective-id one-way. To do this we define a sibling $\overline{\mathsf{LF}}[1, \mathbb{Z}_q^{\hat{n}} \setminus \{\mathbf{0}\}, \mathsf{C}_{\mathrm{FRD}}]$. It preserves the key-generation, evaluation and inversion algorithms of $\overline{\mathsf{F}}[1, \mathbb{Z}_q^{\hat{n}} \setminus \{\mathbf{0}\}, \mathsf{C}_{\mathrm{FRD}}]$ and alters parameter generation to

Algorithm $\overline{\mathsf{LF}}[1, \mathbb{Z}_q^{\hat{n}} \setminus \{\mathbf{0}\}, \mathsf{C}_{\mathrm{FRD}}].\mathsf{Pg}(id)$:
$\mathbf{y} \leftarrow (-\mathsf{C}_{\mathrm{FRD}}(id), \mathbf{1}_{\hat{n}\times\hat{n}})$; $(pars, msk) \xleftarrow{\$} \overline{\mathsf{L}}[1, \mathbb{Z}_q^{\hat{n}} \setminus \{\mathbf{0}\}, \mathsf{C}_{\mathrm{FRD}}].\mathsf{Pg}(\mathbf{y})$; Return $(pars, msk)$.

The following says that our IBTDF is 1-lossy under the LWE assumption with lossiness $\ell = 2m$.

Theorem 5.8 *Let $m = c_2 n > c_1 n = \hat{n}$ and $\ell = 2m$. Let $\mathsf{L} = \overline{\mathsf{L}}[1, \mathbb{Z}_q^{\hat{n}} \setminus \{\mathbf{0}\}, \mathsf{C}_{\mathrm{FRD}}]$ be the IBTDF associated by our construction to parameters $\mu = 1$ and $\mathsf{IDSp} = \mathbb{Z}_q^{\hat{n}} \setminus \{\mathbf{0}\}$. Let $\mathsf{LF} = \overline{\mathsf{LF}}[1, \mathbb{Z}_q^{\hat{n}} \setminus \{\mathbf{0}\}, \mathsf{C}_{\mathrm{FRD}}]$ be the sibling associated to it as above. Let $\delta = 1$ and let be A a selective-id adversary. Then there is an adversary B such that*

$$\mathbf{Adv}^{\delta\text{-los}}_{\mathsf{L},\mathsf{LF},\ell}(A) \;\le\; 2 \cdot \mathbf{Adv}^{\mathrm{lwe}}_{n,\alpha}(B) + negl\;. \tag{27}$$

The running time of B is that of A plus overhead.

Proof: On input id, let algorithm Aux return $(-\mathsf{C}_{\mathrm{FRD}}(id), \mathbf{1}_{\hat{n}\times\hat{n}})$. Let $\mathrm{RL}_0, \mathrm{RL}_1$ be the games of Figure 5 with $\mu = 1$, $\mathsf{IDSp} = \mathbb{Z}_q^{\hat{n}} \setminus \{\mathbf{0}\}$ and this Aux. Then we claim

$$\Pr\left[\,\mathrm{Real}^A_{\mathsf{L}}\,\right] = \Pr\left[\,\mathrm{RL}^A_0\,\right] \quad \text{and} \quad \Pr\left[\,\mathrm{Lossy}^A_{\mathsf{L},\mathsf{LF},\ell}\,\right] = \Pr\left[\,\mathrm{RL}^A_1\,\right]\;. \tag{28}$$

To justify this let id^* be the identity queried by A to both **Initialize** and **Ch**. (These queries are the same because A is selective-id.) Then $\mathbf{y}_0 = (-\mathsf{C}_{\mathrm{FRD}}(id^*), \mathbf{1}_{\hat{n}\times\hat{n}})$ so $f(\mathbf{y}_0, id) = \mathsf{C}_{\mathrm{FRD}}(id) - \mathsf{C}_{\mathrm{FRD}}(id^*)$. Since $\mathsf{C}_{\mathrm{FRD}}$ is a full-rank difference encoding, this is of full rank iff $id \neq id^*$. This means that the conjunct $(id^* \notin IS) \wedge \textsc{Win}$ is always true. The claim of Equation (28) is now true because game RL_0 generates parameters with the real auxiliary input $\mathbf{y}_1 = (-\mathsf{C}_{\mathrm{FRD}}(\mathbf{0}), \mathbf{1}_{\hat{n}\times\hat{n}}) \in (\mathbb{Z}_q^{\hat{n}\times\hat{n}})^2$ that, via $\overline{\mathsf{L}}[1, \mathbb{Z}_q^{\hat{n}} \setminus \{\mathbf{0}\}, \mathsf{C}_{\mathrm{FRD}}]$, defines F. However game RL_1 generates parameters with auxiliary input $\mathbf{y}_0$. Since $f(\mathbf{y}_0, id^*) = \mathbf{0}_{\hat{n}\times\hat{n}}$, the dependency of $\mathbf{c}$ on $\mathbf{B}$ vanishes when $id = id^*$. More concretely, $\mathbf{c}$ is an LWE evaluation with matrix $\mathbf{H} = \mathcal{H}(id^*) = [\mathbf{A} \mid \mathbf{A} \cdot R(\mathbf{R}, id^*)]$. As discussed in Remark 5.6, matrix $\mathbf{H}$ has lossy structure, i.e., $\mathbf{H}^T = \left[\bar{\mathbf{H}}^T \mid \bar{\mathbf{H}}^T\mathbf{S} + \mathbf{E}\right]$ with $\mathbf{E} = \left[\begin{smallmatrix}\mathbf{I}\\ R(\mathbf{R}, id^*)]^T\end{smallmatrix}\right] \mathbf{E}'$ for some Gaussian $\mathbf{E}'$ of parameter αq. By Remark 5.6 and the choice of the parameters from Equation (20) and Equation (21) it only leaves to verify the bound $s = O(n^{3/2})$ on $s_1(\mathbf{E})$. This is correct since $s_1(\mathbf{E}) = s_1(\mathbf{E}')(1 + s_1(R(\mathbf{R}, id^*)) = O(n) \cdot O(\sqrt{n}) = O(n^{3/2})$. (With high probability (over the choice of $\mathbf{R}[i]$ and $\mathbf{E}'$.) ∎

5.6 Full Security

We consider IBTDF $\overline{\mathsf{L}}[\mu, \{0,1\}^\mu, \mathsf{C}_f]$, the instance of our construction with $\mathsf{IDSp} = \{0,1\}^\mu$, auxiliary input $\mathbf{y} = (\mathbf{1}_{\hat{n}\times\hat{n}}, \mathbf{0}_{\hat{n}\times\hat{n}}, \ldots, \mathbf{0}_{\hat{n}\times\hat{n}})$ and $\mathsf{C}_f : \{0,1\}^\mu \to \mathbb{Z}_q^{\hat{n}\times\hat{n}}$ maps $\mathbf{x} \in \{0,1\}^\mu$ into a vector $\mathbf{X}$ of matrices such that $\mathbf{X}[i] = (-1)^{\mathbf{x}[i]} \cdot \mathbf{1}_{\hat{n}\times\hat{n}} \in \mathbb{Z}_q^{\hat{n}\times\hat{n}}$. Note that our scheme satisfies the correct inversion requirement because $f(\mathbf{y}, id) = \mathbf{1}_{\hat{n}\times\hat{n}}$ is of full rank for all $id \in \mathsf{IDSp}$.

We show that this IBTDF is adaptive-id δ-lossy for $\delta = (8Q)^{-1}$ where Q is the number of key-derivation queries of the adversary. By Theorem 3.2 this means $\overline{\mathsf{F}}[\mu, \{0,1\}^\mu, \mathsf{C}_f]$ is adaptive-id one-way. To do this we define a sibling $\overline{\mathsf{LF}}_Q[\mu, \{0,1\}^\mu, \mathsf{C}_f]$. It preserves the key-generation, evaluation and inversion algorithms of $\overline{\mathsf{F}}[\mu, \{0,1\}^\mu, \mathsf{C}_f]$ and alters parameter generation to

$$\begin{array}{l}\text{Algorithm } \overline{\mathsf{LF}}[\mu, \{0,1\}^\mu, \mathsf{C}_f].\mathsf{Pg}(id) : \\ \mathbf{y} \xleftarrow{\$} \mathsf{Aux}\,;\ (pars, msk) \xleftarrow{\$} \overline{\mathsf{L}}[\mu, \{0,1\}^\mu, \mathsf{C}_f].\mathsf{Pg}(\mathbf{y})\,;\ \text{ Return } (pars, msk)\ .\end{array}$$

where Aux is a randomized algorithm from [2, 21] that generates $\mathbf{y} \in (\mathbb{Z}_q^{\hat{n}\times\hat{n}})^{\mu+1}$ such that the image of $f(\mathbf{y}, \cdot)$ is either $\mathbf{0}_{n\times n}$ or of full rank and $f(\mathbf{y}, \cdot)$ is pairwise independent, i.e, for all $id \neq id'$, $\Pr_{\mathsf{Aux}}[f(\mathbf{y}, id) = \mathbf{0}_{n\times n} \mid f(\mathbf{y}, id') = \mathbf{0}_{n\times n}] = 1/(2Q)$. The following says that our IBTDF is δ-lossy under the LWE assumption with lossiness $\ell = 2m$.

Theorem 5.9 *Let $m = c_2 n > c_1 n = \hat{n}$ and $\ell = 2m$. Let $\mathsf{L} = \overline{\mathsf{L}}[\mu, \{0,1\}^\mu, \mathsf{C}_f]$ be the IBTDF associated by our construction to parameters μ and $\mathsf{IDSp} = \{0,1\}^\mu$. Let A be an adaptive-id adversary that makes a maximal number of Q queries and let $\delta = (8Q)^{-1}$. Let $\mathsf{LF} = \overline{\mathsf{LF}}_Q[\mu, \{0,1\}^\mu, \mathsf{C}_f]$ be the sibling associated to L, A as above. Then there is an adversary B such that*

$$\mathbf{Adv}^{\delta\text{-los}}_{\mathsf{L},\mathsf{LF},\ell}(A) \ \le\ 2 \cdot \mathbf{Adv}^{\mathrm{lwe}}_{n,\alpha}(B) + negl(n)\ . \tag{29}$$

The running time of B is that of A plus polynomial overhead.

Proof: (Sketch) Let Q be the number of queries made by A and let algorithm Aux be defined as above. Let $\mathrm{RL}_0, \mathrm{RL}_1$ be the games of Figure 7 with $\mathsf{IDSp} = \{0,1\}^\mu$ and this Aux. Let $\mathrm{E}(IS, id^*)$ denote the event that when $\mathbf{Finalize}(d')$ is called in RL_0^A the flag $\textsc{Win} \leftarrow \mathsf{false}$ is set and $id^* \notin IS$. (Note that $\eta(IS, id^*)$ only depends on IS, id^* since $\mathbf{y}_0$ is exclusively used to set $\textsc{Win} \leftarrow \mathsf{false}$.) Let $\eta(IS, id^*)$ be the probability that $\mathrm{E}(IS, id^*)$ happens. In [2], it was shown that $\lambda_{\mathrm{low}} := \frac{1}{4Q} \le \eta(IS, id^*) \le \frac{1}{2Q} := \lambda_{\mathrm{up}}$. Since RL_0^A and $\mathrm{Real}_{\mathsf{L}}^A$ are only different when $\mathrm{E}(IS, id^*)$ happens, one would like to argue that $\lambda_{\mathrm{low}} \cdot \Pr\left[\mathrm{Real}_{\mathsf{L}}^A\right] = \Pr\left[\mathrm{RL}_0^A\right]$ but this is not true since $\mathrm{E}(IS, id^*)$ and $\mathrm{Real}_{\mathsf{L}}^A$ may not be independent. To get rid of this unwanted dependence we consider a modification of RL_0 and RL_1 which adds some artificial abort such that in total it always sets $\textsc{Win} \leftarrow \mathsf{false}$ with probability around $1 - \lambda_{\mathrm{low}}$, independent of the view of the adversary. (Since, given IS, id^*, the exact value of $\eta(IS, id^*)$ cannot be computed efficiently, it needs to be approximated using sampling.) Concretely, games $\hat{\mathrm{RL}}_0$ and $\hat{\mathrm{RL}}_1$ are defined as RL_0 and RL_1, respectively, the only difference being **Finalize** which is defined as follows.

$$\begin{array}{l}\underline{\textbf{proc Finalize}(d')} \quad /\!/\ \hat{\mathrm{RL}}_0, \hat{\mathrm{RL}}_1 \\ \text{Compute an approximation } \eta'(IS, id^*) \text{ of } \eta(IS, id^*) \\ \text{If } \eta'(IS, id^*) > \lambda_{\mathrm{low}} \text{ then set } \textsc{Win} \leftarrow \mathsf{false} \text{ with probability } 1 - \lambda_{\mathrm{low}}/\eta'(IS, id^*) \\ \text{Return } ((d' = 1) \text{ and } (id^* \notin IS) \text{ and } \textsc{Win})\end{array}$$

One can again show that with a polynomial number of samples to compute approximation $\eta'(IS, id^*)$,

$$\delta \cdot \Pr\left[\mathrm{Real}_{\mathsf{L}}^A\right] = \Pr\left[\hat{\mathrm{RL}}_0^A\right], \tag{30}$$

where $\delta = \lambda_{\mathrm{low}}/2$ is as in the theorem statement. Similar to the proof of Theorem 5.8, we can show that

$$\Pr\left[\mathrm{Lossy}_{\mathsf{L},\mathsf{LF},\ell}^A\right] = \Pr\left[\hat{\mathrm{RL}}_1^A\right]\ . \tag{31}$$

Now Equation (29) follows from Equation (1), Equation (30), Equation (31) and Lemma 5.7. ∎

Acknowledgments

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A Anonymous IBE

In this section we describe an IBE scheme that is similar to IBE from Section 4 with the difference that it encrypts group elements (rather than bits) and it is slightly more efficient. We associate to any integer $\mu \geq 1$ and any identity space $\mathsf{IDSp} \subseteq \mathbb{Z}_p^\mu$ an IBE scheme $\mathsf{IBE}'[\mu, \mathsf{IDSp}]$ that has message space $\mathbb{G}_T^*$ and algorithms as follows:

1. Parameters: Algorithm $\mathsf{IBE}[\mu, \mathsf{IDSp}].\mathsf{Pg}$ lets $g \xleftarrow{\$} \mathbb{G}^*$; $t, z \xleftarrow{\$} \mathbb{Z}_p^*$; $\hat{g} \leftarrow g^t$; $Z \leftarrow \mathbf{e}(g,g)^z$. It then lets $H, \hat{H}, U \xleftarrow{\$} \mathbb{G}$; $\mathbf{U} \xleftarrow{\$} \mathbb{G}^{\mu+1}$. It returns $pars = (g, \hat{g}, H, U, \hat{H}, \mathbf{U}, Z)$ as the public parameters and $msk = (t, z)$ as the master secret key.

2. Key generation: Given parameters $(g, \hat{g}, H, U, \mathbf{U}, Z)$, master secret (t, z) and identity $id \in \mathsf{IDSp}$, algorithm $\mathsf{IBE}'[\mu, \mathsf{IDSp}].\mathsf{Kg}$ returns decryption key (D_1, D_2, D_3, D_4) computed by letting $r, \hat{r} \xleftarrow{\$} \mathbb{Z}_p$ and setting

$$D_1 \leftarrow g^z \cdot \mathcal{H}(\mathbf{U}, id)^{tr} \cdot H^{t\hat{r}} \; ; \; D_2 \leftarrow U^r \cdot H^{\hat{r}} \; ; \; D_3 \leftarrow g^{-tr} \; ; \; D_4 \leftarrow g^{-t\hat{r}} \; .$$

3. Encryption: Given parameters $(g, \hat{g}, H, U, \mathbf{U}, Z)$, identity $id \in \mathsf{IDSp}$ and message $M \in \mathbb{G}_T^*$, algorithm $\mathsf{IBE}[\mu, \mathsf{IDSp}].\mathsf{Enc}$ returns ciphertext $(C_1, C_2, C_3, C_4, C_5)$ computed as follows. It lets $s, \hat{s} \xleftarrow{\$} \mathbb{Z}_p$ and

$$C_1 \leftarrow g^s \; ; \; C_2 \leftarrow \hat{g}^{\hat{s}} \; ; \; C_3 \leftarrow \mathcal{H}(\mathbf{U}, id)^s \cdot U^{\hat{s}} \; ; \; C_4 \leftarrow H^{s+\hat{s}} \; ; \; C_5 \leftarrow Z^{-s} \cdot M \; .$$

4. Decryption: Given parameters $(g, \hat{g}, H, U, \mathbf{U}, Z)$, identity $id \in \mathsf{IDSp}$, decryption key (D_1, D_2, D_4, D_4) for id and ciphertext $(C_1, C_2, C_3, C_4, C_5)$, algorithm $\mathsf{IBE}[\mu, \mathsf{IDSp}].\mathsf{Dec}$ returns

$$M = \mathbf{e}(C_1, D_1)\mathbf{e}(C_2, D_2)\mathbf{e}(C_3, D_3)\mathbf{e}(C_4, D_4)C_5 \ .$$

Compared to $\mathsf{IBE}[\mu, \mathsf{IDSp}]$ from Section 4 , the efficiency improvement consists of replacing $\mathcal{H}(\hat{\mathbf{U}}, id)$ by U in the computation of D_2 and C_3 and of setting $\hat{H} := H$. Using the techniques of the ciphertext pseudorandomness lemma (Lemma 4.1) one can show that the elements (C_1, C_2, C_3, C_4) of the ciphertext are pseudorandom. (Here the reduction knows the secret z.) In a final similar hybrid step one can also show that, under the Bilinear Diffie-Hellman assumption (which is implied by the DLIN assumption), the element C_5 is also pseudorandom. (Here is reduction knows the secret t.) As our main ID-based TDF result uses anonymous IBE techniques, the main ideas of this systems security is implicit in our main proof. A formal proof of the above stand alone system is deferred to the full version.

B Applications

We expand first on the application to achieving deterministic IBE and then on achieving hedged IBE.

D-PKE. Deterministic PKE (D-PKE) cannot achieve IND-CPA security. Bellare, Boldyreva and O'Neill [6] defined a target notion PRIV for it that captures the best possible security under the condition that encryption is deterministic. D-PKE provides a way to do fast (logarithmic time) search on encrypted data. PEKS [19] offers higher security but takes linear time, and trading some security for a significant increase in searching speed is attractive for large databases.

Achieving PRIV for D-PKE has been (and remains) a challenge. It is possible in the RO model [6]. The best results without ROs are due to Boldyreva, Fehr and O'Neill [15], who show how to achieve PRIV without random oracles for message sequences which are blocksources, meaning each message has some min-entropy even given the previous ones. Using the Leftover Hash Lemma (LHL) [14, 37], they show that any LTDF is a D-PKE scheme that is PRIV-secure for blocksources as long as the lossy branch is a universal hash function.

D-IBE. We introduce deterministic IBE (D-IBE). The PRIV definition is easily extended to this setting. D-IBE offers, over D-PKE, the same advantages that IBE offers over PKE, for example that there are no certificates and encryption depends only on the identity of the receiver. Again, D-IBE can be achieved in the RO model by setting the coins of an IBE scheme to the RO-hash of the message. (This is how PKE is turned into D-PKE in the RO model in [8, 6].) We ask what can be done without ROs.

We show that our constructions of DLIN-based lossy IB-TDFs have the properties necessary to obtain PRIV-secure D-IBE schemes for blocksources under the paradigm of [15] in the selective case. We start by observing that the lossy branches are universal hash functions. This can be seen from Equations (3), (4), (5) and (6). In the lossy case, $f(\mathbf{y}, id) = 0$, and the function has a range R of size p^2. Now if x_1, x_2 are distinct inputs, then the outputs of the function on them collide exactly when $(\langle \mathbf{s}, x_1 \rangle, \langle \hat{\mathbf{s}}, x_1 \rangle) = (\langle \mathbf{s}, x_2 \rangle, \langle \hat{\mathbf{s}}, x_2 \rangle)$. The probability that this happens when $\mathbf{s}, \hat{\mathbf{s}}$ are chosen at random from $\mathbb{Z}_p^n$ is $1/p^2 = 1/|R|$.

Hedged IBE. The definitions and methods of [7] can be extended to the identity-based setting in a straightforward way in the selective setting once we have universal lossy IB-TDFs. There are two approaches. One is generic composition of an IBE scheme with a IB-TDF. The other is to first pad the message with randomness and then apply the IB-TDF.

Adaptive setting. It remains open to achieve deterministic or hedged IBE in the adaptive security setting.