On a generalized combinatorial conjecture involving addition mod $2^k - 1$

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Abstract

In this note, we give a simple proof of the combinatorial conjecture proposed by Tang, Carlet and Tang, based on which they constructed two classes of Boolean functions with many good cryptographic properties. We also give more general properties about the generalization of the conjecture they propose.

1 Introduction

In a very recent paper inspired by the previous work of Tu and Deng [6], Tang, Carlet and Tang [5] constructed an infinite family of Boolean functions with many good cryptographic interesting properties depending on the validity of the following combinatorial property:

Conjecture 1.

$$\forall k \geq 2, \max_{t \in (\mathbb{Z}/(2^k-1)\mathbb{Z})^*} \#\left\{(a,b) \in \left(\mathbb{Z}/(2^k-1)\mathbb{Z}\right)^2 \mid a-b=t; w(a)+w(b) \leq k-1\right\} \leq 2^{k-1} \ .$$

They verified it experimentally for $k \leq 29$, as well as the following generalized property for $k \leq 15$ where $u \in \mathbb{Z}/(2^k-1)\mathbb{Z}$ is such that $\gcd(u, 2^k-1) = 1$:

Conjecture 2.

$$\forall k \geq 2, \max_{t \in (\mathbb{Z}/(2^k-1)\mathbb{Z})^*} \#\left\{(a,b) \in \left(\mathbb{Z}/(2^k-1)\mathbb{Z}\right)^2 \mid ua \pm b=t; w(a)+w(b) \leq k-1\right\} \leq 2^{k-1} \ .$$

This generalized conjecture includes the original conjecture proposed by Tu and Deng [6].

From now on, let us denote by $S_{k,t,\pm,u}$ the quantity of interest:

$$S_{k,t,\pm,u} = \#\left\{(a,b) \in \left(\mathbb{Z}/(2^k-1)\mathbb{Z}\right)^2 \mid ua \pm b=t; w(a)+w(b) \leq k-1\right\} \ .$$

In Section 2, we give some general properties about such sets and their cardinalities. In Section 3, we give the proof of Conjecture 1. In Section 4, we conjecture a recursive formula for $2^{k-1} - S_{k,t,-,1}$.

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2 General properties

Here we follow the approach of the previous attempts to prove the original conjecture of Tu and Deng [2, 1].

We recall the elementary results:

Lemma 1. *For $k \geq 1$,*

- $\forall a \in \mathbb{Z}/(2^k-1)\mathbb{Z}$, $w(2a) = w(a)$;
- $\forall a \in \left(\mathbb{Z}/(2^k-1)\mathbb{Z}\right)^*$, $w(-a) = k - w(a)$.

We first remark that for a given $a \in \mathbb{Z}/(2^k-1)\mathbb{Z}$, b must be equal to $\pm(t-ua)$, whence the following lemma.

Lemma 2. *For $k \geq 2$,*

$$S_{k,t,\pm,u} = \#\left\{a \in \mathbb{Z}/(2^k-1)\mathbb{Z} \mid w(a) + w(\pm(t-ua)) \leq k-1\right\} \ .$$

We now show that is enough to study the conjecture for one t, but also one u, in each cyclotomic class.

Lemma 3. *For $k \geq 2$,*

$$S_{k,t,\pm,u} = S_{k,2t,\pm,u} \ .$$

Proof. Indeed $a \mapsto 2a$ is a permutation of $\mathbb{Z}/(2^k-1)\mathbb{Z}$ so that

$$\begin{aligned} S_{k,2t,\pm,u} &= \#\left\{a \in \mathbb{Z}/(2^k-1)\mathbb{Z} \mid w(a) + w(\pm(2t-ua)) \leq k-1\right\} \\ &= \#\left\{a \in \mathbb{Z}/(2^k-1)\mathbb{Z} \mid w(2a) + w(\pm 2(t-ua)) \leq k-1\right\} \\ &= \#\left\{a \in \mathbb{Z}/(2^k-1)\mathbb{Z} \mid w(a) + w(\pm(t-ua)) \leq k-1\right\} \\ &= S_{k,t,\pm,u} \ . \end{aligned}$$

□

Lemma 4. *For $k \geq 2$,*

$$S_{k,t,\pm,u} = S_{k,t,\pm,2u} \ .$$

Proof. Using the previous lemma:

$$\begin{aligned} S_{k,t,\pm,2u} &= S_{k,2t,\pm,2u} \\ &= \#\left\{a \in \mathbb{Z}/(2^k-1)\mathbb{Z} \mid w(a) + w(\pm(2t-2ua)) \leq k-1\right\} \\ &= \#\left\{a \in \mathbb{Z}/(2^k-1)\mathbb{Z} \mid w(a) + w(\pm(t-ua)) \leq k-1\right\} \\ &= S_{k,t,\pm,u} \ . \end{aligned}$$

□

We now show a more elaborate relation.

Lemma 5. *For $k \geq 2$ and $\gcd(u, 2^k-1) = 1$,*

$$S_{k,t,\pm,u} = S_{k,\pm u^{-1}t,\pm,u^{-1}} \ .$$

Proof. We use the fact that $a \mapsto u^{-1}(\mp a + t)$ is a permutation of $\mathbb{Z}/(2^k-1)\mathbb{Z}$.

$$\begin{aligned} S_{k,t,\pm,u} &= \#\left\{a \in \mathbb{Z}/(2^k-1)\mathbb{Z} \mid w(a) + w(\pm(t-ua)) \leq k-1\right\} \\ &= \#\left\{a \in \mathbb{Z}/(2^k-1)\mathbb{Z} \mid w(u^{-1}(\mp a + t)) + w(a) \leq k-1\right\} \\ &= \#\left\{a \in \mathbb{Z}/(2^k-1)\mathbb{Z} \mid w(\pm(\pm u^{-1}t - u^{-1}a)) + w(a) \leq k-1\right\} \\ &= S_{k,\pm u^{-1}t,\pm,u^{-1}} \ . \end{aligned}$$

□

3 Proof of the conjecture

We now prove Conjecture 1, and so Conjecture 2 for u equal to any power of 2 according to Lemma 4.

First, we note that for $u = 1$ and the sign $-$, Lemma 5 becomes

$$S_{k,t,-,1} = S_{k,-t,-,1} \ .$$

Second, for the specific values of $a = 0$, t, we have that

- $w(0) + w(-t) = w(-t) \leq k-1$,
- and $w(t) + w(0) = w(t) \leq k-1$,

so that we always have

$$\{0,t\} \subset \left\{a \in \mathbb{Z}/(2^k-1)\mathbb{Z} \mid w(a) + w(-(t-a)) \leq k-1\right\} \ .$$

Finally, for $a \neq 0$, t, we have that

$$\begin{aligned} w(a) + w(-(t-a)) &= k - w(-a) + k - w(t-a) \\ &= 2k - (w(-a) + w(t-a)) \ . \end{aligned}$$

Then using the fact that $a \mapsto -a$ is a permutation of $\mathbb{Z}/(2^k-1)\mathbb{Z}$:

$$\begin{aligned} S_{k,t,-,1} &= 2 + \#\left\{a \in \mathbb{Z}/(2^k-1)\mathbb{Z} \setminus \{0,t\} \mid w(a) + w(-(t-a)) \leq k-1\right\} \\ &= 2 + \#\left\{a \in \mathbb{Z}/(2^k-1)\mathbb{Z} \setminus \{0,t\} \mid w(-a) + w(t-a) \geq k+1\right\} \\ &= 2 + \#\left\{a \in \mathbb{Z}/(2^k-1)\mathbb{Z} \mid w(-a) + w(t-a) \geq k+1\right\} \\ &= 2 + \#\left\{a \in \mathbb{Z}/(2^k-1)\mathbb{Z} \mid w(a) + w(t+a) \geq k+1\right\} \\ &= 2 + (2^k - 1 - \#\left\{a \in \mathbb{Z}/(2^k-1)\mathbb{Z} \mid w(a) + w(t+a) \leq k\right\}) \\ &\leq 2 + (2^k - 1 - \#\left\{a \in \mathbb{Z}/(2^k-1)\mathbb{Z} \mid w(a) + w(t+a) \leq k-1\right\}) \\ &\leq 2^k + 1 - S_{k,-t,-,1} \\ &\leq 2^k + 1 - S_{k,t,-,1} \ . \end{aligned}$$

Hence

$$2S_{k,t,-,1} \leq 2^k + 1 \ ,$$

but we know that $S_{k,t,-,1}$ is an integer, which concludes the proof of Conjecture 1.

We also note that for $t = 0$,

$$\begin{aligned} S_{k,0,-,1} &= \#\left\{a \in \mathbb{Z}/(2^k-1)\mathbb{Z} \mid 2w(a) \leq k-1\right\} \\ &= \sum_{w=0}^{\lfloor \frac{k-1}{2} \rfloor} \#\left\{a \in \mathbb{Z}/(2^k-1)\mathbb{Z} \mid w(a) = w\right\} \\ &= \sum_{w=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{w} \ , \end{aligned}$$

which is equal to $2^{k-1} - \binom{k}{(k+1)/2}$ if k is odd, and $2^{k-1} - \binom{k}{k/2-1} - \binom{k/2}{k}/2$ if k is even. Therefore the conjecture can be naturally extended to include the case $t = 0$.

4 Computing the exact gap

If we rewrite the above reasoning more carefully, we find that

$$S_{k,t,-,1} = 2^{k-1} + (1 - \#\left\{a \in \mathbb{Z}/(2^k-1)\mathbb{Z} \mid w(a) + w(t+a) = k\right\})/2 \ .$$

It is an interesting problem to find a closed-form formula for the value of

$$\begin{aligned} M_{k,t} &= \#\left\{a \in \left(\mathbb{Z}/(2^k-1)\mathbb{Z}\right)^* \mid w(a) + w(t+a) = k\right\} \ , \\ M_k &= \min_{t \in \mathbb{Z}/(2^k-1)\mathbb{Z}} M_{k,t} \ . \end{aligned}$$

We denote by Δ_k the following value

$$\Delta_k = \frac{M_k - 1}{2} \ ,$$

so that $S_{k,t,-,1} = 2^{k-1} - \Delta_k$.

The experimental results of Tang, Carlet and Tang suggest that the following recursive formula is verified:

$$\Delta_{k+1} = \begin{cases} 2\Delta_k + 1 & \text{if } k \text{ even}, \\ 2\Delta_k + 1 - \Gamma_{(k-1)/2} & \text{if } k \text{ odd}, \end{cases}$$

where

$$\Gamma_n = 1 + \sum_{w=0}^{n-1} C_w$$

and $C_w = \binom{2w}{w}/(w+1)$ is the w-th Catalan number. Γ_n is the sequence A155587 in OEIS [3].

Further experimental investigations made with Sage [4] show that the minimal value M_k seems to be attained for $t = 1$ if k is even and $t = 3$ if k is odd. In fact, the next proposition gives explicit formulae for $M_{k,1}$ and $M_{k,3}$.

We recall that $r(a,t) = w(a+t) - w(a) - w(t)$ can be interpreted as the number of carries occurring while adding a and t. Then we can describe $M_{k,t}$ as

$$\begin{aligned} M_{k,t} &= \#\left\{a \in \left(\mathbb{Z}/(2^k-1)\mathbb{Z}\right)^* \mid w(a) + w(t+a) = k\right\} \\ &= \#\left\{a \in \left(\mathbb{Z}/(2^k-1)\mathbb{Z}\right)^* \mid 2w(a) + w(t) - r(a,t) = k\right\} \\ &= \#\left\{a \in \left(\mathbb{Z}/(2^k-1)\mathbb{Z}\right)^* \mid r(a,t) = -k + w(t) + 2w(a)\right\} \ . \end{aligned}$$

Proposition 1. *For $k \geq 2$,*

$$M_{k,1} = \sum_{w=1}^{\lfloor (k+1)/2 \rfloor} \binom{2w-2}{w-1} .$$

Proof. We know that $M_{k,1} = M_{k,-1}$, so we enumerate the set of a's verifying $r(a,-1) = 2w(a)-1$ according to $w(a)$ or equivalently $r(a,-1)$. The binary expansion of -1 is `1---10`.

First, for any number $t \in \mathbb{Z}/(2^k-1)\mathbb{Z}$, $0 \leq r(a,t) \leq k$, so we deduce that a must verify $1 \leq w(a) \leq \lfloor (k+1)/2 \rfloor$.

Second, for a given number of carries r, a number a verifying $r(a,-1) = r$ must be of the following form

$$\begin{aligned} -1 &= \texttt{1---1---10} \ , \\ a &= \underbrace{\texttt{????}}_{r}\texttt{10---0} \ . \end{aligned}$$

Such a description is valid even if $r(a,-1) = k$. So, for a given weight w, a number a verifying $w(a) = w$ and $r(a,-1) = 2w-1$ must be of the following form

$$\begin{aligned} -1 &= \texttt{1---1---10} \ , \\ a &= \underbrace{\texttt{????}}_{2w-1}\texttt{10---0} \ , \end{aligned}$$

with the other $w-1$ bits equal to `1` anywhere among the $2w-2$ first bits. Hence there are $\binom{2w-2}{w-1}$ differents a's of weight w verifying $r(a,-1) = 2w-1$.

Finally, summing up on $1 \leq w \leq \lfloor (k+1)/2 \rfloor$, we get that $M_{k,1} = \sum_{w=1}^{\lfloor (k+1)/2 \rfloor} \binom{2w-2}{w-1}$. □

Proposition 2. *For $k \geq 3$,*

$$M_{k,3} = 1 + 2 \sum_{w=1}^{\lfloor k/2 \rfloor} \binom{2w-2}{w-1} .$$

Proof. We proceed as in the proof of Proposition 1. The arguments are only slightly more technical.

We know that $M_{k,3} = M_{k,-3}$, so we enumerate the set of a's verifying $r(a,-3) = 2w(a)-2$ according to $w(a)$ or equivalently $r(a,-3)$. The binary expansion of -3 is `1---100`.

First, from $r(a,-3) = 2w(a)-2$, we deduce that $1 \leq w \leq \lfloor k/2 \rfloor + 1$.

Second, for a given number of carries r, there are now different possibilities.

For any $t \in \mathbb{Z}/(2^k-1)\mathbb{Z}$, there are exactly $\sum_{w=0}^{k-w(t)-1} \binom{k-w(t)}{w}$ different a's producing no carries. Indeed, such a's are characterized by the facts that they have no bits equal to `1` in front of any bit of t equal to `1` and that they can not have only `1`'s in front of the bits of t equal to `0`. For $t = -3$, the such a's are exactly 0, 1 and 2 and both 1 and 2 have weight 1.

Then, for a given number of carries $1 \leq r < 2\lfloor k/2 \rfloor$, a number a verifying $r(a,-3) = r$ cannot have its two last bits (in front of the two bits of -3 equal to `0`) equal to `1`. Otherwise it would produce k carries. So it must be of one of the following forms

$$\begin{aligned} -3 &= \texttt{1---1---100} \ , \\ a &= \underbrace{\texttt{????}}_{r}\texttt{10--0?0} \ , \\ a &= \underbrace{\texttt{???}}_{r-1}\texttt{10----01} \ . \end{aligned}$$

So for a given weight w, a number a verifying $w(a) = w$ and $r(a, -3) = 2w - 2$ must be of one of the following forms

$$\begin{aligned} -3 &= \texttt{1---1---100} \ , \\ a &= \underbrace{\texttt{????10--0?0}}_{2w-2} \ , \\ a &= \underbrace{\texttt{???10----01}}_{2w-3} \ , \end{aligned}$$

with the other $w - 1$ bits set to `1` anywhere among the $2w - 2$ remaining bits in the first case, and the other $w - 2$ bits set to `1` anywhere among the $2w - 4$ first bits in the second one. Hence there are $\binom{2w-2}{w-1} + \binom{2w-4}{w-2}$ differents a's of weight w.

Finally, if k is odd and $w(a) = \lfloor k/2 \rfloor + 1$, then $r(a, t) = k - 1$ and a must be of the following form

$$\begin{aligned} -3 &= \texttt{1---100} \ , \\ a &= \texttt{????101} \ . \end{aligned}$$

There are $\binom{2w-4}{w-2}$ different such a's. And, if k is even and $w(a) = \lfloor k/2 \rfloor + 1$, then $r(a, t) = k$ and a must be of the following form

$$\begin{aligned} -3 &= \texttt{1---100} \ , \\ a &= \texttt{?????11} \ . \end{aligned}$$

There are also $\binom{2w-4}{w-2}$ different such a's.

Therefore, we find that

$$\begin{aligned} M_{k,3} &= 2 + \sum_{w=2}^{\lfloor k/2 \rfloor} \binom{2w-2}{w-1} + \sum_{w=2}^{\lfloor k/2 \rfloor + 1} \binom{2w-4}{w-2} \\ &= 1 + 2 \sum_{w=1}^{\lfloor k/2 \rfloor} \binom{2w-2}{w-1} \ . \end{aligned}$$

□

We now prove recurrence relations for $M_{k,1}$ and $M_{k,3}$.

Corollary 1. *If k is even, then*

$$2M_{k,1} + 1 = M_{k+1,3} \ .$$

If k is odd, then

$$M_{k,3} - \Gamma_{(k-1)/2} = (M_{k+1,1} - 1)/2 \ .$$

Proof. The first equality is a simple consequence of the fact $\lfloor k/2 \rfloor = \lfloor (k+1)/2 \rfloor$ when k is even.

For the second one, we write

$$\begin{aligned} M_{k,3} - \Gamma_{(k-1)/2} &= 1 + 2 \sum_{w=1}^{\lfloor k/2 \rfloor} \binom{2w-2}{w-1} - 1 - \sum_{w=0}^{(k-3)/2} \binom{2w}{w} / (w+1) \\ &= 2 \sum_{w=1}^{(k-1)/2} \binom{2w-2}{w-1} - \sum_{w=0}^{(k-3)/2} \binom{2w}{w} / (w+1) \\ &= 1 + \sum_{w=1}^{(k-3)/2} (2 - 1/(w+1)) \binom{2w}{w} \ , \end{aligned}$$

and

$$(M_{k+1,1}-1)/2 = \sum_{w=2}^{\lfloor k/2\rfloor+1} \binom{2w-2}{w-1}/2$$
$$= \sum_{w=1}^{(k-1)/2} \binom{2w}{w}/2 \ ,$$

so that we can equivalently show that

$$\binom{k-1}{(k-1)/2} - 2 = \sum_{w=1}^{(k-3)/2} (3-2/(w+1))\binom{2w}{w} \ ,$$

which follows from a simple induction. For $k=3$, this reduces to $0=0$ which is indeed true; for $k>3$ odd, we have

$$\binom{k+1}{(k+1)/2} - 2 = 4k/(k+1)\binom{k-1}{(k-1)/2} - 2$$
$$= (4-1/(k+1))\binom{k-1}{(k-1)/2} - 2 \ ,$$

and

$$\sum_{w=1}^{(k-1)/2} (3-2/(w+1))\binom{2w}{w} = \left[\sum_{w=1}^{(k-3)/2} (3-2/(w+1))\binom{2w}{w}\right] + (3-1/(k+1))\binom{k-1}{(k-1)/2} \ .$$

□

To conclude this section, let us note that $M_{k,1} \leq M_{k,3}$ if k is even and $M_{k,3} \leq M_{k,1}$ if k is odd. So, if we assume that these are indeed the minimal values M_k according to the parity of k, then Δ_k is given by

$$\Delta_k = \begin{cases} (M_{k,1}-1)/2 & \text{if } k \text{ even,} \\ (M_{k,3}-1)/2 & \text{if } k \text{ odd,} \end{cases}$$

and the recursive formulae are proved.

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